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Harmonic dimensions related to Dirichlet integrals

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It is well known that the X-harmonic dimension (the cardinal number of normalized X-minimal harmonic functions) of any subsurface S of an open Riemann surface R of X-harmonic dimension one is at most one for $X=B$ and D. Since $D^{\sim}$ is, in a sense, an intermediate property between B and D, one might feel that the same is true of $D^{\sim}$. The purpose of this paper is, contrary to the above intuition, to prove the following

MAIN THEOREM. *There exists an open Riemann surface R of $D^{\sim}$-harmonic dimension one such that the $D^{\sim}$-harmonic dimension of every subsurface S of R covers every countable cardinal number.*

The surface R we are going to construct bears a sharper property than $D^{\sim}$-harmonic dimension one; R is actually in the null class $O_{HD}-O_{HB}$ and thus R is closely related to the so called Tôki surface. For the purpose we first discuss in nos. 3-14 a method of forming a new Riemann surface by welding from a given family of Riemann surfaces. The X-harmonic dimension of the resulting surface will be calculated in no. 17. The required surface R will be sought in nos. 19-23 among, which we call, generalized Tôki surfaces. The proof of the above theorem will be completed in no. 24. A short comment on the existence of surfaces with given X-harmonic dimensions will be added in no. 25.

Classes of harmonic functions.

1. Consider an *end* W of an open Riemann surface R, i.e. there exists a regular subregion Ω such that $W=R-\bar{\Omega}$. We do not exclude the case $\Omega=\emptyset$, i.e. the case $W=R$. We denote by $H(W;\partial W)$ the class of harmonic functions u on W with boundary values zero on the relative boundary ∂W of W. We also denote by $H(R)$ the class of harmonic functions u on R and thus $H(W;\partial W) \subset H(W)$. We consider the subclass $HX(W;\partial W)$ ($HX(W)$, resp.) consisting u in $H(W;\partial W)$ ($H(W)$, resp.) with a property X. As for X we consider P meaning the nonnegativeness $u\geqq 0$, B meaning the finiteness of the supremum norm

$$\|u\|_W=\sup_W |u| ,$$

and D meaning the finiteness of the Dirichlet integral

$$D_W(u)=\int_W du\wedge * du .$$

We also consider a property $D^{\sim}$ which is slightly different from P, B, and D in nature. We say that a function u has the property $D^{\sim}$ on W if there exists a decreasing sequence $\{u_n\}$ $(n=1, 2, \cdots)$ of nonnegative harmonic functions u_n on W with $D_W(u_n)<\infty$ such that $u=\lim_{n\to\infty} u_n$. For two properties X and Y we mean by XY the property both of X and Y.

2. We denote by O_{HX} the class of Riemann surfaces R such that the class $HX(R)$ consists of only constants. We also denote by O_G the class of parabolic Riemann surfaces R, i.e. there does not exist the harmonic Green function on R. The basic relation of the classification theory of Riemann surfaces is the following strict inclusion relations (cf. e.g. Sario-Nakai [**11**]):

$$O_G<O_{HP}<O_{HB}<O_{HD}=O_{HBD} .$$

Riemann surfaces formed by welding.

3. Consider an open Riemann surface $\hat{W}$ and a parametric 'disk' $U: 0<\tau<|z-\zeta|\leqq\infty$ in $\hat{W}$. We denote by $U(t)$ the concentric parametic disk $\tau\leqq t<|z-\zeta|\leqq\infty$. Let $\{\Omega\}$ be the directed net of regular subregions Ω of $\hat{W}$ and $w_\Omega(\cdot, t)$ be the harmonic function on $\Omega-\overline{U(t)}$ for Ω containing $\overline{U(t)}$ with boundary values 1 on $\partial\Omega$ and 0 on $\partial U(t)$. We extend $w_\Omega(\cdot, t)$ to the whole $\hat{W}$ by setting 1 on $\hat{W}-\Omega$ and 0 on $\overline{U(t)}$. The maximum principle assures that $\{w_\Omega(\cdot, t)\}$ is a decreasing net as Ω exhausts $\hat{W}$. As a consequence we deduce the existence of

$$w(\cdot, t)=\lim_{\Omega\to\hat{W}} w_\Omega(\cdot, t) \tag{1}$$

uniformly on each compact subset of $\hat{W}$. The function $w(\cdot, t)$, referred to as the *harmonic measure* of the ideal boundary of $\hat{W}$ with respect to $\hat{W}-\overline{U(t)}$, is continuous on $\hat{W}$, harmonic on $\hat{W}-\overline{U(t)}$, and 0 on $\overline{U(t)}$. By the Green formula

$$D_{\hat{W}}(w_\Omega(\cdot, t)-w_{\Omega'}(\cdot, t))=D_{\hat{W}}(w_\Omega(\cdot, t))-D_{\hat{W}}(w_\Omega(\cdot, t)) \tag{2}$$

for $\Omega\subset\Omega'$ and a fortiori

$$\lim_{\Omega\to\hat{W}} D_{\hat{W}}(w(\cdot, t)-w_\Omega(\cdot, t))=0 \tag{3}$$

and in particular

$$D_{\hat{W}}(w(\cdot, t))=\lim_{\Omega\to\hat{W}} D_{\hat{W}}(w_\Omega(\cdot, t))<\infty . \tag{4}$$

By the maximum principle, the property $w(\cdot, t)\equiv 0$ on $\hat{W}$ does not depend on the choice of U and $U(t)$. A Riemann surface $\hat{W}$ with this property is nothing but a surface being *parabolic*. Hereafter we assume that $\hat{W}$ is *hyperbolic*, i.e. $\hat{W}\notin O_G$. Then $1-w(\cdot, t)$ is a *potential* on $\hat{W}$, i.e. $1-w(\cdot, t)$ is superharmonic on $\hat{W}$ with vanishing greatest harmonic minorant (cf. e.g. Sario-Nakai [**11**]). Hence in particular $\inf_{\hat{W}}(1-w(\cdot, t))=0$ or equivalently

(5) $$\sup_{\hat{W}} w(\cdot, t)=1.$$

4. We compare $w(\cdot, t)$ for different choices of t. Let $\tau\leqq s<t$. By the maximum principle, $w_\Omega(\cdot, s)\leqq w_\Omega(\cdot, t)$ on $\hat{W}$ and thus

(6) $$0\leqq w(\cdot, s)\leqq w(\cdot, t)<1$$

on $\hat{W}$, i.e. $\{w(\cdot, t)\}$ $(t\geqq\tau)$ is an increasing net as $t\to\infty$. A fortiori $h=\lim_{t\to\infty} w(\cdot, t)$ is a bounded harmonic function on $\hat{W}$ less the center of U, and then on $\hat{W}$. By (6), $0\leqq 1-w(\cdot, t)\leqq 1-w(\cdot, s)$ on $\hat{W}$ and therefore $0\leqq 1-h\leqq 1-w(\cdot, t)$ on $\hat{W}$. Since $1-w(\cdot, t)$ is a *potential* on $\hat{W}$, $1-h\equiv 0$ on $\hat{W}$, i.e.

(7) $$\lim_{t\to\infty} w(\cdot, t)=1$$

uniformly on each compact subset of $\hat{W}$ less the center of U. By the Green formula

$$D_{\hat{W}}(w_\Omega(\cdot, s)-w_\Omega(\cdot, t))=D_{\hat{W}}(w_\Omega(\cdot, s))-D_{\hat{W}}(w_\Omega(\cdot, t)).$$

By (3) we deduce

$$D_{\hat{W}}(w(\cdot, s)-w(\cdot, t))=D_{\hat{W}}(w(\cdot, s))-D_{\hat{W}}(w(\cdot, t)).$$

This with (4) and (7) implies that

(8) $$\lim_{t\to\infty} D_{\hat{W}}(w(\cdot, t))=0.$$

5. Let N be a countable cardinal number greater than 1, i.e. N is a positive integer >1 or $N=\infty$. Consider a family $\{\hat{W}_k\}$ $(1\leqq k<N)$ of hyperbolic Riemann surfaces $\hat{W}_k$. For each k we fix a parametric 'disk' $\hat{U}_k: 0<\tau<|z-\zeta_k|\leqq\infty$ and set $\hat{U}_k(t): \tau\leqq t<|z-\zeta_k|\leqq\infty$. We fix an analytic Jordan curve α_k in $\hat{W}_k-\bar{U}_k$ homologous to $-\partial\hat{U}_k$. Let $w_k(\cdot, t)$ be the harmonic measure of the ideal boundary of $\hat{W}_k$ with respect to $\hat{W}_k-\overline{\hat{U}_k(t)}$. By (7) and (8) we can find $t_k>\tau$ such that

(9) $$D_{W_k}(w_k)<1/2^k$$

and also

(10) $$\inf_{\alpha_k} w_k>1/2$$

where $W_k=\hat{W}_k-\bar{U}_k$ with $U_k=\hat{U}_k(t_k)$ and $w_k(z)=w_k(z, t_k)$. We may rechoose

$\{\zeta_k\}$ $(1\leqq k<N)$ so as to satisfy the condition that ζ_k $(1\leqq k<N)$ are positive real numbers and

(11) $$\zeta_k-\zeta_{k-1}>t_k+t_{k-1}$$

for every k with $2\leqq k<N$. We denote by β_k the circle $|z-\zeta_k|=t_k$ in the extended complex plane $\hat{C}$ oriented anticlockwise. We also denote by C the surface

(12) $$\hat{C}-\overline{\bigcup_{1\leqq k<N}\{|z-\zeta_k|\leqq t_k\}}\,.$$

From the surfaces C and $\{W_k\}$ $(1\leqq k<N)$ we form a new open Riemann surface

(13) $$R=[C,\ \{W_k\}]$$

as follows: Weld each W_k $(1\leqq k<N)$ to C by identifying ∂W_k with β_k. We will describe $HX(R)$ in terms of $HX(W_k;\partial W_k)$ $(1\leqq k<N)$ for $X=B$, BD and $BD^{\sim}$.

6. For each k we define an operator $L_k: C(\beta_k)\rightarrow HB(W_k)\cap C(\overline{W}_k)$ as follows. Let $f\in C(\beta_k)$ and Ω any regular subregion of $\hat{W}_k$ with $\Omega\supset\bar{U}_k$. We denote by f_Ω the harmonic function on $W_k\cap\Omega$ with boundary values f on $\beta_k=\partial W_k$ and zero on $\partial\Omega$. It is readily seen that the net $\{f_\Omega\}$ converges to a bounded harmonic function uniformly on each compact subset of $\overline{W}_k$. We define

(14) $$L_kf=\lim_{\Omega\rightarrow\hat{W}_k}f_\Omega\,.$$

Then L_k is a linear operator from $C(\beta_k)$ into $HB(W_k)\cap C(\overline{W}_k)$ such that $L_kf|\beta_k=f$ and $\|L_kf\|_{W_k}\leqq\|f\|_{\beta_k}$ for every $f\in C(\beta_k)$ where $\|\cdot\|_E$ is the supremum norm taken over E. The latter follows from the sharper inequality

(15) $$|L_kf|\leqq\|f\|_{\beta_k}(1-w_k)$$

on $\overline{W}_k$. We denote by

(16) $$\prod_{1\leqq k<N}{}^{*}HB(W_k;\partial W_k)$$

the subspace of the product space $\prod_{1\leqq k<N}HB(W_k;\partial W_k)$ consisting of those $\boldsymbol{v}=(v_1,v_2,\cdots)$ $(v_k\in HB(W_k;\partial W_k))$ with

$$\|\boldsymbol{v}\|=\sup_{1\leqq k<N}\|v_k\|_{W_k}<\infty\,.$$

The subspace (16) coincides with the whole product space if and only if $N<\infty$. We define the *order* $\boldsymbol{u}\leqq\boldsymbol{v}$ by $u_k\leqq v_k$ for every k $(1\leqq k<N)$ where $\boldsymbol{u}=(u_1,u_2,\cdots)$. Then (16) is an ordered Banach space and so is the space $HB(R)$ where $R=[C,\{W_k\}]$. Using L_k we define an operator

$$\tau:\ HB(R)\longrightarrow\prod_{1\leqq k<N}{}^{*}HB(W_k;\partial W_k)$$

given by $\tau u=(u_1, u_2, \cdots)$ with

$$u_k=u\,|\,W_k-L_k(u\,|\,\beta_k) \tag{17}$$

for every k $(1\leqq k<N)$. The basic relation in our study is the following

THEOREM. *The operator* τ *is order preserving, linear, isometric, and bijective, i.e. as ordered Banach spaces*

$$HB(R)\cong \prod_{1\leqq k<N}^{*} HB(W_k\,;\,\partial W_k)\,. \tag{18}$$

7. From the definition of τ it instantly follows that τ is a linear operator from $HB(R)$ into (16) such that $\|\tau u\|\leqq\|u\|_R$ for every $u\in HB(R)$ and $u\geqq 0$ on R implies $\tau u\geqq \boldsymbol{o}=(0, 0, \cdots)$. First we prove that τ is *surjective.* We denote by Z_k the annular part of W_k bounded by α_k and β_k and set

$$Z=C\cup(\bigcup_{1\leqq k<N}(\alpha_k\cup Z_k))\,.$$

Then Z is a subregion of R with $\partial Z=\bigcup_{1\leqq k<N}\alpha_k$. The region Z can also be viewed as a subregion of $\hat{\boldsymbol{C}}$ bounded by Jordan curves α_k $(1\leqq k<N)$ if $N<\infty$ and as a subregion of $\hat{\boldsymbol{C}}$ whose boundary consists of α_k $(1\leqq k<N)$ and the point at infinity if $N=\infty$. We denote by X the subspace of the product space $\prod_{1\leqq k<N} C(\alpha_k)$ consisting of those $\boldsymbol{f}=(f_1, f_2, \cdots)$ with $\|\boldsymbol{f}\|=\sup_{1\leqq k<N}\|f_k\|_{\alpha_k}<\infty$. The order in X is given by $f_k\geqq g_k$ on α_k $(1\leqq k<N)$ for $(f_1, f_2, \cdots)\geqq(g_1, g_2, \cdots)$. Then X is an ordered Banach space. For any $\boldsymbol{f}=(f_1, f_2, \cdots)\in X$ we consider the class $\{s\}$ of superharmonic functions s on Z such that

$$\liminf_{z\in Z\ z\to\zeta} s(z)\geqq f_k(\zeta)$$

for every $\zeta\in\alpha_k$ $(1\leqq k<N)$, and moreover if $N=\infty$,

$$\liminf_{z\in Z\ z\to\infty} s(z)\geqq \limsup_{k\to\infty}\,(\sup_{\alpha_k} f_k)\,,$$

where we consider Z being embedded in $\hat{\boldsymbol{C}}$. Set

$$\kappa f=\inf_{s\in\{s\}} s$$

pointwise on Z. By the Perron-Brelot method (cf. e.g. [**11**]) we see that κ is an isometric positive linear operator from X into $HB(Z)\cap C(Z\cup\partial Z)$ with $\kappa\boldsymbol{f}\,|\,\alpha_k=f_k$ $(1\leqq k<N)$. For any bounded continuous function g on $\bigcup_{1\leqq k<N}\beta_k$ we set

$$\lambda g=(L_1(g\,|\,\beta_1),\ L_2(g\,|\,\beta_2),\ \cdots)\,.$$

Finally we define an operator $\gamma: X\to X$ by

$$\gamma f=\lambda(\kappa\boldsymbol{f})\,,$$

which is clearly a bounded positive linear operator. By (15) and (10)

$$\|L_k(g|\beta_k)\|_{\alpha_k} \leqq 2^{-1}\|g\|_{\beta_k} \qquad (1 \leqq k < N)$$

and therefore

$$\|\gamma \boldsymbol{f}\| = \|\lambda(\kappa \boldsymbol{f})\| \leqq 2^{-1} \sup_{1 \leqq k < N} \|\kappa f\|_{\beta_k} \leqq 2^{-1}\|\boldsymbol{f}\|,$$

i. e. the operator norm $\|\gamma\| \leqq 2^{-1}$.

8. Given an arbitrary $\boldsymbol{v}=(v_1, v_2, \cdots)$ in (16). We need to find a $u \in HB(R)$ such that $\tau u = \boldsymbol{v}$. Considering the restriction on $\alpha_k (1 \leqq k < N)$ we may view as $\boldsymbol{v}$ being in X. Then the Fredholm equation

$$(\iota - \gamma)\boldsymbol{f} = \boldsymbol{v}$$

has a unique solution $\boldsymbol{f} \in X$ given by the Neumann series

(19) $$\boldsymbol{f} = \sum_{n=0}^{\infty} \gamma^n \boldsymbol{v}$$

(cf. e. g. Dunford-Schwartz [4]) where ι is the identity operator on X. We set

(20) $$u(z) = \begin{cases} (\kappa \boldsymbol{f})(z) & (z \in Z); \\ (L_k(\kappa f) + v_k)(z) & (z \in W_k) \end{cases}$$

for $1 \leqq k < N$. We maintain that $u \in HB(R)$. For this purpose we only have to show that $\kappa \boldsymbol{f} = L_k(\kappa \boldsymbol{f}) + v_k$ on $Z \cap W_k$ for $1 \leqq k < N$. On β_k, $L_k(\kappa \boldsymbol{f}) + v_k = \kappa \boldsymbol{f} + 0 = \kappa \boldsymbol{f}$, and on α_k, $L_k(\kappa \boldsymbol{f}) + v_k = (\gamma \boldsymbol{f} + \boldsymbol{v})_k = (\boldsymbol{f})_k = (\kappa \boldsymbol{f})|\alpha_k$ where $(\cdot)_k$ indicates the k^{th}-component. Thus the required identity holds on $\partial(W_k \cap Z)$ and hence on $W_k \cap Z$. In view of (20), $u = \kappa \boldsymbol{f}$ on $\beta_k (1 \leqq k < N)$ and then $u - L_k u = v_k$ on $W_k (1 \leqq k < N)$, i. e. $(\tau u)_k = v_k (1 \leqq k < N)$, or $\tau u = \boldsymbol{v}$. Therefore we have proven that τ is surjective.

9. Next we prove that τ is *order preserving*. By the definition of τ (and L_k) it is easily seen that $u \geqq 0$ implies $\tau u \geqq 0$. Conversely let $\boldsymbol{v}$ be in (16) with $\boldsymbol{v} \geqq \boldsymbol{o} = (0, 0, \cdots)$ and $\tau u = \boldsymbol{v}$. Since κ and λ are positive operators, γ and then $\gamma^n (n=1, 2, \cdots)$ are also positive operators. Hence by (19) $\boldsymbol{v} \geqq \boldsymbol{o}$ implies that $\boldsymbol{f} \geqq \boldsymbol{o}$. By (20) we must conclude $u \geqq 0$.

To prove the *isometry* of τ, let $\boldsymbol{e} = (1 - w_1, 1 - w_2, \cdots)$. Then clearly $\tau 1 = \boldsymbol{e}$. Take an arbitrary $u \in HB(R)$ and set $\boldsymbol{v} = \tau u$. Observe that

$$\tau(\|\boldsymbol{v}\| \pm u) = \|\boldsymbol{v}\| \boldsymbol{e} \pm \boldsymbol{v} \geqq \boldsymbol{o},$$

and therefore $\|\boldsymbol{v}\| \pm u \geqq 0$, i. e. $\|\tau u\| \geqq \|u\|_R$. This with the trivial inequality $\|\tau u\| \leqq \|u\|_R$ assures that $\|\tau u\| = \|u\|_R$. In particular, τ is injective and thus τ is *bijective*.

10. We proceed to the description of the class $HBD(R)$ in terms of $HBD(W_k; \partial W_k) (1 \leqq k < N)$. For the purpose we consider

(21) $$\prod_{1 \leqq k < N}^{*} HBD(W_k; \partial W_k)$$

the subspace of the product space $\prod_{1\leqq k<N} HBD(W_k\,;\,\partial W_k)$ consisting of $\boldsymbol{v}=(v_1, v_2,\cdots)(v_k\in HBD(W_k\,;\,\partial W_k),\ 1\leqq k<N)$ such that

$$\|\boldsymbol{v}\|=\sup_{1\leqq k<N}\|v_k\|_{W_k}<\infty$$

and moreover

$$\boldsymbol{D}(\boldsymbol{v})=\sum_{1\leqq k<N} D_{W_k}(v_k)<\infty\,.$$

The subspace (21) coincides with the whole product space $\prod_{1\leqq k<N} HBD(W_k\,;\,\partial W_k)$ if and only if $N<\infty$. Observe that the space (21) is a subspace of (16). Since $HBD(R)$ is a subspace of $HB(R)$, we can consider

$$\tau_D=\tau\,|\,HBD(R)\,,$$

which is an operator from $HBD(R)$ into (16). As a counter part to Theorem 6 we will prove the following

THEOREM. *The restriction τ_D is a bijective linear order-preserving operator from $HBD(R)$ to* (21), *i.e. as ordered linear spaces*

$$HBD(R)\cong \prod_{1\leqq k<N}^{*} HBD(W_k\,;\,\partial W_k)\,. \tag{22}$$

11. In view of Theorem 6 we only have to show that τ maps $HBD(R)$ *surjectively* to (21). Take an arbitrary $u\in HBD(R)$ and let $\tau u=(v_1, v_2,\cdots)$ which is in (16). First we show that τu really belongs to (21). For this purpose we only have to prove that $\boldsymbol{D}(\tau u)<\infty$. Recall that $w_k=w_k(\cdot, t_k)$ is the limit of $w_\Omega(\cdot, t_k)$ as regular subregions Ω of $\hat{W}_k$ with $\bar{U}_k\subset\Omega$ exhaust $\hat{W}_k$ (cf. nos. 3 and 4). Let v_Ω be the harmonic function on $W_k\cap\Omega$ with boundary values zero on ∂W_k and u on $\partial\Omega$. We extend v_Ω to W_k by setting $v_\Omega=u$ on $W_k-\Omega$. Then

$$v_k=u\,|\,W_k-L_k(u\,|\,\beta_k)=\lim_{\Omega\to\hat{W}_k} v_\Omega\,.$$

By the Dirichlet principle and the Fatou lemma

$$D_{W_k}(v_k)\leqq \liminf_{\Omega\to\hat{W}_k} D_{W_k}(v_\Omega)\leqq \liminf_{\Omega\to\hat{W}_k} D_{W_k}(uw_\Omega)\,.$$

By the Schwarz inequality

$$D_{W_k}(uw_\Omega)^{1/2}\leqq \|u\|_{W_k}D_{W_k}(w_\Omega)^{1/2}+\|w_\Omega\|_{W_k}D_{W_k}(u)^{1/2}\,.$$

By (4) we deduce

$$\liminf_{\Omega\to\hat{W}_k} D_{W_k}(uw_\Omega)^{1/2}\leqq \|u\|_R D_{W_k}(w_k)^{1/2}+D_{W_k}(u)^{1/2}\,.$$

Therefore we obtain

$$D_{W_k}(v_k)\leqq 2\,\|u\|_R^2 D_{W_k}(w_k)+2D_{W_k}(u)\,.$$

By (9) we deduce that

$$\begin{aligned}\boldsymbol{D}(\tau u) &= \sum_{1\leqq k<N} D_{W_k}(v_k)\\ &\leqq 2\|u\|_R^2 \sum_{1\leqq k<N} D_{W_k}(w_k)+2\sum_{1\leqq k<N} D_{W_k}(u)\\ &\leqq 2\|u\|_R^2+2D_R(u)<\infty\,,\end{aligned}$$

i.e. τ_D maps $HBD(R)$ *into* (21).

12. Conversely let $\boldsymbol{v}=(v_1, v_2, \cdots)$ be in (21). Since $\boldsymbol{v}$ is also in (16) there exists a unique $u\in HB(R)$ such that $\tau u=\boldsymbol{v}$. We will show that $u\in HBD(R)$. We consider a function s on R such that $s=1-w_k$ on $W_k(1\leqq k<N)$ and $s=1$ on $\bar{C}$. Then s is a superharmonic function on R. Take an arbitrary $h\in H(R)$ with $0\leqq h\leqq s$ on R. Observe that $h\in HB(R)$. Clearly $h_k\leqq 1-w_k=s$ on W_k, where $\tau h=(h_1, h_2, \cdots)$, and then $h_k\leqq(1-w_k)-(1-w_\Omega)$ on $W_k\cap\Omega$, where $w_\Omega=w_\Omega(\cdot, t_k)$. Since $w_\Omega\to w_k$, we have $h_k=0$, i.e. $\tau h=\boldsymbol{o}$, and a fortiori $h=0$. This shows that s is a potential on R. Next we consider a function v on R defined by $v=v_k$ on W_k and $v=0$ on $\bar{C}$. The definition of τ assures that $|u-v|\leqq cs$ on R with a suitable constant c. Since $D_R(v)=\boldsymbol{D}(\boldsymbol{v})<\infty$, the harmonic decomposition can be applied to v (cf. e.g. [**11**]):

$$v=\phi+g$$

where $\phi\in HBD(R)$ and $|g|$ is dominated by a potential. Then

$$|u-\phi|\leqq|u-v|+|v-\phi|\leqq cs+|g|$$

and therefore the subharmonic function $|u-\phi|$ on R is dominated by a potential. Hence $|u-\phi|=0$ on R, i.e. $u=\phi\in HBD(R)$.

13. Based on (22) we finally study $HBD^{\sim}(R)$ in terms of $HBD^{\sim}(W_k\,;\partial W_k)$. For the purpose we consider the half linear space

$$\prod_{1\leqq k<N}{}^{*} HBD^{\sim}(W_k\,;\partial W_k) \tag{23}$$

consisting of the elements $\boldsymbol{v}=(v_1, v_2, \cdots)\in \prod_{1\leqq k<N} HBD^{\sim}(W_k\,;\partial W_k)$ such that

$$\|\boldsymbol{v}\|=\sup_{1\leqq k<N}\|v_k\|_{W_k}<\infty\,.$$

As before (23) coincides with the whole product space if and only if $N<\infty$, and (23) is a subset of (16). Since $HBD^{\sim}(R)\subset HB(R)$, we can consider

$$\tau_{D^{\sim}}=\tau\,|\,HBD^{\sim}(R)\,.$$

As a consequence of Theorem 6 and 10 we maintain the following

THEOREM. *The mapping $\tau_{D^{\sim}}$ is a bijective half-linear order-preserving operator from $HBD^{\sim}(R)$ to* (23), *i.e. as ordered half-linear spaces*

$$HBD^{\sim}(R)\cong \prod_{1\leqq k<N}{}^{*} HBD^{\sim}(W_k\,;\partial W_k)\,. \tag{24}$$

14. We only have to show that τ maps $HBD^{\sim}(R)$ *surjectively* to (23). First we take an arbitrary u in $HBD^{\sim}(R)$ and will show that τu belongs to (23). There exists a decreasing sequence $\{u_m\}(m=1, 2, \cdots)$ in $HBD(R)$ such that $u=\lim_{m\to\infty} u_m$ on R. Let $\tau u_m=(v_{m1}, v_{m2}, \cdots)$. By Theorem 10, $\{v_{mj}\}(m=1, 2, \cdots)$ is a decreasing sequence in $HBD(W_j; \partial W_j)$ and a fortiori $v_j=\lim_{m\to\infty} v_{mj}\in HBD^{\sim}(W_j; \partial W_j)$ with $\|v_j\|_{W_j}\leqq\|u\|_R$. Therefore $\boldsymbol{v}=(v_1, v_2, \cdots)$ belongs to (23). Since $\tau u_m\geqq\tau u$, we deduce $\boldsymbol{v}\geqq\tau u$. By Theorem 6, there exists a $u_0\in HB(R)$ with $\boldsymbol{v}=\tau u_0$. Then $\tau u_0=\boldsymbol{v}\leqq\tau u_m$ implies that $u_0\leqq u_m$ and then $u_0\leqq u$. Thus $\boldsymbol{v}\leqq\tau u$ as a consequence of $\tau u_0\leqq\tau u$. Hence we conclude that $\tau u=\boldsymbol{v}$ belongs to (23). Next we take an arbitrary $\boldsymbol{v}=(v_1, v_2, \cdots)$ in (23) and let $\boldsymbol{v}=\tau u$ with $u\in HB(R)$. We will prove that $u\in HBD^{\sim}(R)$. Since $v_j\in HBD^{\sim}(W_j; \partial W_j)$, there exists a decreasing sequence $\{v_{mj}\}(m=1, 2, \cdots)$ in $HBD^{\sim}(W_j; \partial W_j)$ such that $\lim_{m\to\infty} v_{mj}=v_j$. Here by considering $v_{mj}\wedge\|\boldsymbol{v}\|w_j$ (the greatest harmonic minorant of v_{mj} and $\|\boldsymbol{v}\|w_j$) instead of v_{mj}, we may assume that $v_{mj}\leqq\|\boldsymbol{v}\|w_j$. Let

$$\boldsymbol{v}_m=(v_{m1}, v_{m2}, \cdots, v_{m,N-1})$$

if $N<\infty$ and

$$\boldsymbol{v}_m=(v_{m1}, v_{m2}, \cdots, v_{mm}, \|\boldsymbol{v}\|w_{m+1}, \|\boldsymbol{v}\|w_{m+2}, \cdots)$$

if $N=\infty$. By (9), $\boldsymbol{v}_m$ belongs to (21). By (22) there exists a $u_m\in HBD(R)$ such that $\boldsymbol{v}_m=\tau u_m$. It is easy to see that $\{\boldsymbol{v}_m\}(m=1, 2, \cdots)$ is decreasing. Therefore $\{u_m\}$ is decreasing and $u_0=\lim_{m\to\infty} u_m\in HBD^{\sim}(R)$. From $\boldsymbol{v}_m\geqq\boldsymbol{v}$, $u_m\geqq u$ follows and a fortiori $u_0\geqq u$. On the other hand, $u_m\geqq u_0$ implies that $\boldsymbol{v}_m=\tau u_m\geqq\tau u_0$, which implies that $\boldsymbol{v}=\tau u\geqq\tau u_0$. Thus $u\geqq u_0$. We thus conclude that $u=u_0\in HBD^{\sim}(R)$.

Harmonic dimensions.

15. A function u in $HX(W; \partial W)(HX(W)$, resp.) is said to be *X-minimal* on $(W: \partial W)(W$, resp.) if $u>0$ and $u\geqq v\geqq 0$ implies the constancy of v/u for any v in $HX(W; \partial W)(HX(W)$, resp.). Two X-minimal functions u_1 and u_2 are said to be equivalent if u_1/u_2 is a constant. Let W_1 and W_2 be two ends of R such that $R-W_j\neq\emptyset(j=1, 2)$. It is well known that there exists a bijective homogeneous additive order-preserving mapping between $HX(W_1; \partial W_1)$ and $HX(W_2; \partial W_2)$ and, if $R\notin O_G$, between $HX(R)$ and $HX(W; \partial W)$ for any end W of R where $X=P, B, D, D^{\sim}, BD$, and $BD^{\sim}$(cf. e. g. [**11**], Ozawa [**9**], Rodin-Sario [**10**]). The cardinal number of the set of equivalence classes of X-minimal functions on $(W; \partial W)$, where W is an end of R with $R-W\neq\emptyset$, is referred to as the *X-harmonic dimension* of R for $X=P, B, D, D^{\sim}, BD$, and $BD^{\sim}$. We will denote by $x(R)$ the X-harmonic dimension of R, where $x=p, b, d, d^{\sim}, bd$, and $bd^{\sim}$, respectively and $X=P, B, D, D^{\sim}, BD$, and $BD^{\sim}$, respectively. By the above, $x(R)$ does not depend on the choice of W. The P-harmonic dimension, usually simply called just harmonic dimension, was studied by many authors such as Heins, Ozawa, Kuramochi, among others. We are especially interested in the existence of R with a single ideal boundary component with any integral $p(R)$ shown by Heins [**5**], with countably infinite $p(R)$ by Kuramochi [**6**], with uncountably infinite $p(R)$ by Constantinescu-Cornea [**2**]. In this paper we do not discuss $p(R)$.

16. An X-minimal function ($X=D$ and $D^{\sim}$) is automatically bounded (cf. e. g. [**11**]) and therefore

$$bd(R)=d(R), \qquad bd^{\sim}(R)=d^{\sim}(R). \tag{25}$$

Another basic fact is that $x(R)$ is at most countably infinite and $x(R)=0$ for $R\in O_G$(cf. e. g. [**11**]) where $x=b$, d, and $d^{\sim}$. As a consequence of the study of $b(R)$ and $d(R)$ in terms of the Wiener and Royden compactifications (cf. e. g. [**11**]) we have

$$b(S)\leqq b(R), \qquad d(S)\leqq d(R) \tag{26}$$

for any subsurface S of R. That this is no longer true for $d^{\sim}$ will be the main conclusion of this paper. The first systematic study of the B-, D-, and $D^{\sim}$-harmonic dimensions was carried by Constantinescu-Cornea [**1**] (see also [**3**]) developing the earlier works by many authors such as Sario, Kuroda, Tôki, Mori, among others, and especially by Kuramochi. For subsequent and related works we refer to the reference of the monograph [**11**] and also the one at the end of this paper.

17. Let $R=[C, \{W_k\}]$ be a Riemann surface formed by a welding as in no. 5 from $\{\hat{W}_k\}(1\leqq k<N)$. Observe that $x(\hat{W}_k)=x(W_k)(x=b, d$, and $d^{\sim})$. We maintain

THEOREM. *The X-harmonic dimension $x(R)$ of R is related to the X-harmonic dimensions $x(W_k)$ of $W_k(1\leqq k<N)$ as follows:*

$$x(R)=\sum_{1\leqq k<N} x(W_k) \tag{27}$$

where $X=B$, D, and $D^{\sim}$, respectively and $x=b$, d, and $d^{\sim}$, respectively.

18. We prove (27) for $x=d$. The other cases can be shown by exactly the same fashion. We say an element $\boldsymbol{v}$ in (21) is *minimal* if $\boldsymbol{v}\geqq\boldsymbol{O}$, $\boldsymbol{v}\neq\boldsymbol{O}$, and $\boldsymbol{v}\geqq\boldsymbol{w}\geqq\boldsymbol{O}$ implies the existence of a constant c such that $\boldsymbol{w}=c\boldsymbol{v}$ for every $\boldsymbol{w}$ in (21). By (22) and (25), $d(R)$ is nothing but the cardinal number of minimal elements $\boldsymbol{v}$ in (21) with the normalization $\|\boldsymbol{v}\|=1$. Let $\boldsymbol{v}$ be a minimal element with $v_j>0$ where $\boldsymbol{v}=(v_1, v_2, \cdots)$. Let $\boldsymbol{v}_j=(v_{j1}, v_{j2}, \cdots)$ be such that $v_{jj}=v_j$ and $v_{jk}=0$ $(k\neq j)$. Since $v_k\geqq 0$ $(1\leqq k<N)$, $\boldsymbol{v}\geqq\boldsymbol{v}_j$. Therefore $v_k=0$ $(k\neq j)$ and $\|v_j\|_{W_j}=1$. Let v be any element in $HBD(W_j; \partial W_j)$ with $v_j\geqq v\geqq 0$. Then $\bar{\boldsymbol{v}}=(\bar{v}_1, \bar{v}_2, \cdots)$ with $\bar{v}_j=v$ and $\bar{v}_k=0$ $(k\neq j)$ is dominated by $\boldsymbol{v}$ and a fortiori there exists a constant c such that $\bar{\boldsymbol{v}}=c\boldsymbol{v}$, i. e. $v=cv_j$. Then a minimal element $\boldsymbol{v}$ in (21) has

the form: there exists a component v_j of $\boldsymbol{v}=(v_1, v_2, \cdots)$ such that v_j is D-minimal on W_j with $\|v_j\|_{W_j}=1$ and $v_k=0$ $(k\neq j)$. Conversely, such a $\boldsymbol{v}$ is clearly a normalized minimal function. From these observation the desired conclusion (27) follows.

Tôki surfaces.

19. In our former paper [8] we introduced a notion of *Tôki surfaces:* A Riemann surface T is referred to as a Tôki surface if the following three conditions are satisfied:

α) *T is an infinite unbounded (unlimited) covering surface (T, U, π) with the unit disk U: $|z|<1$ as its base surface and π the projection of T onto U;*

β) *There exists a radial slits disk $V=U-\bigcup_\nu \sigma_\nu$ with σ_ν the radial slits in U accumulating only to the circumference of U such that $T-\pi^{-1}(\bigcup_\nu \sigma_\nu)=\sum_{n=1}^{\infty} V_n$ (disjoint union) where $V_n (n=1, 2, \cdots)$ are copies of V;*

γ) *There exists a bounded harmonic function $\hat{h}$ on U for any given bounded harmonic function h on T such that $h=\hat{h}\circ\pi$.*

A typical example of Tôki surfaces is the one constructed by Tôki [**12**], in which the condition

$$0\in V \tag{28}$$

can also be assumed where 0 is the origin of $\boldsymbol{C}$. Hereafter we consider only those Tôki surfaces T with (28). Let K be a concentric closed disk $|z|\leqq\rho$ $(0<\rho<1)$ contained in V, and consider the subsurface

$$S=T-\pi^{-1}(K)$$

of T, which is one of the *admissible subsurfaces* of T in the terminology of [8]. We have shown in [8], as a localization of the property γ), the following relation

$$HB(S;\partial S)=HB(U-K;\partial(U-K))\circ\pi \tag{29}$$

where $\partial(U-K)$ is the relative boundary of $U-K$ relative to U. Another result in [8] which we will repeatedly make use of is the following

$$d(S)=0, \qquad d^{\sim}(S)=1. \tag{30}$$

20. For our present setting it is convenient to take the base surface U of T as the 'unit disk' about the point at infinity ∞, i.e. $U: 1<|z|\leqq\infty$. The condition α), β), and γ) of Tôki surfaces are modified accordingly in an obvious manner. As a counter part to (28) we always assume that

$$\infty\in V. \tag{31}$$

We fix such a (modified) Tôki surface T. Let $K_t: t\leqq|z|\leqq\infty$ with $t\in[\tau, \infty)$ where $\tau>1$ is chosen so large that $K_\tau\subset V$, and set $K_{t,n}=V_n\cap\pi^{-1}(K_t)(n=1, 2, \cdots)$. Let w_t be the harmonic measure of the ideal boundary of T relative to the region $T-K_{t,1}$. By no. 5, there exists an increasing divergent sequence $\{t_k\}$ $(k=1, 2, \cdots)$ in (τ, ∞) such that

$$\sup_{z\in\alpha} w_{t_k}(z)>1/2 \tag{32}$$

where $\alpha=V_1\cap\pi^{-1}(|z|=\tau)$ and

$$D_{T-K_{t_k,1}}(w_{t_k})<1/2^k \tag{33}$$

for every $k=1, 2, \cdots$.

Let N be an at most countably infinite cardinal number with $N>1$ and $\{c_k\}$ $(1\leqq k<N)$ be a sequence of positive real numbers such that

$$c_k-c_{k-1}>t_k+t_{k-1} \qquad (2\leqq k<N). \tag{34}$$

We denote by (T^k, U^k, π^k) the translation of (T, U, π) by $z\to z+c_k(1\leqq k<N)$, i.e. T^k is the covering surface of $U^k: 1<|z-c_k|\leqq\infty$ and $\pi^k(\cdot)=\pi(\cdot)+c_k$; the images of V, V_n, K_t, $K_{t,n}$, σ_ν, α, etc. in (T^k, U^k, π^k) under this translation are denoted by V^k, V_n^k, K_t^k, $K_{t,n}^k$, σ_ν^k, α^k, etc., respectively. We denote by B_N the extended plane $\hat{C}: |z|\leqq\infty$ less the closed disks $|z-c_k|\leqq 1$ $(1\leqq k<N)$, and the point at infinity ∞ if $N=\infty$. We consider a Riemann surface T_N constructed as follows:

a) *T_N is an infinite unbounded (unlimited) covering surface (T_N, B_N, π_N) with the base surface B_N and π_N the projection of T_N onto B_N;*

b) *$\pi_N^{-1}(1<|z-c_k|<\tau)$ is identical with $(\pi^k)^{-1}(1<|z-c_k|<\tau)$ so that there exists a slits region $A=B_N-\bigcup_{1\leqq k<N}\bigcup_\nu\sigma_\nu^k$ such that $T_N-\pi_N^{-1}(\bigcup_{1\leqq k<N}\bigcup_\nu\sigma_\nu^k)=\sum_{n=1}^\infty A_n$ (disjoint union) where A_n $(n=1, 2, \cdots)$ are copies of A.*

These two properties a) and b) may be considered as counter parts to α) and β) in no. 19. We will prove that the following condition c) is satisfied by T_N which is a counter part to γ), and thus we may call T_N a *generalized Tôki surface.* Namely

c) *There exists a bounded harmonic function $\hat{h}$ on B_N for any given bounded harmonic function h on T_N such that $h=\hat{h}\circ\pi_N$, i.e.*

$$HB(T_N)=HB(B_N)\circ\pi_N. \tag{35}$$

21. To prove c) we may assume that $h>0$. Let $a=\sup_{T_N} h$ and $\hat{s}_k$ be harmonic on $1<|z-c_k|<\tau$ with boundary values 0 on $|z-c_k|=1$ and a on $|z-c_k|=\tau$. Let $\hat{s}=\hat{s}_k$ on $1<|z-c_k|<\tau$ $(1\leqq k<N)$ and $\hat{s}=a$ elsewhere on B_N. Set $s=\hat{s}\circ\pi_N$ and $s_k=\hat{s}_k\circ\pi_N$. By the Perron-Brelot method we can construct a harmonic function b_m on $\pi_N^{-1}(1+1/m<|z-c_k|<\tau)$ for integral $m>1/(\tau-1)$ such that $b_m=0$ on $\tau_N^{-1}(|z-c_k|=1+1/m)$ and $b_m=h$ on $\pi_N^{-1}(|z-c_k|=\tau)$ for

$1 \leqq k < N$. By the Perron-Brelot procedure we have $0 \leqq b_m \leqq b_{m+1} \leqq \max(h, s_k)$ and thus the limit $f_k = \lim_{m\to\infty} b_m$ exists and $0 \leqq f_k \leqq \max(h, s_k)$ on $\pi_N^{-1}(1<|z-c_k|<\tau)$ and $f_k = h$ on $\pi_N^{-1}(1<|z-c_k|<\tau)$. Therefore $h - f_k \in HB(S^k; \partial S^k)$ where $S^k = \pi_N^{-1}(1<|z-c_k|<\tau)$. By (20) we can find a bounded harmonic function $\hat{u}_k$ on $1<|z-c_k|<\tau$ with vanishing boundary values on $|z-c_k|=\tau$ such that $h-f_k = \hat{u}_k \circ \pi_N$ $(1 \leqq k < N)$. Let $\hat{u}$ be defined on B_N such that $\hat{u} = \hat{u}_k$ on $1<|z-c_k|<\tau$ $(1 \leqq k < N)$ and $\hat{u}=0$ elsewhere on B_N. Finally let $\hat{h}$ be the least harmonic majorant of $\hat{u}$ on B_N. Then $\hat{g} = \hat{h} - \hat{u}$ is a potential on B_N. Let $f = f_k$ on S^k $(1 \leqq k < N)$ and $f = h$ elsewhere on T_N. Then $f \leqq s = \hat{s} \circ \pi_N$ and $h - f = \hat{u} \circ \pi_N$. Therefore

$$|h - \hat{h} \circ \pi_N| \leqq (\hat{g} + \hat{s}) \circ \pi_N . \tag{36}$$

Clearly $\hat{s}$ is a potential on B_N and hence $\hat{p} = \hat{g} + \hat{s}$ is a potential on B_N. Let v be harmonic on T_N such that $0 \leqq v \leqq \hat{p} \circ \pi_N$ on T_N. Observe that except at most countable number of points c in B_N, $\pi_N^{-1}(|z-c|<\varepsilon_c) = \sum_{n=1}^{\infty} D_n$ (disjoint union) with D_n $(n=1, 2, \cdots)$ copies of $|z-c|<\varepsilon_c$ for a suitable $\varepsilon_c > 0$ for any given $c \in B_N$. Then it is easy to check that

$$v_0(z) = \sup_{\zeta \in \pi_N^{-1}(z)} v(\zeta)$$

is subharmonic on B_N, and $0 \leqq v_0 \leqq \hat{p}$ on B_N. Therefore $v_0 = 0$ on B_N and a fortiori $v=0$ on T_N. This means that $(\hat{g}+\hat{s}) \circ \pi_N = \hat{p} \circ \pi_N$ is a potential on T_N. By (36), since the subharmonic function $|h - \hat{h} \circ \pi_N|$ is dominated by a potential, we conclude that $h = \hat{h} \circ \pi_N$ on T_N.

22. As a direct consequence of (35) we obtain the following:

$$T_N \in O_{HD} - O_{HB} . \tag{37}$$

In view of $B_N \notin O_{HB}$, we clearly have $T_N \notin O_{HB}$. Let $u \in HBD(T_N)$. By b) and c), or (35),

$$\infty > D_{T_N}(u) = D_{\pi^{-1}(A)}(u) = \sum_{n=1}^{\infty} D_{A_n}(u \mid A_n) = \infty \cdot D_A(\hat{u})$$

where $u = \hat{u} \circ \pi_N$. Thus $\hat{u}$ and hence u is constant on T_N. Since $O_{HD} = O_{HBD}$, we conclude that $T_N \in O_{HD}$.

23. Consider a subsurface S_N of T_N defined as follows:

$$S_N = T_N - \pi_N^{-1}(B_{N,\tau}) \cap \left(\sum_{n=2}^{\infty} A_n\right) \tag{38}$$

where $B_{N,\tau}$ is the extended plane $\hat{C}$ less the closed disks $|z-c_k| \leqq \tau$ $(1 \leqq k < N)$, and the point at infinity ∞ if $N = \infty$. We represent S_N as a welding of convenient surfaces. As in no. 20 let $K_{t_k}^k$ be the closed disk $t_k \leqq |z-c_k| \leqq \infty$ and set

(39) $$C_N=\hat{\boldsymbol{C}}-\overline{\bigcup_{1\leqq k<N}(\boldsymbol{C}-K_{t_k}^k)}.$$

We also set

(40) $$\hat{W}_k=T^k-(\pi^k)^{-1}(\tau\leqq|z-c_k|\leqq\infty)\cap(\sum_{n=2}^{\infty}V_n^k)$$

$(1\leqq k<N)$. Observe that $Z_k=(\pi^k)^{-1}(t_k<|z-c_k|\leqq\infty)\cap V_1^k$ is a parametric 'disk' in $\hat{W}_k$. Finally set

(41) $$W_k=\hat{W}_k-\bar{Z}_k$$

$(1\leqq k<N)$. Then, as in no. 5, we have

(42) $$S_N=[C_N,\ \{W_k\}]$$

where $\partial W_k=-\partial Z_k$ is identified with $-\partial K_{t_k}^k$ for every k with $1\leqq k<N$. By (30) and (27) we deduce

(43) $$d^{\sim}(S_N)=N-1$$

where $\infty-1=\infty$.

24. PROOF OF THE MAIN THEOREM. We take the T_∞, i.e. T_N with $N=\infty$, as the required R. Since $R\in O_{HD}-O_{HB}$ (cf. (37)), $d^{\sim}(R)=d(R)=1$. For each integer $m\geqq 1$ let $d_m\in(c_m+t_m,\ c_{m+1}-t_{m+1})$ and Y_m be the 'disk' $d_m<|z|\leqq\infty$. Considering $\bar{Y}_m\subset C_{m+1}$, Y_m can be viewed as a parametric 'disk' in $S_{m+1}=[C_{m+1},\ \{W_k\}]$. Let $S^m=S_{m+1}-\bar{Y}_m$. Then by (43)

(44) $$d^{\sim}(S^m)=d^{\sim}(S_{m+1})=m\,.$$

It can also be viewed that S^m is a subsurface of R. Clearly

(45) $$d^{\sim}(S^\infty)=\infty\,,$$

the countably infinite cardinal number, where $S^\infty=S_\infty$, i.e. S_N with $N=\infty$, is a subsurface of R. Any compact subsurface S^0 of R satisfies

(46) $$d^{\sim}(S^0)=0\,.$$

This completes the proof of the main theorem.

Riemann surfaces with given harmonic dimensions.

25. We denote by $R_{bdd^{\sim}}$ a Riemann surface such that $x(R_{bdd^{\sim}})=x$ for $x=b,d$, and $d^{\sim}$. We have seen in [8] that R_{000}, R_{001}, R_{011}, R_{101}, R_{111} exist. For $\hat{W}_1=R_{lmn}$ and $\hat{W}_2=R_{l'm'n'}$, (27) assures that $[C_3,\ \{W_1,\ W_2\}]$ is an $R_{l+l',m+m',n+n'}$: By this, we can define an operation

(47) $$R_{lmn}\oplus R_{l'm'n'}=R_{l+l',m+m',n+n'}\,.$$

More generally, for $\hat{W}_k=R_{l_km_kn_k}$ $(1\leqq k<N)$, (27) implies that $[C_N,\ \{W_k\}]$ is an

$R_{\sum_{1\leq k<N} l_k, \sum_{1\leq k<N} m_k, \sum_{1\leq k<N} n_k}$ and we can write this as

$$(48) \qquad \bigoplus_{1\leq k<N} R_{l_k m_k n_k} = R_{\sum_{1\leq k<N} l_k, \sum_{1\leq k<N} m_k, \sum_{1\leq k<N} n_k}.$$

Using (47), (48), and surfaces R_{000}, R_{001}, R_{011}, R_{101}, R_{111}, the following conclusion can be instantly derived (cf. [8]):

THEOREM. *For any triple* $(b, d, \tilde{d})$ *of countable cardinal numbers with* $\max(b, d) \leqq \tilde{d}$, *there exists a Riemann surface* $R_{bd\tilde{d}}$.

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