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On a construction of a recurrent potential kernel by mean of time change and killing

By Yoichi ŌSHIMA

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§1. Introduction.

Let E be a locally compact Hausdorff space with countable base, $\mathcal{E}$ be the σ-field of Borel subsets of E and $X=(\Omega, \mathcal{F}, \mathcal{F}_t)_{t\geqq 0}, (X_t)_{t\geqq 0}, (\theta_t)_{t\geqq 0}, (P^x)_{x\in E})$ be a Hunt process on $(E, \mathcal{E})$. The constructions of the (weak) potential kernel of X were given by many authors ([6], [9], [11], [13]). In this paper we shall give a construction by mean of time change and killing. Let $A=(A_t)_{t\geqq 0}$ be a non-trivial non-negative continuous additive functional of X such that $A_t<\infty$ a.s. for all $t<\infty$. Let $K^0_{P,C}$ and $G^0_{P,C}$ be the resolvent of the time changed process corresponding to the additive functional A^C and the potential kernel of the subprocess of X corresponding to the multiplicative functional $(e^{-PA^C_t})_{t\geqq 0}$, respectively, where A^C is defined by

$$A^C_t=\int_0^t I_C(X_s)dA_s$$

for a Borel subset C of E. Then for a suitably chosen Borel subset C of E there exists a potential kernel K_C of $K^0_{1,C}$ restricted to $C\times C$ and the kernel defined by

$$K(x, dy)=G^0_{1,C}(x, dy)+K^0_{1,C}K_CG^0_{1,C}(x, dy)$$

is a potential kernel of X. If there exists a dual Hunt process $\hat{X}$ of X relative to the invariant measure μ of X then the kernels K and $\hat{K}$ defined as above by A^C and $\hat{A}^C$ are in dual relative to μ, where $\hat{A}$ is the dual continuous additive functional of A. By these method, we can construct, explicitly, the potential kernel of one dimensional non-singular diffusion processes.

§2. Construction of a potential kernel.

Throughout in this paper we shall assume that X is a recurrent Hunt process on $(E, \mathcal{E})$, that is, it satisfies the following equivalent conditions (Azema-Duflo-Revuz [1], Blumenthal-Getoor [5] problems II.4.17-4.20).

(i) *If* $B\in\mathcal{E}^n$ *then either* $U(x, B)=E^x\left[\int_0^\infty I_B(X_t)dt\right]\equiv 0$ *or* $U(x, B)\equiv\infty$.

(ii) *If* $B\in\mathcal{E}^n$ *then either* $P^x[T_B<\infty]\equiv 0$ *or* $P^x[T_B<\infty]\equiv 1$.

(iii) *The only excessive functions are constants.*

Here $\mathcal{E}^n$ is the σ-field of the nearly Borel measurable subsets of E. Then, from Azema-Duflo-Revuz [**2**], there exists a unique (up to the multiplicative constants) σ-finite invariant measure μ of $\boldsymbol{X}$ which is equivalent to $U^p(x, \cdot)=E^x\left[\int_0^\infty e^{-pt}I_{\{\cdot\}}(X_t)dt\right]$ for all $x\in E$ and $p\geqq 0$. Hence, in particular, any p-excessive function is $\mathcal{E}$-measurable ([**5**] proposition V.1.4).

Assume, furthermore, that we are given a non-negative continuous additive functional $\boldsymbol{A}=(A_t)_{t\geqq 0}$ of $\boldsymbol{X}$ satisfying $A_t<\infty$ a. s. for all $t<\infty$ and $P^x[A_\infty=0]\neq 1$ for some $x\in E$. Then, from [**1**], $P^x[A_\infty=\infty]=1$ for all $x\in E$. For any Borel subset C of E define a continuous additive functional $\boldsymbol{A}^C$ as in § 1. Then $\boldsymbol{A}^C$ vanishes on the complement of C. Denote $b\mathcal{E}$ and $b\mathcal{E}^+$ the classes of all bounded $\mathcal{E}$-measurable and all non-negative bounded $\mathcal{E}$-measurable functions, respectively. For r, $p\geqq 0$ and $f\in b\varepsilon$, define

$$K^r_{p,C}f(x)=E^x\left[\int_0^\infty e^{-pA^C_t-rt}f(X_t)dA^C_t\right] \tag{2.1}$$

$$G^r_{p,C}f(x)=E^x\left[\int_0^\infty e^{-pA^C_t-rt}f(X_t)dt\right]. \tag{2.2}$$

Then $G^r_{0,C}=U^r$. Set $K^r_p=K^r_{p,E}$ and $G^r_p=G^r_{p,E}$.

THEOREM 2.1. (Nagasawa-Sato [**10**] Theorems 2.1 and 2.2). *For any* p, $q>0$, r, $s\geqq 0$, $f\in b\mathcal{E}$ *and* $C\in\mathcal{E}$,

$$K^r_{p,C}f-K^s_{q,C}f+(p-q)K^r_{p,C}K^s_{q,C}f+(r-s)G^r_{p,C}K^s_{q,C}f=0\,, \tag{2.3}$$

$$G^p_{r,C}f-G^q_{s,C}f+(p-q)G^p_{r,C}G^q_{s,C}f+(r-s)K^p_{r,C}G^q_{s,C}f=0\,. \tag{2.4}$$

In particular, if $K^{r_0}_{0,C}1$ *(resp.* $G^0_{r_0,C}(\cdot, B)$*) is bounded for some* $r_0\geqq 0$, *then* $K^r_{0,C}f$ *(resp.* $G^0_{r,C}f$*) is bounded for arbitrary* $r>0$ *and* $f\in b\mathcal{E}$ *and, furthermore,* (2.3) *(rep.* (2.4)*) holds for all* $f\in b\mathcal{E}$ *(resp.* $f\in b\mathcal{E}$, $f=0$ *on* B^C*) and* p, q, r, $s\geqq 0$ *such that* $p+r>0$ *and* $q+s>0$.

Let ν be the measure associated with $\boldsymbol{A}$ (see [**2**], [**12**]), that is, $\nu(\cdot)=\mu K^1_0(\cdot)$. Then,

LEMMA 2.2. *There exists an increasing sequence* $\{E_n\}_{n\geqq 1}$ *of Borel subsets of* E *satisfying* $\bigcup_n E_n=E$, $\nu(E_n)<\infty$ *and* $K^1_{0,E_n}1$ *is bounded for every* n.

PROOF. The proof is similar to the proof of Revuz [**12**], Theorem III.1. Set $C=E$, $r=0$, $p=q=s=1$ at (2.4) then we have

$$K^1_0G^1_1f=U^1f-G^1_1f\leqq U^1f$$

$$\mu K_0^1 G_1^1 f \leqq \langle \mu, f \rangle \equiv \int_E f(x)\mu(dx)$$

for any $f \in b\mathcal{E}_+$. If f is a strictly positive μ-integrable function then $0 < G_1^1 f \leqq \|f\| < \infty$ and, since $U^1 f$ and $U^1 f - G_1^1 f$ are 1-excessive ([**5**], Corollary III.4.10), $G_1^1 f = U^1 f - (U^1 f - G_1^1 f)$ is a fine continuous Borel measurable function on E. Therefore the sets $E_n = \{x : G_1^1 f(x) \geqq 1/n\}$ are fine closed Borel subsets which increase to E as $n \uparrow \infty$ and satisfy $K_0^1(x, E_n) \leqq n\|f\|$ and

$$\nu(E_n) \leqq n \mu K_0^1 G_1^1 f \leqq \langle \mu, f \rangle < \infty .$$

LEMMA 2.3. *The measure ν is the unique invariant measure of K_1^0 and which is equivalent to $K_1^0(x, \cdot)$ for any $x \in E$.*

PROOF. The invariance and uniqueness were proved by Revuz [**12**] proposition III.4. Clearly the measures $K_1^0(x, \cdot)$ and $K_0^1(x, \cdot)$ are equivalent, so we shall prove the equivalence of ν and $K_0^1(x, \cdot)$. Obviously $K_0^1(x, B) \equiv 0$ implies $\nu(B) = 0$. Suppose, on the contrary, that $K_0^1(x, B) > 0$ for some $x \in E$ and $B \in \mathcal{E}$ then, since $K_0^1(\cdot, B)$ is 1-excessive ([**5**] Proposition IV.2.4), there exists a fine neighbourhood W of x such that $K_0^1(y, B) \geqq a > 0$ for all $y \in W$. Therefore,

$$\nu(B) = \int \mu(dy) K_0^1(y, B) \geqq a\mu(W) > 0$$

since $U^1(x, W) > 0$.

By Lemma 2.3, there exists an $\mathcal{E} \times \mathcal{E}$-measurable density $g_1^0(x, y)$ of $K_1^0(x, \cdot)$ relative to ν since $\mathcal{E}$ is countably generated. For a set $B \in \mathcal{E}$ such that $0 < \nu(B) < \infty$, a positive integer n and a real number $r \in (1/2, 1)$, define

$$K(B, r, n) = \left\{ x \in B : \nu\left\{ y \in B : g_1^0(x, y) > \frac{1}{n} \right\} > r\nu(B) \right\} \tag{2.5}$$

then $B = \bigcup_{n=1}^{\infty} K(B, r, n)$.

LEMMA 2.4. *If C is a Borel subset of $K(B, r_0, n)$ such that $\nu(C) > 2(1 - r_0)\nu(B)$ for some $n \geqq 1$ and $r_0 \in (1/2, 1)$ then $C = K(C, r, n)$ for any $r \in (1/2, 1 - (1 - r_0)\nu(B)/\nu(C))$.*

LEMMA 2.5. *If $C \in \mathcal{E}$ satisfies $0 < \nu(C) < \infty$ and $C = K(C, r, n)$ for some $n \geqq 1$ and $r \in (1/2, 1)$ then*

$$\sup_{x \in C, y \in C} \frac{1}{2} \| K_{1,C}^0(x, \cdot) - K_{1,C}^0(y, \cdot) \| \equiv k_C < 1 .$$

The proofs of Lemmas 2.4 and 2.5 are similar to the proof of Theorem II.1 of Revuz [**13**], where he proved these results in the case $A_t = \int_0^t f(X_s) ds$ $(f \in b\mathcal{E}_+)$ and $\nu = \mu$, so the proofs are omitted.

Take a positive integer k satisfying $0 < \nu(E_k) < \infty$, where $E_k \in \mathcal{E}$ is the set introduced in Lemma 2.2. Let $B = E_k$ at Lemma 2.4, then there is a subset

$C\in\mathcal{E}$ of E_k satisfying $0<\nu(C)<\infty$ and $C=K(C, r, n)$ for some $r\in(1/2, 1)$ and $n\geqq 1$. Obviously $A_\infty^C=\infty$ a. s. by $\nu(C)>0$ and A^C vanishes outside of C. Moreover, since $\nu_C(\cdot)=\nu(\cdot\cap C)$ is the measure associated with A^C, ν_C is the invariant measure of $K_{1,C}^0$. Hence, again similarly to Lemma III.2 of [**13**],

LEMMA 2.6. *Under the conditions of Lemma* 2.5,

$$\sup_{x\in C}\left\|(K_{1,C}^0)^n(x, \cdot)-\frac{\nu_C(\cdot)}{\nu(C)}\right\|\leqq 2k_C^n .$$

THEOREM 2.7. $\mu=constant\times\nu_C G_{1,C}^0$.

PROOF. From Theorem 2.1,

$$G_{1,C}^0 K_{1,C}^1 f=K_{1,C}^0 f-K_{1,C}^1 f\leqq K_{1,C}^0 f\leqq\|f\|$$

for any $f\in b\mathcal{E}_+$. Since $A_\infty^C=\infty$ a. s. and $A_t^C<\infty$ a. s. for all $t<\infty$, for a strictly positive bounded measurable function f, $K_{1,C}^1 f(x)>0$ for all $x\in E$. Then, by the similar argument to the proof of Lemma 2.2, there exists a sequence $\{F_n\}_{n\geqq 1}$ which increases to E such that $G_{1,C}^0(x, F_n)$ is bounded for every n. Hence $\nu_C G_{1,C}^0$ is a σ-finite measure. Since $G_{1,C}^0(x, F_n)$ is bounded for all n, again by Theorem 2.1, for any $f\in b\mathcal{E}$ which vanishes outside of some F_n,

$$G_{1,C}^0 f-U^1 f-G_{1,C}^0 U^1 f+K_{1,C}^0 U^1 f=0 .$$

Integrating by ν_C it follows that

$$\nu_C G_{1,C}^0 f=\nu_C G_{1,C}^0 U^1 f$$

since ν_C is an invariant measure of $K_{1,C}^0$. Therefore $\nu_C G_{1,C}^0$ is an invariant measure of $\boldsymbol{X}$, hence, by the uniqueness of the invariant measure of $\boldsymbol{X}$, the theorem follows.

For simplicity, we shall assume, in the following, that the constant of the Theorem 2.7 equals to 1, that is, $\mu=\nu_C G_{1,C}^0$.

COROLLARY. *If there exists a local time* $\boldsymbol{A}$ *for* $\boldsymbol{X}$ *at* $x_0\in E$ *then the measure* $G_1^0(x_0, \cdot)$ *is the invariant measure of* $\boldsymbol{X}$, *where* G_1^0 *is the kernel defined by the local time* $\boldsymbol{A}$ *as before.*

For $x\in C$ and $B\in\mathcal{E}$ define a kernel K_C by

$$K_C(x, B)=I(x, B\cap C)+\sum_{n=1}^{\infty}\left[(K_{1,C}^0)^n(x, B)-\frac{\nu_C(B)}{\nu(C)}\right] \tag{2.6}$$

where I is the identity kernel. From Lemma 2.6, K_C is well defined and $K_C(x, \cdot)$ is a bounded signed measure which vanishes outside of C for all $x\in C$.

LEMMA 2.8. *For any* $f\in b\mathcal{E}$,

$$(I-K_{1,C}^0)K_C f=f-\frac{1}{\nu(C)}\langle\nu_C, f\rangle \qquad on\ \ C . \tag{2.7}$$

Let $\boldsymbol{D}$ be the set of all $f \in b\mathcal{E}$ such that $G_{1,C}^{0}|f|$ is bounded. Then $\boldsymbol{D}$ contains all functions $f \in b\mathcal{E}$ such that $f=0$ on F_n^C for some n, where F_n are the sets in the proof of Lemma 2.6. In particular, if $\boldsymbol{X}$ is strong Feller, any bounded measurable function which vanishes outside a compact set is contained in $\boldsymbol{D}$ ([5] III.5.16, 5.18). Define a kernel K on E by

$$K(x, B) = G_{1C}^{0}(x, B) + K_{1,C}^{0} K_C G_{1,C}^{0}(x, B). \tag{2.8}$$

THEOREM 2.9. *If $f \in \boldsymbol{D}$ then Kf is bounded and satisfies*

$$(I - pU^p)Kf = U^p f - \frac{K_{0,C}^{p}(\cdot, C)}{\nu(C)} \langle \mu, f \rangle. \tag{2.9}$$

If, moreover, $\langle \mu, f \rangle = 0$ then

$$(I - pU^p)Kf = U^p f \tag{2.10}$$

that is, K is a potential kernel of $\boldsymbol{X}$.

PROOF. If $f \in \boldsymbol{D}$ then the boundedness of Kf is obvious. Moreover, from Theorem 2.7, f is μ-integrable. Let $r=q=0$ and $s=1$ at (2.4), then

$$pU^p G_{1,C}^{0} f = K_{0,C}^{p} G_{1,C}^{0} f + G_{1,C}^{0} f - U^p f. \tag{2.11}$$

Similarly, let $p=s=0$, $q=1$ and p for r, then

$$pU^p K_{1,C}^{0} g = K_{0,C}^{p} K_{1,C}^{0} g + K_{1,C}^{0} g - K_{0,C}^{p} g \tag{2.12}$$

for any $g \in b\mathcal{E}$. Since $K_{0,C}^{p}(x, \cdot)$ vanishes outside of C, by setting $g = K_C G_{1,C}^{0} f$, we obtain from Theorem 2.7 and Lemma 2.8,

$$\begin{aligned} pU^p K_{1,C}^{0} K_C G_{1,C}^{0} f &= K_{0,C}^{p} K_{1,C}^{0} K_C G_{1,C}^{0} f + K_{1,C}^{0} K_C G_{1,C}^{0} f \\ &\quad - K_{0,C}^{p} K_C G_{1,C}^{0} f \\ &= K_{0,C}^{p}\Big(K_C G_{1,C}^{0} f - G_{1,C}^{0} f + \frac{1}{\nu(C)} \langle \nu_C, G_{1,C}^{0} f \rangle\Big) \\ &\quad + K_{1,C}^{0} K_C G_{1,C}^{0} f - K_{0,C}^{p} K_C G_{1,C}^{0} f \\ &= -K_{0,C}^{p} G_{1,C}^{0} f + \frac{K_{0,C}^{p}(\cdot, C)}{\nu(C)} \langle \mu, f \rangle + K_{1,C}^{0} K_C G_{1,C}^{0} f. \end{aligned} \tag{2.13}$$

From (2.11) and (2.13) we obtain

$$\begin{aligned} pU^p Kf &= pU^p G_{1,C}^{0} f + pU^p K_{1,C}^{0} K_C G_{1,C}^{0} f \\ &= Kf - U^p f + \frac{K_{0,C}^{p}(\cdot, C)}{\nu(C)} \langle \mu, f \rangle. \end{aligned}$$

§ 3. The case when the dual process exists.

In this section we shall assume that X is a recurrent Hunt process on E with the invariant measure μ and that the dual Hunt process $\hat{X}$ of X relative to μ exists. Let $\hat{U}^p(dx, y)$ be the resolvent of $\hat{X}$ then for any $f \in b\mathcal{E}_+$ and $p>0$,

$$\langle \mu, f\hat{U}^p \rangle = \iint f(x)\hat{U}^p(dx, y)\mu(dy) = \int f(x)U^p 1(x)\mu(dy) = \frac{1}{p}\langle \mu, f \rangle .$$

Hence μ is an invariant measure of $\hat{X}$. It is well known that, for any $p>0$, there exists a bimeasurable function $u^p(x, y)$ satisfying

(i) $U^p(x, dy) = u^p(x, y)\mu(dy)$ *and* $\hat{U}^p(dx, y) = \mu(dx)u^p(x, y)$,

(ii) $u^p(\cdot, y)$ *and* $u^p(x, \cdot)$ *are a p-excessive and p-coexcessive functions for any y and x, respectively.*

Set $u(x, y) = \lim_{p \to 0} u^p(x, y)$ then $u(\cdot, y)$ and $u(x, \cdot)$ are an excessive and a coexcessive functions, respectively.

LEMMA 3.1.

$$u(x, y) = \infty \qquad \text{for all} \quad x, y .$$

PROOF. Let y be an arbitrary point in E then, for any non-empty nearly Borel cofine neighbourhood W of y, $\mu(W)>0$ since $\hat{U}^p(W, y)>0$ for any $p>0$. Moreover, we may assume that $\mu(W)<\infty$ for all small W (see [2] IV.1). For such W, the recurrence of X implies that

$$\int_W u(x, y)\mu(dy) = v(x, W) = \infty \qquad \text{for all} \quad x \in E .$$

Hence from the cofine continuity of $u(x, \cdot)$ the lemma follows.

From Lemma 3.1, $\hat{X}$ is a recurrent Hunt process. Let A be the continuous additive functional of X given in § 1 and ν be the measure associated with A in § 2. Then by Revuz [12] VII.1 (cf. [4] and [7]), there exists a polar set P and a continuous additive functional $\hat{A}$ of $\hat{X}$ restricted to $E-P$ satisfying $A_t<\infty$ a. s. for any $t<\infty$ and which is associated with ν. By the recurrence of $\hat{X}$, $\hat{A}_\infty=\infty$ a. s. Since μ and ν do not charge for any polar set, $\mu(P)=\nu(P)=0$.

Define $\hat{K}^r_{p,C}(B, y)$ and $\hat{G}^r_{p,C}(B, y)$ as (2.1) and (2.2) by $(\hat{X}, \hat{A})$, then there exists a Borel subset C of $E-P$ satisfying $0<\nu(C)<\infty$, $C=K(C, r, n)=\hat{K}(C, r, n)$ for some $r \in (1/2, 1)$ and $n \geqq 1$ from Lemma 2.4, where $\hat{K}(C, r, n)$ is defined as (2.5) by $(\hat{X}, \hat{A})$. For such C define $\hat{K}_C(B, y)$ and $\hat{K}(B, y)$ by means of $\hat{K}^0_{1,C}$ and $\hat{G}^0_{1,C}$ as (2.6) and (2.8) for $B \subset E-P$, $y \in E-P$. Set $\hat{K}(B, y)=0$ for $B \subset P$, $y \notin P$ and

$$\hat{K}(B, y) = p\hat{K}\hat{U}_p(B, y) + \hat{U}_p(B, y)$$

for $B\subset E$, $y\in P$. Then $\hat{K}$ is a potential kernel of $\hat{X}$. Let $\hat{D}$ be the set of all functions $f\in b\mathcal{E}$ such that $\hat{G}_{1,C}^{0}|f|$ is bounded.

THEOREM 3.1. *For any* f, $g\in D\cap\hat{D}$,

$$\int_{E}g(x)Kf(x)\mu(dx)=\int_{E}g\hat{K}(y)f(y)\mu(dy)\,. \tag{3.1}$$

PROOF. Let f, $g\in D\cap\hat{D}$, then by the definitions of K and $\hat{K}$, it is enough to prove $\langle\mu, g\rangle\langle\nu_C, G_{1,C}^{0}f\rangle=\langle\nu_C, g\hat{G}_{1,C}^{0}\rangle\langle\mu, f\rangle$ and

$$\int_{E}g(x)(K_{1,C}^{0})^{n}G_{1,C}^{0}f(x)\mu(dx)=\int_{E}g\hat{G}_{1,C}^{0}(\hat{K}_{1,C}^{0})^{n}(y)f(y)\mu(dy) \tag{3.2}$$

for all $n\geqq 0$. It is known by [12] theorem VII.2 that there exists a bimeasurable function $g_{1,C}^{0}(x, y)$ such that

$$G_{1,C}^{0}(x, dy)=g_{1,C}^{0}(x, y)\mu(dy)\,,\qquad \hat{G}_{1,C}^{0}(dx, y)=\mu(dx)g_{1,C}^{0}(x, y)$$

$$K_{1,C}^{0}(x, dy)=g_{1,C}^{0}(x, y)\nu_C(dy)\quad\text{and}\quad \hat{K}_{1,C}^{0}(dx, y)=\nu_C(dx)g_{1,C}^{0}(x, y)$$

for all $x\in E$ and $y\in E-P$. Hence we have

$$\int g(x)K_{1,C}^{0}G_{1,C}^{0}f(x)\mu(dx)=\iiint g(x)g_{1,C}^{0}(x, y)g_{1,C}^{0}(y, z)f(z)\mu(dx)\nu_C(dy)\mu(dz)$$

$$=\int g\hat{G}_{1,C}^{0}\hat{K}_{1,C}^{0}(z)f(z)\mu(dz)\,.$$

Similarly (3.2) holds for arbitrary $n\geqq 0$. From Theorem 2.7,

$$\langle\mu, g\rangle\langle\nu_C, G_{1,C}^{0}f\rangle=\langle\nu_C, g\hat{G}_{1,C}^{0}\rangle\langle\mu, f\rangle=\langle\mu, g\rangle\langle\mu, f\rangle\,.$$

Hence the theorem follows.

§ 4. Examples.

Let X be a non-singular recurrent diffusion process on $Q=(-\infty, \infty)$ with natural scale and speed measure m. Let A be the local time of X at 0 and h_1 (resp. h_2) be the strictly positive non-decreasing (resp. non-increasing) solution of the following equations.

$$dh^{+}(x)=0 \qquad \text{for}\quad x\neq 0\,, \tag{4.1}$$

$$h^{+}(0)-h^{+}(0-)=h(0)\,, \tag{4.2}$$

$$h_1^{+}h_2-h_1h_2^{+}\equiv 1\,. \tag{4.3}$$

Then the Green function $g_{1,}^{0}(x, y)$ is given by

$$g_1^{0}(x, y)=g_1^{0}(y, x)=h_1(x)h_2(y)\quad\text{and}\quad x\leqq y \tag{4.4}$$

(see [8], 5.6 and problem 4.11.9). Hence for any continuous function f with compact support,

$$G_1^0 f(x) = \int_Q g_1^0(x, y) f(y) m(dy) \tag{4.5}$$

$$= \int_Q \frac{|x| + |y| - |x-y| + 2}{2} f(y) m(dy),$$

which shows that $G_1^0(0, dy) = m(dy)$ is the invariant measure and

$$K(x, dy) = \frac{|x| + |y| - |x-y| + 4}{2} m(dy) \tag{4.6}$$

is a potential kernel of $\boldsymbol{X}$ by the Corollary of Theorem 2.7 and Theorem 3.9.

If $\boldsymbol{X}$ is a recurrent non-singular diffusion process on $Q = [0, \infty)$ or $Q = [0, 1]$ with natural scale and speed measure m, then by solving (4.1), (4.3) and

$$g_1^+(0) = g_1(0) \tag{4.2$'$}$$

we can see that (4.6) is a potential kernel of $\boldsymbol{X}$.

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Yoichi Ōshima
Department of Mathematics
Faculty of Engineering
Kumamoto University
Kurokami-machi
Kumamoto, Japan