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Mean ergodic theorems for semigroups of positive linear operators

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Introduction.

The mean ergodic theorem was given first by von Neumann [**10**], and has been generalized to semigroups of operators more general than the discrete semigroup $\{T^n : n\geqq 0\}$ by Alaoglu and Birkhoff [**1**], Eberlein [**2**], and many others, see [**6**, VIII. 10]. Let $\mathfrak{S}$ be a semigroup of bounded linear operators on a Banach space. Then the mean ergodic theorem is concerned with the existence and uniqueness of a fixed point of $\mathfrak{S}$ in the closed convex hull of the orbit under $\mathfrak{S}$. The main technique in the mean ergodic theorem is based on various weak compactness properties of orbits, and the (left) amenability condition for semigroups is useful in view of Day's fixed point theorem [**4**].

In this paper, let $(X, \mathcal{F}, m)$ be a σ-finite measure space. We shall study mean ergodic properties of semigroups of bounded linear operators on $L_1(X) =L_1(X, \mathcal{F}, m)$, and determine the structure of those semigroups for which the mean ergodic theorem holds. In particular, we shall consider amenable semigroups of uniformly bounded positive linear operators on $L_1(X)$, and also consider general semigroups of positive linear contractions on $L_1(X)$. In §1 we shall obtain three decomposition theorems from the viewpoint of the mean ergodic theory, applying Takahashi [**11, 12**] and Nagel [**9**]. In §2 several criteria will be given, in connection with the decompositions in §1, which are equivalent to the condition that the mean ergodic theorem holds on the whole space $L_1(X)$. In §3 other necessary and sufficient conditions will be given for the k-parameter semigroup and the discrete semigroup, by means of weak compactness properties of orbits.

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§1. Decomposition theorems.

Throughout this paper, let $(X, \mathcal{F}, m)$ be a σ-finite measure space and let $L_1(X)=L_1(X, \mathcal{F}, m)$ and $L_\infty(X)=L_\infty(X, \mathcal{F}, m)$ be the usual Banach spaces of (equivalence classes of) real-valued $\mathcal{F}$-measurable functions on X whose norms are denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively. The space $L_\infty(X)$ is the dual Banach space of $L_1(X)$ by the bilinear form $\langle f, u\rangle=\int f\cdot u\,dm$, where $f\in L_1(X)$ and $u\in L_\infty(X)$. Let $L_1^+(X)$ denote the class of nonnegative functions in $L_1(X)$. For $A\in\mathcal{F}$, 1_A denotes the characteristic function of A, and $L_1(A)$ $[L_\infty(A)]$ denotes the class of functions f in $L_1(X)$ $[L_\infty(X)]$ such that $f=0$ a. e. on $X\backslash A$. Let $f_A=f\cdot 1_A$ for $f\in L_1(X)$. A linear operator T on $L_1(X)$ or on $L_\infty(X)$ is called positive if $f\geqq 0$ a. e. implies $Tf\geqq 0$ a. e., and a contraction if $\|T\|\leqq 1$.

Let $\mathfrak{S}$ be an abstract semigroup and let $m(\mathfrak{S})$ be the Banach space of all bounded real-valued functions on $\mathfrak{S}$ with the supremum norm. A bounded linear functional μ of $m(\mathfrak{S})$ is called a *mean* on $m(\mathfrak{S})$ if $\|\mu\|=\mu(1)=1$. If μ is a mean on $m(\mathfrak{S})$, then

$$\inf\{\xi(s): s\in\mathfrak{S}\}\leqq\mu(\xi)\leqq\sup\{\xi(s): s\in\mathfrak{S}\}\,,\qquad \xi\in m(\mathfrak{S})\,.$$

A mean μ is called *left* [*right*] *invariant* if $\mu({}_s\xi)=\mu(\xi)$ $[\mu(\xi_s)=\mu(\xi)]$ for all $\xi\in m(\mathfrak{S})$ and $s\in\mathfrak{S}$, where ${}_s\xi(t)=\xi(st)$ and $\xi_s(t)=\xi(ts)$. An *invariant mean* is a left and right invariant mean. A semigroup $\mathfrak{S}$ is called *left* [*right*] *amenable* if there exists a left [right] invariant mean on $m(\mathfrak{S})$, and *amenable* if there exists an invariant mean on $m(\mathfrak{S})$. It is well known [**3**, p. 516] that a commutative semigroup is amenable.

Let $\mathfrak{S}$ be a semigroup of bounded linear operators on $L_1(X)$, which is called uniformly bounded if

$$M=\sup\{\|T\|: T\in\mathfrak{S}\}<\infty\,,$$

M will always denote this supremum. The orbit of $f\in L_1(X)$ under $\mathfrak{S}$ is denoted by $\mathfrak{S}f$, i. e., $\mathfrak{S}f=\{Tf: T\in\mathfrak{S}\}$, and its [closed] convex hull by $\mathrm{co}\,\mathfrak{S}f$ $[\overline{\mathrm{co}}\,\mathfrak{S}f]$. A function $f\in L_1(X)$ is called $\mathfrak{S}$-invariant if $Tf=f$ for all $T\in\mathfrak{S}$. A set $A\in\mathcal{F}$ is called $\mathfrak{S}$-closed if $f\in L_1(A)$ implies $Tf\in L_1(A)$ for all $T\in\mathfrak{S}$. Taking adjoints, $\mathfrak{S}^*=\{T^*: T\in\mathfrak{S}\}$ is a semigroup of bounded linear operators on $L_\infty(X)$. If $\mathfrak{S}$ is amenable, then μ will always denote an invariant mean on $m(\mathfrak{S})$.

In this section we shall give three decomposition theorems. Theorems 1.1 and 1.2 are decomposition theorems for a semigroup of (uniformly bounded) positive linear operators on $L_1(X)$, and Theorem 1.3 is for a semigroup of linear contractions on $L_1(X)$.

THEOREM 1.1. *Let $\mathfrak{S}$ be a semigroup of positive linear operators on $L_1(X)$. Then the space X decomposes uniquely* (*up to an equivalence*) *into two disjoint measurable sets P and N with the following properties:*

(1) *there exists an $\mathfrak{S}$-invariant function $g\in L_1^+(X)$ such that $P=\{g>0\}$*;

(2) *if $f\in L_1^+(X)$ is $\mathfrak{S}$-invariant, then $f\in L_1(P)$*;

(3) *if* $f \in L_1(P)$, *then* $Tf \in L_1(P)$ *for any* $T \in \mathfrak{S}$.

PROOF. Let G be the set of all $\mathfrak{S}$-invariant $f \in L_1^+(X)$, and P the supremum of $\{f>0\}$ $(f \in G)$ which is taken as an element in the measure algebra. Then there exists a sequence $\{g_n\}$ in G such that $P=\cup\{g_n>0\}$. Putting $g=\sum \|g_n\|_1^{-1}2^{-n}g_n$, we obtain a $g \in G$ such that $P=\{g>0\}$. Thus the existence of a set P with properties (1) and (2) is proved and the uniqueness is clear. To prove (3), let $f \in L_1^+(P)$. Then $\|\min(f, ng)-f\|_1 \to 0$ as $n \to \infty$ by Lebesgue's convergence theorem, and so $\|T(\min(f, ng))-Tf\|_1 \to 0$ for any $T \in \mathfrak{S}$. Since $T(\min(f, ng)) \leqq T(ng)=0$ on $N=X \backslash P$, it follows that $Tf \in L_1(P)$ for all $T \in \mathfrak{S}$.

Q. E. D.

THEOREM 1.2. *Let* $\mathfrak{S}$ *be a semigroup of uniformly bounded positive linear operators on* $L_1(X)$. *Then the space* X *decomposes uniquely (up to an equivalence) into two disjoint measurable sets* W *and* V *with the following properties:*

(1) *there exists a function* $g \in L_1^+(X)$ *such that* $W=\{g>0\}$ *and* $\mathfrak{S}g$ *is weakly sequentially compact;*

(2) *if* $f \in L_1^+(X)$ *and* $\mathfrak{S}f$ *is weakly sequentially compact, then* $f \in L_1(W)$;

(3) *if* $f \in L_1(W)$, *then* $Tf \in L_1(W)$ *for any* $T \in \mathfrak{S}$.

Furthermore assume that $\mathfrak{S}$ *is amenable. Then there exists a linear projection* Q *of* $L_1(W)$ *onto the subspace of* $\mathfrak{S}$*-invariant functions such that, for each* $f \in L_1(W)$, Qf *is a unique* $\mathfrak{S}$*-invariant function contained in* $\overline{\mathrm{co}}\,\mathfrak{S}f$, *and such that* $Q=TQ=QT$ *on* $L_1(W)$ *for all* $T \in \mathfrak{S}$.

PROOF. Let G be the set of all $f \in L_1^+(X)$ such that $\mathfrak{S}f$ is weakly sequentially compact, and W the supremum of $\{f>0\}$ $(f \in G)$ in the measure algebra. Take a sequence $\{g_n\}$ in G such that $W=\cup\{g_n>0\}$, and put $g=\sum \|g_n\|_1^{-1}2^{-n}g_n$. To show $g \in G$, let $\{T_k\}$ be any sequence from $\mathfrak{S}$. Since, for each n, $\{T_k g_n\}_k$ has a weakly convergent subsequence, we can extract a subsequence $\{S_j\}$ of $\{T_k\}$ such that $\{S_j g_n\}_j$ converges weakly to some $h_n \in L_1(X)$ with $\|h_n\|_1 \leqq M\|g_n\|_1$ for each n. Putting $h=\sum \|g_n\|_1^{-1}2^{-n}h_n$, it is easy to see that $S_j g \to h$ (weakly). Thus the existence of a set W with properties (1) and (2) is proved and the uniqueness is clear. We continue the proof as in Takahashi [**11**, pp. 140-141]. Let $f \in L_1(W)$ and $\varepsilon>0$ be given. By Lebesgue's convergence theorem, there exists an $n>0$ such that

$$\| |f| - \min(|f|, ng)\|_1 < \varepsilon/2M.$$

Putting $h=\min(|f|, ng)$, we have

$$\begin{aligned} |\langle Tf, 1_A\rangle| &\leqq \langle T|f|, 1_A\rangle \\ &= \langle Th, 1_A\rangle + \langle T|f|-Th, 1_A\rangle \\ &\leqq \langle T(ng), 1_A\rangle + \|T(|f|-h)\|_1 \end{aligned}$$

$$\leqq n\langle Tg, 1_A\rangle+\varepsilon/2\,, \qquad T\in\mathfrak{S},\ A\in\mathfrak{F}\,.$$

This shows that the countable additivity of the integrals $\langle Tf, 1_A\rangle$ is uniform with respect to $T\in\mathfrak{S}$. Thus it follows that $\mathfrak{S}f$ is weakly sequentially compact (cf. [6, p. 292]), so that $\overline{\text{co}}\,\mathfrak{S}f$ is weakly compact (cf. [6, p. 430 and p. 434]). Now we have $L_1^+(W)=G$. If $f\in G$, then, for any $T\in\mathfrak{S}$, it follows from $\mathfrak{S}(Tf)\subset\mathfrak{S}f$ that $Tf\in G$. Hence (3) is obtained.

Furthermore assume $\mathfrak{S}$ to be amenable, and let $f\in L_1(W)$. Since $\mathfrak{S}$ is left amenable, Day's fixed point theorem implies that $\overline{\text{co}}\,\mathfrak{S}f$ contains an $\mathfrak{S}$-invariant function. For an $\mathfrak{S}$-invariant function h in $\overline{\text{co}}\,\mathfrak{S}f$, since $\mathfrak{S}$ is also right amenable, it is easily verified that $\langle h, u\rangle=\mu_T\langle Tf, u\rangle$ for any $u\in L_\infty(X)$, where $\mu_T\langle Tf, u\rangle$ denotes $\mu(\xi)$ of $\xi(T)=\langle Tf, u\rangle$. Thus we conclude that $\overline{\text{co}}\,\mathfrak{S}f$ contains a unique $\mathfrak{S}$-invariant function Qf which is determined by the equation $\langle Qf, u\rangle=\mu_T\langle Tf, u\rangle$ for any $u\in L_\infty(X)$. Now the stated properties of Q are straightforward. Q. E. D.

THEOREM 1.3. *Let $\mathfrak{S}$ be a semigroup of linear contractions on $L_1(X)$. Then the space X decomposes uniquely (up to an equivalence) into two disjoint measurable sets P and N with the following properties:*

(1) *there exists an $\mathfrak{S}$-invariant function $g\in L_1(X)$ such that $P=\{g\neq 0\}$;*

(2) *if $f\in L_1(X)$ is $\mathfrak{S}$-invariant, then $f\in L_1(P)$;*

(3) *if $f\in L_1(P)$, then $Tf\in L_1(P)$ for any $T\in\mathfrak{S}$.*

Moreover there exists a linear projection Q of $L_1(P)$ onto the subspace of $\mathfrak{S}$-invariant functions such that, for each $f\in L_1(P)$, Qf is a unique $\mathfrak{S}$-invariant function contained in $\overline{\text{co}}\,\mathfrak{S}f$, and such that $Q=TQ=QT$ on $L_1(P)$ for all $T\in\mathfrak{S}$.

PROOF. Let $|T|$ be the linear modulus of $T\in\mathfrak{S}$ (cf. [5]). Let G be the set of all $\mathfrak{S}$-invariant $f\in L_1(X)$, and P the supremum of $\{f\neq 0\}$ $(f\in G)$. For each $f\in G$, since $|T|\,|f|\geqq|Tf|=|f|$ and $|T|$ is a contraction, it follows that $|T|\,|f|=|f|$ for all $T\in\mathfrak{S}$, so that $\{f\neq 0\}$ is $\mathfrak{S}$-closed. To show that there exists a $g\in G$ such that $P=\{g\neq 0\}$, it suffices to show that if $f, h\in G$, then $f+h_{\{f=0\}}$ is in G. Put $A=\{f\neq 0\}$ and $B=\{h\neq 0\}$. Since A and B are $\mathfrak{S}$-closed, it follows that $C=A\cap B$ is also $\mathfrak{S}$-closed. Now it is easy to see that

$$|T|(|h|_{B\setminus C})=|h|_{B\setminus C}\,, \qquad T\in\mathfrak{S}\,,$$

so that $B\setminus C$ is $\mathfrak{S}$-closed. This shows that $h_{\{f=0\}}=h_{B\setminus C}$ is in G and hence $f+h_{\{f=0\}}$ is in G. Thus we have shown that the properties (1)-(3) hold.

Now put $h=|g|$ and $v=|g|^{-1}g$ on P. For $T\in\mathfrak{S}$, define a linear contraction T' on $L_1(P)$ by

$$T'f=v^{-1}T(vf)\,, \qquad f\in L_1(P)\,.$$

Then $T'h=h$. For $f\in L_1(P)$ with $0\leqq f\leqq nh$, we have

$$\|nh\|_1 \leqq \|T'f\|_1 + \|nh - T'f\|_1$$
$$\leqq \|f\|_1 + \|nh - f\|_1 = \|nh\|_1 ,$$

so that

$$nh = |T'f| + |nh - T'f| .$$

This implies that $T'f \geqq 0$. Thus it is seen that T' is a positive contraction, in fact $T' = |T|$ on $L_1(P)$. Hence we conclude that $\mathfrak{S}' = \{T' : T \in \mathfrak{S}\}$ is a semigroup of positive linear contractions on $L_1(P)$. By Nagel [**9**, p. 83], there exists a linear projection Q' of $L_1(P)$ such that $Q'f$ is a unique $\mathfrak{S}'$-invariant function in $\overline{\mathrm{co}}\, \mathfrak{S}'f$ for each $f \in L_1(P)$. Define a linear projection Q of $L_1(P)$ by

$$Qf = vQ'(v^{-1}f) , \qquad f \in L_1(P) .$$

Then it is readily verified that Qf is a unique $\mathfrak{S}$-invariant function in $\overline{\mathrm{co}}\, \mathfrak{S}f$ for each $f \in L_1(P)$, and the desired properties of Q are clear. Q. E. D.

REMARK. If $\mathfrak{S}$ is a semigroup of positive linear contractions on $L_1(X)$, then the decompositions in Theorems 1.1 and 1.3 are the same, because $\mathfrak{S}$-invariance of $f \in L_1(X)$ implies that of $|f|$.

§ 2. Conditions for the mean ergodicity.

The following theorem is concerned with the mean ergodic properties of a semigroup of positive linear operators on $L_1(X)$ which has no nonzero invariant function.

THEOREM 2.1. *Let $\mathfrak{S}$ be a semigroup of positive linear operators on $L_1(X)$. Then the following conditions are equivalent:*

(i) *for each $f \in L_1(X)$, $\overline{\mathrm{co}}\, \mathfrak{S}f$ contains* 0;

(ii) *for each $f \in L_1(X)$*, $\inf \{\|Tf\|_1 : T \in \mathfrak{S}\} = 0$;

(iii) *the weak* closure of $\{T^*1 : T \in \mathfrak{S}\}$ contains* 0.

PROOF. It is trivial that (ii) implies (i).

(i) $\Rightarrow$ (ii). Let $f \in L_1(X)$ and $\varepsilon > 0$ be given. We can choose $T_1, \cdots, T_n \in \mathfrak{S}$ and $\alpha_1, \cdots, \alpha_n > 0$ with $\sum \alpha_i = 1$ such that $\|\sum \alpha_i T_i |f|\|_1 < \varepsilon$. Since $|T_i f| \leqq T_i |f|$, we have

$$\sum \alpha_i \|T_i f\|_1 \leqq \sum \alpha_i \|T_i |f|\|_1$$
$$= \|\sum \alpha_i T_i |f|\|_1 < \varepsilon .$$

It follows that $\|T_i f\|_1 < \varepsilon$ for some i, and thus (i) implies (ii).

(ii) $\Rightarrow$ (iii). Let $f_1, \cdots, f_n \in L_1(X)$ and $\varepsilon > 0$ be given. There exists a $T \in \mathfrak{S}$ such that $\|T(\sum |f_i|)\|_1 < \varepsilon$, so that we have

$$\sum |\langle f_i, T^*1 \rangle| \leqq \sum \|Tf_i\|_1$$
$$\leqq \|T(\sum |f_i|)\|_1 < \varepsilon .$$

Thus (ii) implies (iii).

(iii) $\Rightarrow$ (ii). Since $\|Tf\|_1 \leqq \langle |f|, T^*1 \rangle$, this implication is clear. Q. E. D.

Let $\mathfrak{S}$ be a semigroup of bounded linear operators on $L_1(X)$. Let $A \in \mathcal{F}$ and $B = X \setminus A$, and suppose that A is $\mathfrak{S}$-closed. For each $T \in \mathfrak{S}$, define a bounded linear operator T_B on $L_1(B)$ by

$$T_B f = (Tf)_B \quad (= (Tf) \cdot 1_B), \qquad f \in L_1(B).$$

It is easily checked that $T_B f_B = (Tf)_B$ for any $f \in L_1(X)$, and $S_B T_B = (ST)_B$ for any $S, T \in \mathfrak{S}$. Thus it is seen that $\mathfrak{S}_B = \{T_B : T \in \mathfrak{S}\}$ is a semigroup of bounded linear operators on $L_1(B)$.

Let $\mathfrak{S}$ be a semigroup of uniformly bounded positive linear operators on $L_1(X)$, and let $X = P + N = W + V$ be the decompositions in Theorems 1.1 and 1.2. We can define semigroups $\mathfrak{S}_N$ and $\mathfrak{S}_V$ of uniformly bounded positive linear operators on $L_1(N)$ and $L_1(V)$ respectively. Similarly, for a semigroup $\mathfrak{S}$ of linear contractions on $L_1(X)$, taking the decomposition $X = P + N$ in Theorem 1.3, we obtain a semigroup $\mathfrak{S}_N$ of linear contractions on $L_1(N)$. Then we have

THEOREM 2.2. *Let $\mathfrak{S}$ be a semigroup of linear contractions on $L_1(X)$. Let $X = P + N$ be the decomposition given in Theorem* 1.3. *Then the following conditions are equivalent:*

(i) *for each $f \in L_1(X)$, $\overline{\mathrm{co}}\,\mathfrak{S}f$ contains an $\mathfrak{S}$-invariant function;*

(ii) *for each $f \in L_1(N)$, $\overline{\mathrm{co}}\,\mathfrak{S}_N f$ contains* 0.

PROOF. (i) $\Rightarrow$ (ii). For each $f \in L_1(N)$ and $\varepsilon > 0$, we can choose $T_1, \cdots, T_n \in \mathfrak{S}$, $\alpha_1, \cdots, \alpha_n > 0$ with $\sum \alpha_i = 1$ and an $\mathfrak{S}$-invariant function h such that

$$\|\sum \alpha_i T_i f - h\|_1 < \varepsilon.$$

Since $h \in L_1(P)$, it follows that

$$\|\sum \alpha_i (T_i)_N f\|_1 = \|(\sum \alpha_i T_i f)_N\|_1 < \varepsilon.$$

Thus (i) implies (ii).

(ii) $\Rightarrow$ (i). Let $f \in L_1(X)$. We shall construct two sequences $\{g_n\} \subset L_1(P)$ and $\{h_n\} \subset L_1(N)$ by induction. Set $g_1 = f_P$ and $h_1 = f_N$. If g_{n-1} and h_{n-1} have been chosen, then we define g_n and h_n as follows. By assumption, we can choose $T_1, \cdots, T_k \in \mathfrak{S}$ and $\alpha_1, \cdots, \alpha_k > 0$ with $\sum \alpha_i = 1$ such that

$$\|(\sum \alpha_i T_i h_{n-1})_N\|_1 = \|\sum \alpha_i (T_i)_N h_{n-1}\|_1 < 1/n.$$

Define

$$g_n = (\sum \alpha_i T_i (g_{n-1} + h_{n-1}))_P = \sum \alpha_i T_i g_{n-1} + (\sum \alpha_i T_i h_{n-1})_P,$$

$$h_n = (\sum \alpha_i T_i (g_{n-1} + h_{n-1}))_N = (\sum \alpha_i T_i h_{n-1})_N.$$

Now let Q be a linear projection of $L_1(P)$ obtained in Theorem 1.3. It follows that $\{Qg_n\}$ is a Cauchy sequence in $L_1(P)$. Indeed, if $m>n$, then $g_m+h_m\in \mathrm{co}\,\mathfrak{S}(g_n+h_n)$, and hence we have

$$\|Qg_m-Qg_n\|_1\leqq\|h_n\|_1<1/n\,.$$

Thus $\{Qg_n\}$ converges strongly to an $\mathfrak{S}$-invariant function g. Since $Qg_n\in\overline{\mathrm{co}}\,\mathfrak{S}g_n$ and $g_n+h_n\in\mathrm{co}\,\mathfrak{S}f$, it is easy to see that g is contained in $\overline{\mathrm{co}}\,\mathfrak{S}f$. Q.E.D.

In the following, assume that $\mathfrak{S}$ is a semigroup of uniformly bounded positive linear operators on $L_1(X)$. Let $X=P+N=W+V$ be the decompositions given in Theorems 1.1 and 1.2. Consider the following conditions:

(i) *for each* $f\in L_1(X)$, $\overline{\mathrm{co}}\,\mathfrak{S}f$ *contains an* $\mathfrak{S}$*-invariant function;*

(ii) *for each* $f\in L_1(N)$, $\overline{\mathrm{co}}\,\mathfrak{S}_Nf$ *contains* 0;

(iii) *for each* $f\in L_1(N)$, $\inf\{\|T_Nf\|_1:T\in\mathfrak{S}\}=0$;

(iv) *the weak* closure of* $\{T^*1_N:T\in\mathfrak{S}\}$ *contains* 0;

(v) *for each* $f\in L_1(V)$, $\overline{\mathrm{co}}\,\mathfrak{S}_Vf$ *contains* 0;

(vi) *for each* $f\in L_1(V)$, $\inf\{\|T_Vf\|_1:T\in\mathfrak{S}\}=0$;

(vii) *the weak* closure of* $\{T^*1_V:T\in\mathfrak{S}\}$ *contains* 0.

Over these conditions, we have

THEOREM 2.3. *The following statements* (1°)-(3°) *hold:*

(1°) *If* $\mathfrak{S}$ *is amenable, then all the conditions* (i)-(vii) *are equivalent.*

(2°) *If* $\mathfrak{S}$ *is a semigroup of positive linear contractions, then the conditions* (i)-(iv) *are equivalent.*

(3°) *If* $\mathfrak{S}$ *is a left amenable semigroup of positive linear contractions, then all the conditions* (i)-(vii) *are equivalent.*

PROOF. Since $(T_N)^*=T^*$ on $L_\infty(N)$, Theorem 2.1 gives the equivalence of (ii), (iii) and (iv). Since $N\supset V$, it is clear that (iv) implies (vii). On the other hand, since $(T_V)^*u=(T^*u)\cdot 1_V$ for any $u\in L_\infty(V)$, Theorem 2.1 shows that (v) and (vi) are equivalent and these are implied by (vii). To prove (2°) and (3°), let $\mathfrak{S}$ be a semigroup of positive linear contractions. Then the equivalence of (i) and (ii) is a special case of Theorem 2.2, and (2°) is proved. Moreover assume $\mathfrak{S}$ to be left amenable. To show (3°), it suffices to show that (vi) implies (iii).

(vi) $\Rightarrow$ (iii). Let $f\in L_1(N)$ and $\varepsilon>0$ be given. By assumption, there exists an $S\in\mathfrak{S}$ such that

$$\|S_Vf_V\|_1<\varepsilon/2\,.$$

Put $g=S_Nf-S_Vf_V=(Sf)_N-(Sf)_V$. Since $|g|\in L_1(W)$, it follows as in the proof of Theorem 1.2 that $\overline{\mathrm{co}}\,\mathfrak{S}|g|$ is weakly compact, so that it contains an $\mathfrak{S}$-invariant function $h\in L_1(P)$ by Day's fixed point theorem. Thus there exist $T_1,\cdots,T_n\in\mathfrak{S}$ and $\alpha_1,\cdots,\alpha_n>0$ with $\sum\alpha_i=1$ such that

$$\|\sum\alpha_iT_i|g|-h\|_1<\varepsilon/2\,,$$

so that we have

$$\sum\alpha_i\|(T_i)_N g\|_1 \leqq \|\sum\alpha_i(T_i)_N|g|\|_1 = \|(\sum\alpha_i T_i|g|)_N\|_1 < \varepsilon/2 .$$

Taking an i with $\|(T_i)_N g\|_1 < \varepsilon/2$, we have

$$\inf\{\|T_N f\| : T\in\mathfrak{S}\} \leqq \|(T_i S)_N f\|_1 = \|(T_i)_N(g+S_V f_V)\|_1 < \varepsilon .$$

Thus (vi) implies (iii), and (3°) is proved.

We shall now prove (1°). Let $\mathfrak{S}$ be amenable. It remains to show that (i) implies (iii), and (vi) implies (i).

(i) $\Rightarrow$ (iii). For each $f\in L_1(N)$ and $\varepsilon>0$, we can choose $T_1, \cdots, T_n\in\mathfrak{S}$, $\alpha_1, \cdots, \alpha_n>0$ with $\sum\alpha_i=1$ and an $\mathfrak{S}$-invariant function $h\in L_1(P)$ such that

$$\|\sum\alpha_i T_i|f|-h\|_1 < \varepsilon .$$

As in the above proof of (vi) $\Rightarrow$ (iii), it follows that $\|(T_i)_N f\|_1 < \varepsilon$ for some i. Thus (iii) holds.

(vi) $\Rightarrow$ (i). Let $f\in L_1(X)$. Then there exists, for any $\varepsilon>0$, an $S\in\mathfrak{S}$ such that

$$\|S_V f_V\|_1 < \varepsilon/2M .$$

Putting $g=(Sf)_W$, we can choose $S_1, \cdots, S_k\in\mathfrak{S}$, $\alpha_1, \cdots, \alpha_k>0$ with $\sum\alpha_i=1$ and an $\mathfrak{S}$-invariant function g_0 such that

$$\|g_0-\sum\alpha_i S_i g\|_1 < \varepsilon/2 .$$

Since $S_V f_V = Sf-g$, we have

$$\|g_0-\sum\alpha_i S_i Sf\|_1 \leqq \|g_0-\sum\alpha_i S_i g\|_1 + \|\sum\alpha_i S_i(g-Sf)\|_1 < \varepsilon .$$

Therefore, for each $u\in L_\infty(X)$ with $\|u\|_\infty=1$, we have

$$|\langle g_0, u\rangle - \mu_T\langle Tf, u\rangle| = |\mu_T\langle T(g_0-\sum\alpha_i S_i Sf), u\rangle| \leqq M\varepsilon .$$

Now, letting $\varepsilon=1/n$, there exists a sequence $\{g_n\}$ of $\mathfrak{S}$-invariant functions such that

$$|\langle g_n, u\rangle - \mu_T\langle Tf, u\rangle| \leqq M/n , \qquad \|u\|_\infty = 1 ,$$

and the distance from g_n to $\overline{\text{co}}\,\mathfrak{S}f$ is smaller than $1/n$. Thus it follows that $\{g_n\}$ is a Cauchy sequence in $L_1(X)$ and so converges strongly to an $\mathfrak{S}$-invariant function which is contained in $\overline{\text{co}}\,\mathfrak{S}f$. Hence we obtain (i). Q. E. D.

Now consider the following conditions:

(i) *for each* $f \in L_1(X)$, $\overline{\text{co}}\,\mathfrak{S}f$ *contains an* $\mathfrak{S}$*-invariant function;*

(iv′) *there exists an* $\mathfrak{S}$*-invariant function* $g \in L_1^+(X)$ *such that the weak* closure of* $\{T^* 1_{\{g=0\}} : T \in \mathfrak{S}\}$ *contains* 0;

(vii′) *there exists a* $g \in L_1^+(X)$ *such that* $\mathfrak{S}g$ *is weakly sequentially compact and the weak* closure of* $\{T^* 1_{\{g=0\}} : T \in \mathfrak{S}\}$ *contains* 0.

The conditions (iv′) and (vii′) are restatements of (iv) and (vii) in Theorem 2.3 respectively. Hence the theorem below is obvious.

THEOREM 2.4. *Over the above conditions, the following statements* (1°)-(3°) *hold:*

(1°) *If* $\mathfrak{S}$ *is amenable, then* (i), (iv′) *and* (vii′) *are equivalent.*

(2°) *If* $\mathfrak{S}$ *is a semigroup of positive linear contractions, then* (i) *is equivalent to* (iv′).

(3°) *If* $\mathfrak{S}$ *is a left amenable semigroup of positive linear contractions, then* (i), (iv′) *and* (vii′) *are equivalent.*

COROLLARY. *Let* $\mathfrak{S}$ *be a semigroup of uniformly bounded positive linear operators on* $L_1(X)$ *satisfying the above condition* (i). *Let* $\mathfrak{S}'$ *be a subsemigroup of* $\mathfrak{S}$ *such that* $\mathfrak{S}T \cap \mathfrak{S}' \neq \emptyset$ *for all* $T \in \mathfrak{S}$. *Assume that* $\mathfrak{S}$ *and* $\mathfrak{S}'$ *are amenable, or that* $\mathfrak{S}$ *is a semigroup of positive linear contractions. Then, for each* $f \in L_1(X)$, $\overline{\text{co}}\,\mathfrak{S}'f$ *contains an* $\mathfrak{S}'$*-invariant function.*

PROOF. Let $g \in L_1^+(X)$ be an $\mathfrak{S}$-invariant function given in the above condition (iv′). To prove the corollary, it suffices to show that (iv′) holds also for $\mathfrak{S}'$ with the same g. Let $f \in L_1^+(X)$ and $\varepsilon > 0$ be given. Take a $T \in \mathfrak{S}$ such that $\langle f, T^* 1_A \rangle < \varepsilon$ where $A = \{g=0\}$. If an $S \in \mathfrak{S}$ is chosen so that $ST \in \mathfrak{S}'$, then, since $S^* 1_A \leqq M \cdot 1_A$ a. e. on X, we have

$$\langle f, (ST)^* 1_A \rangle = \langle f, T^* S^* 1_A \rangle$$
$$\leqq M \langle f, T^* 1_A \rangle < M\varepsilon .$$

This shows that the weak* closure of $\{T^* 1_A : T \in \mathfrak{S}'\}$ contains 0. Q. E. D.

REMARKS. (1) If $\mathfrak{S}$ is a [left] amenable group, so is every subgroup (cf. [**3**, p. 516]).

(2) Let $\mathfrak{S}$ be a [left] amenable semigroup and $\mathfrak{S}'$ a subsemigroup of $\mathfrak{S}$. Assume that there exists a [left] invariant mean μ on $m(\mathfrak{S})$ with $\mu(1_{\mathfrak{S}'}) > 0$, where $1_{\mathfrak{S}'}$ is the characteristic function of $\mathfrak{S}'$. Then $\mathfrak{S}'$ is [left] amenable (cf. [**3**, p. 518]). For the amenable case, it also follows that $\mathfrak{S}t \cap \mathfrak{S}' \neq \emptyset$ for all $t \in \mathfrak{S}$. For if $\mathfrak{S}t \cap \mathfrak{S}' = \emptyset$ for some $t \in \mathfrak{S}$, then, since $1_{\mathfrak{S}t} + 1_{\mathfrak{S}'} \leqq 1$, we have

$$\mu(1_{\mathfrak{S}t}) < \mu(1_{\mathfrak{S}t} + 1_{\mathfrak{S}'}) \leqq 1 ,$$

which contradicts

$$\mu(1_{\mathfrak{S}t}) = \mu_s(1_{\mathfrak{S}t}(st)) = \mu(1) = 1 .$$

§ 3. Mean ergodic theorems.

For the k-parameter semigroup, the following theorem asserts that the mean ergodic theorem holds on $L_1(X)$ if and only if every orbit is "almost" weakly sequentially compact.

THEOREM 3.1. *Let* $\mathfrak{S}=\{T(\tau): \tau\in \boldsymbol{R}_k^+\}$ *be a strongly measurable k-parameter semigroup of uniformly bounded positive linear operators on* $L_1(X)$, *where* $\boldsymbol{R}_k^+=\{(t_1, \cdots, t_k): t_1, \cdots, t_k>0\}$. *Then the following conditions are equivalent:*

(i) *for each* $f\in L_1(X)$, *the averages* $(t_1\cdots t_k)^{-1}\int_0^{t_1}\cdots\int_0^{t_k}T(s_1, \cdots, s_k)f\,ds_1\cdots ds_k$ *converge strongly as* $t_1, \cdots, t_k\to\infty$;

(ii) *for each* $f\in L_1(X)$ *and each compact set* $C\subset \boldsymbol{R}_k^+$ *(with the usual topology of* $\boldsymbol{R}_k^+$*), the set* $\{T(\alpha\tau)f: \alpha\geqq 1, \tau\in C\}$ *is weakly sequentially compact in* $L_1(X)$;

(iii) *there exists a* $C\subset \boldsymbol{R}_k^+$ *with the nonempty interior such that, for any* $f\in L_1(X)$, *the set* $\{T(\alpha\tau)f: \alpha\geqq 1, \tau\in C\}$ *is weakly sequentially compact in* $L_1(X)$.

PROOF. We first observe from standard arguments of mean ergodic theory [8, pp. 16-17] that the condition (i) is equivalent to the following:

(i′) for each $f\in L_1(X)$, $\overline{\mathrm{co}}\,\mathfrak{S}f$ contains an $\mathfrak{S}$-invariant function. It is clear that (ii) implies (iii). We shall prove that (i′) implies (ii), and (iii) implies (i′).

(i′) $\Rightarrow$ (ii). Let $f\in L_1(X)$ and C a compact subset of $\boldsymbol{R}_k^+$. To prove the weak sequential compactness of $\{T(\alpha\tau)f: \alpha\geqq 1, \tau\in C\}$, let $\{\tau_n\}$ be any sequence in $\{\alpha\tau: \alpha\geqq 1, \tau\in C\}$. If $\{\tau_n\}$ is bounded, then there is a subsequence $\{\sigma_j\}$ convergent to some $\tau_0\in \boldsymbol{R}_k^+$. Since $T(\tau)f$ is strongly continuous in $\boldsymbol{R}_k^+$ (cf. [7, p. 328]), it follows that $T(\sigma_j)f\to T(\tau_0)f$ (strongly). If $\{\tau_n\}$ is unbounded, then we may assume, by extracting a subsequence if necessary, that $\tau_1<\tau_2<\cdots$, where $(s_1, \cdots, s_k)<(t_1, \cdots, t_k)$ means $s_i<t_i$ for $i=1, \cdots, k$. Let $X=P+N$ be the decomposition in Theorem 1.1, and define a k-parameter semigroup $\mathfrak{S}_N=\{T(\tau)_N: \tau\in \boldsymbol{R}_k^+\}$ by the manner in the preceding section. Since $\tau_n>\tau$ eventually for any fixed $\tau\in \boldsymbol{R}_k^+$ and by Theorem 2.3,

$$\inf\{\|T(\tau)_N f_N\|_1: \tau\in \boldsymbol{R}_k^+\}=0,$$

it follows that

$$\|(T(\tau_n)f)_N\|_1=\|T(\tau_n)_N f_N\|_1\longrightarrow 0.$$

Since

$$\begin{aligned}T(\tau_m)f&=T(\tau_m-\tau_n)T(\tau_n)f\\&=T(\tau_m-\tau_n)(T(\tau_n)f)_P+T(\tau_m-\tau_n)(T(\tau_n)f)_N, \qquad m>n,\end{aligned}$$

and $\{T(\tau_m-\tau_n)(T(\tau_n)f)_P: m>n\}$ is weakly sequentially compact for any fixed n, we conclude that $\{T(\tau_n)f\}$ is weakly sequentially compact (cf. [6, p. 292]). Thus (i′) implies (ii).

(iii) $\Rightarrow$ (i′). Let $C\subset \boldsymbol{R}_k^+$ be taken as in (iii). We may assume that C is convex. Set $D=\{\alpha\tau : \alpha\geqq 1, \tau\in C\}$ and $\mathfrak{S}'=\{T(\tau) : \tau\in D\}$. Then $\mathfrak{S}'$ is an amenable subsemigroup of $\mathfrak{S}$. Let $f\in L_1(X)$. Since, by assumption, $\overline{\text{co}}\,\mathfrak{S}'f$ is weakly compact, Day's fixed point theorem gives an $\mathfrak{S}'$-invariant function g contained in $\overline{\text{co}}\,\mathfrak{S}'f$. We now show that g is also $\mathfrak{S}$-invariant. For $\sigma\in \boldsymbol{R}_k^+$, since $D\cap(\sigma+D)\neq\emptyset$, there is a $\tau\in D$ with $\sigma+\tau\in D$. Then $T(\tau)$ and $T(\sigma+\tau)$ are in $\mathfrak{S}'$, and we have

$$T(\sigma)g=T(\sigma)T(\tau)g=T(\sigma+\tau)g=g\,.$$

Thus (iii) implies (i′). Q. E. D.

For the discrete semigroup, we give the following theorem.

THEOREM 3.2. *Let T be a positive linear operator on $L_1(X)$ with* $\sup\{\|T^n\| : n\geqq 0\}<\infty$. *Then the following conditions are equivalent:*

(i) *for each $f\in L_1(X)$, the averages $n^{-1}\sum_{i=0}^{n-1}T^i f$ converge strongly as $n\to\infty$;*

(ii) *for each $f\in L_1(X)$, the set $\{T^n f : n\geqq 0\}$ is weakly sequentially compact in $L_1(X)$;*

(iii) *there exists a T-invariant function $g\in L_1^+(X)$ such that $T^{*n}1_{\{g=0\}}\to 0$ a. e. on X.*

PROOF. The proof of the equivalence of (i) and (ii) is similar to that of Theorem 3.1, and so we omit the details. To prove the equivalence of (i) and (iii), it suffices, in view of Theorem 2.4, to show that, for $u=1_{\{g=0\}}$ with a T-invariant $g\in L_1^+(X)$, the weak* closure of $\{T^{*n}u : n\geqq 0\}$ contains 0 if and only if $T^{*n}u\to 0$ a. e. on X. Putting $u_0=\limsup T^{*n}u$, we have $u_0\leqq M\cdot T^{*k}u$ for all k where $M=\sup\|T^n\|$. Thus, if the weak* closure of $\{T^{*n}u : n\geqq 0\}$ contains 0, then $u_0=0$ and so $T^{*n}u\to 0$ a. e. on X. The converse is clear. Q. E. D.

REMARK. The following example shows that the condition (ii) in Theorem 3.2 cannot be extended to more general semigroups. Let T be a shift operator on the bilateral l_1 space, and let $\mathfrak{S}=\{T^n : n\geqq 0\}\cup\{0\}$. Then $\mathfrak{S}$ is a commutative semigroup of positive linear contractions on l_1 such that

(1) for each $f\in l_1$, $\overline{\text{co}}\,\mathfrak{S}f$ contains 0;

(2) for each nonzero $f\in l_1$, $\mathfrak{S}f$ is not weakly sequentially compact.

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