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On subdiagonal algebras associated with flows in operator algebras

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Abstract. The noncommutative Hardy spaces $H^{\infty}(\alpha)$ and $H^{1}(\alpha)$ are introduced with respect to a σ-weakly continuous flow $\alpha=\{\alpha_t\}$ of $*$-automorphisms of a von Neumann algebra. In case that the algebra is α-finite the algebra $H^{\infty}(\alpha)$ becomes a maximal subdiagonal algebra. The concept of C^*-subdiagonal algebras will also be given for C^*-algebras as a noncommutative counterpart of the algebras of generalized analytic functions. Examples of maximal C^*-subdiagonal algebras and their structures are discussed.

§ 1. Introduction.

Let B (resp. M) be a C^*-algebra (resp. a von Neumann algebra) and $\{\alpha_t\}$ be a flow by which we mean a strongly continuous (resp. σ-weakly continuous) one parameter group of $*$-automorphisms of the algebra. Let $A(\alpha)$ be the set of all elements of B with nonnegative spectrum with respect to $\{\alpha_t\}$, then it turns out to be a closed subalgebra of B such that the intersection $A(\alpha)\cap A(\alpha)^*$ is the C^*-subalgebra of fixed elements of α_t. The algebra has been studied by several authors, especially in the case where B is a commutative C^*-algebra as a function algebra with an analyticity ([8], [17], [18], [4] etc). A prototype of such algebra is the classical disk algebra on the unit circle, or the algebra of generalized analytic functions, which is determined by the rotation flow on the unit circle (periodic), or by an almost periodic flow on the compact dual of an ordered discrete group. Furthermore, in these cases, the Hardy spaces H^{∞}'s are also considered as such algebras of elements with nonnegative spectrums with respect to the weak $*$ continuous flows of the corresponding L^{∞}-spaces, i.e. of commutative von Neumann algebras. Now, a counterpart of the H^{∞} algebra or more generally of a weak $*$ Dirichlet algebra has been studied by Arveson [2] in the context of von Neumann algebras as a theory of subdiagonal algebras, in which, however, the above rôle of flows has not been discussed. A subdiagonal algebra is a pair $(\mathfrak{A}, \varepsilon)$ where $\mathfrak{A}$ is a subalgebra of a von Neumann algebra M such that $\mathfrak{A}+\mathfrak{A}^*$ is σ-weakly dense in M, and where ε is a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}\cap\mathfrak{A}^*$ which extends in a suitable way to M.

It is the purpose of this paper to show that the above circumstance of analyticity may be carried over to the case of flows of operator algebras. We prove (Theorem 2.4) that for a flow $\{\alpha_t\}$ on a von Neumann algebra M the algebra $H^\infty(\alpha)$ of elements with nonnegative spectrums becomes a maximal subdiagonal algebra provided that M is α-finite. The result provides an abundance of examples of maximal subdiagonal algebras which are not covered by these examples in [**2**], for by the Tomita-Takesaki theory we find enough such flows in general as modular automorphism groups. In this connection, the results of Takesaki [**26**] may be understood as the structure theory of the subdiagonal algebra of a periodic flow of modular automorphism group. In § 3, we shall consider the representations of the algebra $A(\alpha)$ of analytic elements for a flow on a C^*-algebra. As in § 2, our arguments here contain a generalization of a part of Muhly [**17**]. In § 4, by using flows associated with generalized analytic functions we construct a family of examples of $H^\infty(\alpha)$ algebras as representations of C^*-subdiagonal algebras. The results here may be regarded as generalized and sharpened versions of those corresponding results in [**2**]. As we have mentioned above, they are different types of subdiagonal algebras from those algebras derived from modular automorphism groups.

§ 2. Subdiagonal algebras associated with flows.

Let M be a von Neumann algebra acting on a Hilbert space H. Let $\{\alpha_t\}$ be a flow on M, i.e., a σ-weakly continuous one parameter group of $*$-automorphisms of M. We denote by M_* the space of all σ-weakly continuous functionals of M. For each element x of M, a functional φ of M_* and a function f of $L^1(R)$, we define the convolution $x \underset{\alpha}{*} f$ in M and $\varphi \underset{\alpha}{*} f$ in M_* by

$$x \underset{\alpha}{*} f = \int_{-\infty}^{\infty} f(t)\alpha_t(x)dt$$

$$\langle x, \varphi \underset{\alpha}{*} f\rangle = \langle x \underset{\alpha}{*} \tilde{f}, \varphi\rangle ,$$

where $\tilde{f}$ means the function such that $\tilde{f}(t) = f(-t)$. The above integral exists in the sense that

$$\langle x \underset{\alpha}{*} f, \varphi\rangle = \int_R f(t)\langle \alpha_t(x), \varphi\rangle dt$$

for every $\varphi \in M_*$ ([**4**; Proposition 1.6]). Define the ideals of $L^1(R)$, $J(x)$ and $J(\varphi)$ by

$$J(x) = \{f \in L^1(R) \mid x \underset{\alpha}{*} f = 0\} ,$$

$$J(\varphi) = \{f \in L^1(R) \mid \varphi \underset{\alpha}{*} f = 0\} .$$

The hull of the ideal $J(x)$ (resp. $J(\varphi)$) is said to be the spectrum of x (resp. φ) with respect to the flow α and is denoted by $sp_\alpha(x)$ (resp. $sp_\alpha(\varphi)$). The spectrum is a closed subset of the real number space R. In the following, when there arise no confusion we sometimes drop the indiction α. Let E be a closed subset of R. The spectral subspace $M^\alpha(E)$ of M is defined to be the set of all x's in M with $sp_\alpha(x)\subset E$. For the Fourier transform we use the transform

$$\hat{f}(s)=\int_R e^{-ist}f(t)dt\,.$$

We refer the readers to [6], [4] for the elementary properties of spectrums and spectral subspaces. Put $H^\infty(\alpha)=M^\alpha([0, \infty))$. The space $H^\infty(\alpha)$ is σ-weakly closed and moreover it is a subalgebra of M by the following property of the α-spectrum;

$$sp_\alpha(xy)\subseteqq\overline{sp_\alpha(x)+sp_\alpha(y)} \qquad \text{(cf. [6], [14])}\,.$$

We call an element of $H^\infty(\alpha)$ an analytic element of M with respect to the flow α. Let $H^1(\alpha)$ be the set of all σ-weakly continuous analytic functions of M by which we mean the closed subspace of functional of M_* with nonnegative spectrums. We write $H^\infty(\alpha)_0$ the σ-weak closure of the set of elements of M with positive spectrums. The spaces $H^\infty(\alpha)$ and $H^1(\alpha)$ may occupy noncommutative counterpart of the usual Hardy space H^∞ and H^1 in the L^∞ and L^1 spaces. The following (known, but not expricitly formulated before in the literatures) equivalence will also justify this definition. We recall that $H^\infty(R)$ is the class of all functions f's in $L^\infty(R)$ such that $\int_R \mathrm{Im}\left(\frac{1}{t-z}\right)f(t)dt$ is holomorphic in the upper half plane. Let $H^1(R)$ be the corresponding Hardy class in $L^1(R)$.

PROPOSITION 2.1 (cf. [17; Proposition 2.1]). *Let $x\in M$, then x belongs to $H^\infty(\alpha)$ if and only if the function $F(t)=\langle\alpha_{-t}(x), \varphi\rangle$ belongs to $H^\infty(R)$ for every $\varphi\in M_*$.*

The following lemma may also be proved in a similar way to the case of a flows on a C^*-algebra (cf. [4; Proposition 5.1]).

LEMMA 2.2. *Let φ be a σ-weakly continuous functional on M. Then φ is analytic, i.e. $sp(\varphi)\subset[0, \infty)$ if and only if φ vanishes on $H^\infty(\alpha)_0$.*

We have details of the above results to the readers.

It is to be noticed that

$$M(\alpha)\equiv H^\infty(\alpha)\cap H^\infty(\alpha)^*=\{x\in M\,|\,sp(x)\subseteqq\{o\}\}$$

and it is the von Neumann subalgebra of all fixed elements of M by α. Also, we recall that for a functional φ of M_* we have $sp_\alpha(\varphi)\subseteqq\{o\}$ if and only if φ is α-invariant. The importance of the analytic functionals in the context of C^* or von Neumann algebras have been already discussed in the literatures in connection with generalizations of the F and M. Riesz theorem on the unit circle and of the quasi-equivalence for measures ([**6**], [**4**], [**11**]).

Take an element $x \in M$ and $\varphi \in M_*$, and let $F(t) = \langle \alpha_t(x), \varphi \rangle$. The function F is a bounded continuous function on R. For the usual spectrum of F we have

PROPOSITION 2.3. $sp(F) \subset -sp_\alpha(x) \cap sp_\alpha(\varphi)$.

The proposition was formulated in [**6**; p. 50 (30)] for a flow on the space of continuous functions, but the proof there remains valid. We include its proof because of the importance of the proposition in our arguments.

PROOF. Let r be a real number not in $sp_\alpha(\varphi)$. Then there exists a function f in $J(\varphi)$ with $\hat{f}(r)=1$. We have

$$F*f(o) = \int_R F(-t)f(t)dt = \int_R \langle \alpha_{-t}(x), \varphi \rangle f(t)dt$$
$$= \langle x \underset{\alpha}{*} \tilde{f}, \varphi \rangle = \langle x, \varphi \underset{\alpha}{*} f \rangle = 0 .$$

Since the set $J(\varphi)$ is translation invariant we have $(F*f_s)(o)=0$ for every $s \in R$, where $f_s(t)=f(s+t)$. Thus, we get

$$(F*f)(s) = 0 \qquad \text{for every} \quad s \in R ,$$

and f belongs to $J(F)$. This shows that r is not in $sp(F)$. On the other hand, let r be a real number not in $-sp_\alpha(x)$. There exists a function $f \in J(x)$ with $\hat{f}(-r)=1$. We have

$$F*\tilde{f}(o) = \langle x \underset{\alpha}{*} f, \varphi \rangle = 0 .$$

Then, as we have mentioned above,

$$(F*\tilde{f})(s) = 0 \qquad \text{for every} \quad s \in R .$$

Thus $\tilde{f} \in J(F)$ and r is not in $sp(F)$ because $\hat{\tilde{f}}(r) = \hat{f}(-r) = 1$. This completes the proof.

Let ε be a faithful normal projection of norm one in a von Neumann algebra M. Following Arveson [**2**] we call an algebra $\mathfrak{A}$ of M with the unit of M a subdiagonal algebra with respect to the projection ε if $\mathfrak{A}$ satisfies the following conditions;

(i) $\mathfrak{A} + \mathfrak{A}^*$ is σ-weakly dense in M,

(ii) $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$ for $x, y \in \mathfrak{A}$, that is, ε is multiplicative on $\mathfrak{A}$,

(iii) $\varepsilon(\mathfrak{A}) \subset \mathfrak{A} \cap \mathfrak{A}^*$.

We call the self-adjoint subalgebra $\mathfrak{A} \cap \mathfrak{A}^*$ the diagonal of $\mathfrak{A}$. When the diagonal of $\mathfrak{A}$ is reduced to the scalars, $\mathfrak{A}$ is said to be antisymmetric. A subdiagonal algebra $\mathfrak{A}$ with respect to ε is said to be maximal if it is contained properly in no larger subdiagonal algebra of M with respect to ε. For a given subdiagonal algebra $\mathfrak{A}$ for ε the maximal subdiagonal algebra which contains

$\mathfrak{A}$ is determined as $\mathfrak{A}_m=\{x\in M|\varepsilon(I\times\mathfrak{A})=\varepsilon(\mathfrak{A}\times I)=0\}$ where I is the kernel of ε in $\mathfrak{A}$ ([**2**; Theorem 2.2.1]). We notice that for a σ-weakly closed subdiagonal algebra the range of ε is exactly the diagonal $\mathfrak{A}\cap\mathfrak{A}^*$. It is an open question whether or not every σ-weakly closed subdiagonal algebra is already maximal, whereas every maximal subdiagonal algebra is σ-weakly closed. Now recall that M is said to be α-finite when the family of α-invariant normal states of M separates the nonnegative elements of M. In this case, by [**15**; Theorem 1.2] there exists a faithful normal projection of norm one ε of M onto $M(\alpha)$. Moreover, for each element x, $\varepsilon(x)$ is given as the only one element of the intersection $K(x, \alpha)\cap M(\alpha)$, where $K(x,\alpha)$ denotes the σ-weakly closed convex hull of $\{\alpha_t(x)\}$.

THEOREM 2.4. *Let M be a von Neumann algebra with a flow $\{\alpha_t\}$. Then, if M is α-finite, the algebra $H^\infty(\alpha)$ is a maximal subdiagonal with respect to the projection of norm one ε induced by the α-finiteness.*

The theorem shows a systematic way to construct maximal subdiagonal algebras in von Neumann algebras. In one way, the result may be regarded as a noncommutative version of Muhly [**17**; Theorem 1] for a weak $*$ Dirichlet algebra determined by a flow (and actually his theorem follows from the above theorem in commutative cases). In the other way, our theorem introduces a new class of maximal subdiagonal algebras which are not covered by those examples in [**4**], that is, a class of maximal subdiagonal algebras determined by the modular automorphism groups of Tomita-Takesaki theorem [**25**]. In fact, according to the theory each faithful normal state φ of M gives rise the modular automorphism group σ_t^φ for which φ is invariant, that is, M is σ_t^φ-finite. Thus, as we have already mentioned in the introduction, the results of Takesaki [**26**] is the structure theory of the algebra $H^\infty(\alpha)$ associated with a periodic modular automorphism group. The structure of $H^\infty(\alpha)$ for a general periodic flow has been recently given by Saito [**23**] as well as that of the space $H^1(\alpha)$. It is worth noticing, however, that in a commutative von Neumann algebra every modular automorphism group is reduced to the identity, so that flows associated to the usual analyticity fall into another category. We note that in the commutative case the theorem says that for an ergodic flow α the algebra $H^\infty(\alpha)$ is maximal as a weak $*$ Dirichlet algebra, though more general result is known in the literature ([**10**; III, Theorem 2.2]). As for the maximality of a σ-weakly closed subdiagonal algebra, the only systematic result known before was the result of Kamei [**13**; Theorem 3].

PROOF OF THE THEOREM. We show first that $H^\infty(\alpha)$ is subdiagonal. Suppose that a functional φ of M_* vanish on $H^\infty(\alpha)+H^\infty(\alpha)^*$. By Lemma 2.2, we have $sp(\varphi)\subset(0,\infty)$. Moreover, since $sp(x^*)=-sp(x)$ for every $x\in M$, $sp(\varphi)$ is also contained in $(-\infty, 0]$. Hence, $sp(\varphi)\subseteq\{0\}$ and φ is α-invariant. Therefore, since the image $\varepsilon(x)$ belongs to the σ-weakly closed convex hull of $\{\alpha_t(x)\}$ we have that $\varphi\circ\varepsilon=\varphi$. As φ vanishes on the algebra $M(\alpha)$, we conclude that $\varphi=0$ namely, $H^\infty(\alpha)+H^\infty(\alpha)^*$ is σ-weakly dense in M. To show that ε is multiplicative on $H^\infty(\alpha)$, we shall show that

$$\{x\in H^\infty(\alpha)\,|\,\varepsilon(x)=0\}=H^\infty(\alpha)_0\,,$$

an ideal of $H^\infty(\alpha)$. Take an element x with positive spectrum and let φ be an arbitrary normal state of $M(\alpha)$. Then, $\varphi\circ\varepsilon$ is an invariant normal state and hence by Lemma 2.2 one concludes that $\langle x, \varphi\circ\varepsilon\rangle=\langle\varepsilon(x), \varphi\rangle=0$. Hence $\varepsilon(x)=0$ and

$$H^\infty(\alpha)_0\subset\{x\in H^\infty(\alpha)\,|\,\varepsilon(x)=0\}\,.$$

Next suppose that there exist an element a of $H^\infty(\alpha)$ such that $\varepsilon(a)=0$ and $a\notin H^\infty(\alpha)_0$. We can find a functional φ in M_* such that $\langle a, \varphi\rangle=1$ and $\langle H^\infty(\alpha)_0, \varphi\rangle=0$. Let $F(t)=\langle\alpha_t(a), \varphi\rangle$. Then by Lemma 2.2 and Proposition 2.3 we have

$$sp(F)\subset -sp_\alpha(a)\cap sp_\alpha(\varphi)\subset(-\infty, 0]\cap[0, \infty)=\{0\}\,.$$

Hence, by [**22**; 7.8.3 (e)] F is constant in R. That is, $\langle\alpha_t(a), \varphi\rangle=\langle a, \varphi\rangle=1$. Therefore, $\langle x, \varphi\rangle=1$ for every $x\in K(x, \alpha)$ and this implies that $\langle\varepsilon(a), \varphi\rangle=1$. This is a contradiction. Hence $H^\infty(\alpha)_0$ coincides with the kernel of ε in $H^\infty(\alpha)$ and ε is multiplicative on $H^\infty(\alpha)$.

Now suppose that $H^\infty(\alpha)$ is not maximal subdiagonal with respect to ε. Then there exists an element a in the maximal subdiagonal algebra $H^\infty(\alpha)_m$ with $sp(a)\not\subset[0, \infty]$. By [**2**; Theorem 2.2.1] $H^\infty(\alpha)_m$ is an α_t-invariant σ-weakly closed subalgebra. Hence for every f in $L^1(R)$, $a*f$ belongs to $H^\infty(\alpha)_m$. We choose a function f of $L^1(R)$ such that $a*f\neq 0$ and $sp(a*f)\subset(-\infty, 0)$. Then the element $(a*f)^*$ belongs to $H^\infty(\alpha)_0$ and hence

$$\varepsilon((a*f)^*(a*f))=\varepsilon((a*f)^*)\varepsilon(a*f)=0\,,$$

which implies that $a*f=0$, a contradiction. This completes all proofs.

In this paper we do not enter into the discussions for the construction of another noncommutative Hardy spaces $H^p(\alpha)$ except for $H^\infty(\alpha)$ and $H^1(\alpha)$. It would be interesting to investigate their features in the generalized L^p-spaces canonically associated to a semifinite von Neumann algebra as the first candidate in the theory of noncommutative Hardy spaces.

Let $H^1(\alpha)_0$ be the norm closure of the set of elements with positive spectrums. The following is a dual version of Lemma 2.2.

LEMMA 2.5. *An element a of M has nonnegative spectrum if and only if every functional of $H^1(\alpha)_0$ vanishes at a.*

PROOF. Suppose that $\langle a, H^1(\alpha)_0\rangle=0$. Let f be a function of $L^1(R)$ such that $\hat{f}$ vanishes on a neighbourhood of $[0, \infty]$. For any functional φ of M_*, we have

$$\langle a*f, \varphi\rangle=\langle a, \varphi*\tilde{f}\rangle=0$$

because $sp(\varphi*\tilde{f})\subset \mathrm{supp}\,\hat{\tilde{f}}\subset[\varepsilon, \infty)$ for some positive ε. Hence, $a*f=0$ and $sp(a)\subset[0, \infty)$ by [**4**; Proposition 2.2]. Next, suppose that $sp(a)\subset[0, \infty)$ and take a functional φ of M_* with positive spectrum, say $sp(\varphi)\subset[\varepsilon, \infty)$. We may assume that $sp(\varphi)$ is a compact subset of R. Choose a function f of $L^1(R)$ such that $\hat{f}(r)=1$ on a neighborhood of $-sp(\varphi)$ and $\hat{f}(r)=0$ for $r\geqq -\frac{1}{3}\varepsilon$. Then, we have

$$\langle a, \varphi\rangle=\langle a, \varphi*\tilde{f}\rangle=\langle a*f, \varphi\rangle=0\,.$$

This completes the proof.

THEOREM 2.6. *Let M be a von Neumann algebra with a flow $\alpha=\{\alpha_t\}$. Then, if M is α-finite the space $H^1(\alpha)+H^1(\alpha)^*$ is norm dense in M_*.*

PROOF. Let a be an element of M such that

$$\langle a, H^1(\alpha)+H^1(\alpha)^*\rangle=0\,.$$

We assert that $a=0$. From the above lemma we have that $sp(a)\subset[0, \infty)$ and $sp(a^*)\subset[0, \infty)$. Hence, $sp(a)\subset\{0\}$ and a is α-invariant. Let φ be a normal invariant state of M. Then, the functional $L_a{}^*\varphi$ is also α-invariant, where $L_a{}^*\varphi$ is defined as $\langle x, L_a{}^*\varphi\rangle=\langle a^*x, \varphi\rangle$. Hence,

$$\langle a^*a, \varphi\rangle=\langle a, L_a{}^*\varphi\rangle=0\,,$$

which implies that $a=0$.

We conclude this section with the following observation. Recall that a subdiagonal algebra $(\mathfrak{A}, \varepsilon)$ in a finite von Neumann algebra M is said to be finite if there exists a faithful normal trace for which ε is invariant. The structure of a finite subdiagonal algebra has been extensively studied in [**2**].

PROPOSITION 2.7. *Let M be a finite von Neumann algebra on a separable Hilbert space with a flow α and let Z be the center of M. Then if Z is α-finite, $H^\infty(\alpha)$ is a finite maximal subdiagonal algebra.*

PROOF. Let Φ be the faithful normal center-valued trace of M. Then, for every t, $\Phi_t(a)=\alpha_t^{-1}\Phi(\alpha_t(a))$ is also a center-valued trace and hence the unicity of the trace implies that $\alpha_t\Phi(a)=\Phi(\alpha_t(a))$. Let $\{\varphi_\alpha\}$ be a separating family of invariant normal states of Z. The above equality for Φ shows that $\{\varphi_\alpha\circ\Phi\}$ is a separating family of invariant normal traces of M. Therefore we get a faithful α-invariant normal trace and by Theorem 2.4 the algebra $H^\infty(\alpha)$ is a finite maximal subdiagonal algebra.

The proposition includes the case where α is a group of automorphisms leaving the center elementwise fixed such as modular automorphism groups.

§ 3. Dirichlet subalgebras and C^*-subdiagonal algebras.

In this section we shall consider a subalgebra of a C^*-algebra corresponding to a subdiagonal algebra in a von Neumann algebra, as a noncommutative version of the algebra of generalized analytic functions. Let B be a C^*-algebra with unit. We say that a closed subalgebra $\mathcal{A}$ of B with the unit of B is a Dirichlet subalgebra if $\mathcal{A}+\mathcal{A}^*$ is norm dense in B. Let ε be a faithful projection of norm one in B. A Dirichlet subalgebra $\mathcal{A}$ of B is said to be C^*-subdiagonal if the pair $(\mathcal{A}, \varepsilon)$ satisfies the following conditions;

(i) ε is multiplicative on $\mathcal{A}$,

(ii) $\varepsilon(\mathcal{A})=\mathcal{A}\cap\mathcal{A}^*$.

We call the C^*-subalgebra $\mathcal{A}\cap\mathcal{A}^*$ the diagonal of $\mathcal{A}$. Let $I=\{x\in\mathcal{A}\,|\,\varepsilon(x)=0\}$ and put

$$\mathcal{A}_m=\{x\in B\,|\,\varepsilon(Ix\mathcal{A})=\varepsilon(\mathcal{A}xI)=0\}\ .$$

Then as in the case of a subdiagonal algebra, the set $\mathcal{A}_m$ turns out to be the maximal C^*-subdiagonal algebra which contains $\mathcal{A}$. This can be proved along with the lines of the proof of Theorem 2.2.1 in [2] with some technical modifications. We include the result in

THEOREM 3.1. *For a given C^*-subdiagonal algebra $\mathcal{A}$ of B with respect to ε, the set $\mathcal{A}_m$ is the maximal C^*-subdiagonal algebra with respect to ε which contains $\mathcal{A}$.*

PROOF. Let $\mathcal{A}_1$ be a C^*-subdiagonal algebra which contains $\mathcal{A}$. Then, for every $a\in\mathcal{A}$, $b\in I$ and $c\in\mathcal{A}_1$ we have $\varepsilon(acb)=\varepsilon(bca)=0$. Hence, $\mathcal{A}_m\supset\mathcal{A}_1$ and $\mathcal{A}_m+\mathcal{A}_m^*$ is norm dense in B. We shall show that $\mathcal{A}_m$ is an algebra and ε is multiplicative on $\mathcal{A}_m$. Note that ε is a projection of norm one of B to the C^*-subalgebra $\mathcal{A}\cap\mathcal{A}^*$. Let $\{\varphi_\alpha\}$ be a faithful family of states of $\mathcal{A}\cap\mathcal{A}^*$ and put $\hat{\varphi}_\alpha=\varphi_\alpha\circ\varepsilon$. Let π be the Gelfand-Neumark-Segal (GNS) representation of B by $\hat{\varphi}_\alpha$ on the Hilbert space H_α and consider the (isomorphic) representation $\pi=\sum_\alpha\pi_\alpha$ on the space $H=\sum_\alpha H_\alpha$. Let η_α be the canonical map of B into H_α. We define the following closed subspaces of H_α;

$$\mathcal{M}_\alpha=[\eta_\alpha(\mathcal{A})]\,,\qquad \mathcal{M}_\alpha^*=[\eta_\alpha(\mathcal{A}^*)]\,,$$

$$\mathcal{N}_\alpha=[\eta_\alpha(I)]\,,\qquad \mathcal{N}_\alpha^*=[\eta_\alpha(I^*)]\,,$$

and consider the subspaces of H;

$$\mathcal{M}=\sum_\alpha\mathcal{M}_\alpha\,,\qquad \mathcal{M}^*=\sum_\alpha\mathcal{M}_\alpha^*\ ,$$

$$\mathcal{N}=\sum_\alpha\mathcal{N}_\alpha\,,\qquad \mathcal{N}^*=\sum_\alpha\mathcal{N}_\alpha^*\,.$$

Let

$$\mathcal{A}_M=\{x\in B\mid \pi(x)\mathcal{M}\subset\mathcal{M},\ \pi(x)^*\mathcal{M}^*\subset\mathcal{M}^*\}\ .$$

Clearly, $\mathcal{A}_M$ is a closed subalgebra of B. We assert that $\mathcal{A}_m=\mathcal{A}_M$ and ε is multiplicative on $\mathcal{A}_M$. Since, the set $\mathcal{A}+\mathcal{A}^*=\mathcal{A}+I^*=\mathcal{A}^*+I$ is norm dense in B the same argument in the proof of [**2**; Theorem 2.2.1] shows that

$$H_\alpha=\mathcal{M}_\alpha\oplus\mathcal{N}_\alpha^*=\mathcal{M}_\alpha^*\oplus\mathcal{N}_\alpha\ .$$

Hence,

$$H=\mathcal{M}\oplus\mathcal{N}^*=\mathcal{M}^*\oplus\mathcal{N}\ .$$

Take an element $x\in\mathcal{A}_m$, then for every $a\in\mathcal{A}$ and $b\in I^*$ we have that

$$(\pi_\alpha(x)\eta_\alpha(a),\ \eta_\alpha(b^*))=\hat{\varphi}_\alpha(bxa)=\varphi_\alpha\circ\varepsilon(bxa)=0$$

and

$$(\pi_\alpha(x^*)\eta_\alpha(a^*),\ \eta_\alpha(b))=\hat{\varphi}_\alpha(b^*x^*a^*)=\varphi_\alpha\circ\varepsilon(axb)^*=0$$

by the definition of $\mathcal{A}_m$. Hence, $\pi_\alpha(x)\eta_\alpha(\mathcal{A})\perp\eta_\alpha(I^*)$ and $\pi_\alpha(x^*)\eta_\alpha(\mathcal{A}^*)\perp\eta_\alpha(I)$. Therefore,

$$\pi_\alpha(x)\mathcal{M}_\alpha\subset\mathcal{M}_\alpha \quad\text{and}\quad \pi_\alpha(x^*)\mathcal{M}_\alpha^*\subset\mathcal{M}_\alpha^*\ ,$$

which implies that

$$\pi(x)\mathcal{M}\subset\mathcal{M} \quad\text{and}\quad \pi(x^*)\mathcal{M}^*\subset\mathcal{M}^*\ .$$

Thus, $\mathcal{A}_m$ is contained in $\mathcal{A}_M$. Next, let p_α be the projection of H onto the subspace of H_α determined by $\eta_\alpha(\varepsilon(B))$. By [**2**; Proposition 6.11],

$$p_\alpha\eta_\alpha(x)=\eta_\alpha(\varepsilon(x)) \qquad\text{for every}\quad x\in B\ .$$

Now, for every element a and b in $\mathcal{A}$, we have

$$\begin{aligned} p_\alpha\pi(a)^*\eta_\alpha(b^*) &= p_\alpha(\eta_\alpha(a^*b^*))=\eta_\alpha(\varepsilon(a^*b^*)) \\ &= \eta_\alpha(\varepsilon(a^*)\varepsilon(b^*))=\pi(\varepsilon(a^*))p_\alpha\eta_\alpha(b^*)\ , \end{aligned}$$

which says that $p_\alpha\pi(a^*)\mid\mathcal{M}_\alpha^*=\pi(\varepsilon(a^*))p_\alpha\mid\mathcal{M}_\alpha^*$. Therefore, if $a\in\mathcal{A}$, $x\in\mathcal{A}_M$, for every index α we get

$$\begin{aligned} \eta_\alpha((\varepsilon(x)\varepsilon(a))^*) &= \eta_\alpha(\varepsilon(a^*)\varepsilon(x^*))=\pi(\varepsilon(a^*))p_\alpha\eta_\alpha(x^*) \\ &= p_\alpha\pi(a^*)\eta_\alpha(x^*)=\eta_\alpha(\varepsilon(a^*x^*)) \end{aligned}$$

because $\eta_\alpha(x^*)=\pi_\alpha(x^*)\eta_\alpha(1)\in\mathcal{M}_\alpha^*$. Hence,

$$\eta_\alpha((\varepsilon(x)\varepsilon(a))^*-\varepsilon(a^*x^*))=0$$

and

$$\hat{\varphi}_\alpha((\varepsilon(a^*)\varepsilon(x^*)-\varepsilon(a^*x^*))^*(\varepsilon(a^*)\varepsilon(x^*)-\varepsilon(a^*x^*)))=0 \qquad\text{for every}\quad \alpha\ .$$

Therefore, $\varepsilon(a^*)\varepsilon(x^*)=\varepsilon(a^*x^*)$, that is, $\varepsilon(x)\varepsilon(a)=\varepsilon(xa)$. Thus,

$$p_\alpha\pi(x)\eta_\alpha(a)=\eta_\alpha(\varepsilon(xa))=\eta_\alpha(\varepsilon(x)\varepsilon(a))=\pi_\alpha(\varepsilon(x))p_\alpha\eta_\alpha(a) \qquad\text{for every}\quad \alpha\ ,$$

which means that $p_\alpha\pi(x)|\mathcal{M}_\alpha=\pi(\varepsilon(x))p_\alpha|\mathcal{M}_\alpha$. If $x, y\in\mathcal{A}_M$, then $\eta_\alpha(y)=\pi(y)\eta_\alpha(1)\in\mathcal{M}_\alpha$, so that

$$\begin{aligned}\eta_\alpha(\varepsilon(xy)) &= p_\alpha\eta_\alpha(xy)=p_\alpha\pi(x)\eta_\alpha(y)\\ &=\pi(\varepsilon(x))p_\alpha\eta_\alpha(y)=\pi(\varepsilon(x))\eta_\alpha(\varepsilon(y))\\ &=\eta_\alpha(\varepsilon(x)\varepsilon(y)) \qquad \text{for every } \alpha .\end{aligned}$$

Therefore, the same arguments as above show that $\varepsilon(xy)=\varepsilon(x)\varepsilon(y)$. Hence, $\mathcal{A}_M$ is a C^*-subdiagonal algebra which contains $\mathcal{A}$, and $\mathcal{A}_M=\mathcal{A}_m$.

Let $\alpha=\{\alpha_t\}$ be a strongly continuous one parameter group of $*$-automorphisms of B, a flow. We denote by $A(\alpha)$ the algebra of all elements with nonnegative α-spectrums (the same relation for the spectrum $sp_\alpha(xy)$ as in the case of a flow on a von Neumann algebra enables the set $A(\alpha)$ to be a subalgebra of B). Then as in Theorem 2.4 if $A(\alpha)$ turns out to be a C^*-subdiagonal algebra we can show that it is the maximal C^*-subdiagonal algebra by the above theorem. Put $B(\alpha)=A(\alpha)\cap A(\alpha)^*$ and write $A(\alpha)_0$ as the norm closure of the set of elements with positive spectrums.

Let φ be an α-invariant state of B. Then the GNS-representation π of φ becomes a covariant representation in the sense that there exists a strongly continuous one parameter unitary group u_t in H such that $u_t\pi_\varphi(a)u_t^*=\pi(\alpha_t(a))$ for every $a\in B$. Thus $\{u_t\}$ induces a flow β_t on $M=\widetilde{\pi_\varphi(B)}$. Let $H^\infty(\beta)$ be the algebra of analytic element of M for β.

THEOREM 3.2. *The σ-weak closure of $\pi_\varphi(A(\alpha)_0)$ coincides with $H^\infty(\beta)_0$.*

PROOF. Take a positive number ε and an element a of B with $sp_\alpha(a)\subset[\varepsilon, \infty)$. Let f be an arbitrary function of $J(a)$, then $\pi_\varphi(a)\underset{\beta}{*}f=\pi_\varphi(a\underset{\alpha}{*}f)=0$. Hence, $sp_\beta(\pi_\varphi(a))\subseteqq sp_\alpha(a)\subset[\varepsilon, \infty)$. This shows that $\pi_\varphi(A(\alpha)_0)$ is contained in $H^\infty(\beta)_0$. Next, take an element x with $sp_\beta(x)\subset[\varepsilon, \infty)$ and let $\{\pi_\varphi(a_i)\}$ be a net of $\pi_\varphi(B)$ converging σ-weakly to x. Note that x is σ-weakly approximated by $x\underset{\beta}{*}f_r$ for the approximate identity $\{f_r\}$ of $L^1(R)$ whose Fourier transforms have compact supports. We may assume that $sp_\beta(x)$ is compact. Then let f be a function of $L^1(R)$ such that $\hat{f}=1$ on a neighborhood of $sp(x)$ and $\operatorname{supp}\hat{f}\subset\left[\frac{\varepsilon}{2}, \infty\right]$. We have $x=x\underset{\beta}{*}f$ and since the map $y\to y\underset{\beta}{*}f$ is σ-weakly continuous by [**4**; Proposition 1.6], we get

$$x=\lim \pi_\varphi(a_i)\underset{\beta}{*}f=\lim \pi_\varphi(a_i\underset{\alpha}{*}f)$$

where $a_i\underset{\alpha}{*}f\in A(\alpha)_0$. Hence, $H^\infty(\beta)_0$ is contained in $\pi_\varphi(A(\alpha)_0)$.

Let ω be a normal state of M such that $\omega\circ\pi=\varphi$. We have the following

PROPOSITION 3.3. *Suppose that M is β-finite and the σ-weak closure of $\pi_\varphi(B(\alpha))$ coincides with $M(\beta)$. Then, $H^\infty(\beta)$ is the σ-weak closure of $\pi_\varphi(A(\alpha))$.*

The proposition is an easy consequence of the Theorem 1.2 and the fact $\{x\in H^{\infty}(\beta)\mid\varepsilon(x)=0\}=H^{\infty}(\beta)_0$.

Corresponding to the definition in a commutative dynamics, we say that a flow α on a C^*-algebra B is strictly ergodic if there exists only one α-invariant state φ_s on B. The following result is a generalization of Muhly [**17**; Theorem 2].

PROPOSITION 3.4. *If* (B, α) *is strictly ergodic, then the algebra* $A(\alpha)$ *is a Dirichlet subalgebra of* B.

PROOF. Let φ be a bounded functional of B vanishing on $A(\alpha)+A(\alpha)^*$. Then, $sp(\varphi)\subseteqq\{0\}$ and φ is α-invariant. By [**5**; 12.3.4], φ is uniquely decomposed into the sum of nonnegative linear functionals in which

$$\varphi=\varphi_1-\varphi_2+i(\varphi_3-\varphi_4)\,,\qquad \|\varphi_1-\varphi_2\|=\|\varphi_1\|+\|\varphi_2\|$$

and

$$\|\varphi_3-\varphi_4\|=\|\varphi_3\|+\|\varphi_4\|\,.$$

Since φ is invariant, the unicity implies that all φ_i's are invariant. Therefore, we can write as $\varphi_i=\|\varphi_i\|\varphi_s$ $(i=1, 2, 3, 4)$ and hence $\varphi=\lambda\varphi_s$ for some scalar λ. However, since the unit of B is in $A(\alpha)$ one sees that

$$\lambda=\lambda\langle 1, \varphi_s\rangle=\langle 1, \varphi\rangle=0 \quad\text{and}\quad \varphi=0\,.$$

Thus, $A(\alpha)+A(\alpha)^*$ is norm dense in B.

Let π be the GNS-representation of B by φ and write $\varphi=\omega\circ\pi_{\varphi}$. In commutative cases, the functional ω becomes a faithful invariant state on $\pi_{\varphi}(B)$ and one may conclude without difficulty that the algebra $A_{\pi}(\beta)$ of analytic elements of $\pi_{\varphi}(B)$ for the induced flow β from α turns out to be a maximal C^*-subdiagonal algebra with the projection $\varepsilon(\pi_{\varphi}(x))=\langle\pi_{\varphi}(x), \omega\rangle 1=\langle x, \varphi\rangle 1$.

§ 4. Examples.

Let Γ be a discrete abelian group with an archimedian order and let G be the dual of Γ. As it is well known, Γ is isomorphic to the integer group or to a dense subgroup of the additive group of real numbers R. In both cases, every real number t gives rise a character e_t of Γ by $\langle e_t, \gamma\rangle=e^{t+\gamma}$. In the latter case, e_t is a character of Γ which is continuous on Γ with respect to the usual topology of R restricted to Γ. Now the translation $T_t(g)=g+e_t$ defines a flow on the compact dual G. Let $C(G)$ be the algebra of all complex valued continuous functions on G. The flow $\{T_t\}$ induces naturally a strongly continuous one-parameter group $\{\alpha_t\}$ of $*$-automorphisms of $C(G)$ defined by $\alpha_t f(y)=f(g-e_t)$. Here the algebra $A(\alpha)=A(G)$ is simply the disk algebra or the algebra of generalized analytic functions on G (cf. [**9**; Chap. VII]). The flow α is strictly ergodic with the unique invariant faithfull state μ, the normalized Haar measure on G. Hence, as we have mentioned in § 3 the algebra $A(G)$ is a maximal C^*-subdiagonal algebra with respect to the faithful projection $\varepsilon(f)=\langle f, \mu\rangle=\int_G f(g)d\mu$. Let M be a von Neumann algebra on a separable Hilbert space H. We denote by B the C^*-tensor product $C(G)\underset{\alpha}{\otimes} M$ of $C(G)$ and M. Since the cross norm α coincides with the least cross norm λ by [**24**; Proposition 2] the algebra B is also regarded as the algebra $C(G, M)$ of all M-valued continuous functions on G. Set $\tilde{\alpha}_t=\alpha_t\otimes 1$. We get a flow $\tilde{\alpha}=\{\tilde{\alpha}_t\}$ on B. We shall investigate first the structure of the algebra of analytic elements, $A(\tilde{\alpha})$. Let L_φ be the left Fubini mapping associated with a bounded linear functional φ of M (cf. [28]), that is, L_φ is a bounded linear mapping of B to $C(G)$ such that

$$L_\varphi(\sum_{i=1}^n f_i\otimes a_i)=\sum_{i=1}^n\langle a_i, \varphi\rangle f_i\,.$$

LEMMA 4.1. *For each element x of B, we have*

$$sp_{\tilde{\alpha}}(x)=\overline{\bigcup_{\varphi\in M_*} sp_\alpha(L_\varphi(x))}=\overline{\bigcup_{\varphi\in M^*} sp_\alpha(L_\varphi(x))}\,.$$

PROOF. Take a real number γ such that $\gamma\notin\overline{\bigcup_{\varphi\in M_*} sp_\alpha(L_\varphi(x))}$. There exists a function f of $L^1(R)$ such that $\hat{f}$ vanishes on a neighborhood of $\overline{\bigcup_{\varphi\in M_*} sp_\alpha(L_\varphi(x))}$ and $\hat{f}(\gamma)=1$. Then,

$$\begin{aligned}L_\varphi(x\underset{\tilde{\alpha}}{*}f)&=\int_R L_\varphi(\tilde{\alpha}_t(x))f(t)dt=\int_R \alpha_t(L_\varphi(x))f(t)dt\\&=L_\varphi(x)\underset{\alpha}{*}f=0\qquad\text{for every}\quad \varphi\in M_*\,.\end{aligned}$$

It follows that $x\underset{\tilde{\alpha}}{*}f=0$ and $\gamma\notin sp_{\tilde{\alpha}}(x)$. Thus,

$$sp_{\tilde{\alpha}}(x)\subset\overline{\bigcup_{\varphi\in M_*} sp_\alpha(L_\varphi(x))}\subset\overline{\bigcup_{\varphi\in M^*} sp_\alpha(L_\varphi(x))}\,.$$

On the other hand, if $\gamma\notin sp_{\tilde{\alpha}}(x)$ we can find a function $f\in J(x)$ with $\hat{f}(\gamma)=1$. Then, $x\underset{\tilde{\alpha}}{*}f=0$ which implies that $L_\varphi(x)\underset{\alpha}{*}f=0$ for every $\varphi\in M^*$. Hence, $\gamma\notin sp_\alpha(L_\varphi(x))$ and

$$\bigcup_{\varphi\in M_*} sp_\alpha(L_\varphi(x))\subset\bigcup_{\varphi\in M^*} sp_\alpha(L_\varphi(x))\subset sp_{\tilde{\alpha}}(x)\,.$$

Since $sp_{\tilde{\alpha}}(x)$ is closed we get the desired inclusion.

Let $\{p_\tau\}$ be an approximate identity consisting of trigonometric polynomials on G by which we mean a net of polynomials,

$$p_\tau(g)=\sum_{\gamma\in\Gamma}\lambda_\tau(\gamma)\langle g, \gamma\rangle$$

satisfies the following conditions;

$$\text{(4.2)}\qquad \begin{aligned}&\text{(i)}\quad p_\tau \geqq 0,\\ &\text{(ii)}\quad \int_G p_\tau(g)d\mu = 1,\\ &\text{(iii)}\quad \lim \sup_{g\notin U} |p(g)| = 0 \quad \text{for every neighborhood } U \text{ of the identity.}\end{aligned}$$

Take an element x of B and consider its "Fourier transform" $\hat{x}(\gamma)$ which is defined by $\hat{x}(\gamma)=\int_G x(g)\langle g, \gamma\rangle d\mu$ for $\gamma\in\Gamma$.

Put $x_\tau(g)=\sum_{\gamma\in\Gamma}\lambda_\tau(\gamma)\langle g, \gamma\rangle\hat{x}(\gamma)$. Clearly x_τ belongs to B.

LEMMA 4.3. *The function $x_\tau(g)$ converges uniformly to $x(g)$, this is $x_\tau\to x$ in B.*

PROOF. We have

$$\begin{aligned}x(g)&=\sum_{\gamma\in\Gamma}\lambda_\tau(g)\langle g, \gamma\rangle\int_G x(h)\langle h, \gamma\rangle d\mu\\ &=\int_G\sum_{\gamma\in\Gamma}\lambda_\tau(\gamma)\langle g-h, \gamma\rangle x(h)d\mu(n)\\ &=\int_G\sum_{\gamma\in\Gamma}\lambda_\tau(\gamma)\langle h, \gamma\rangle x(g-h)d\mu(n)\\ &=\int_G p_\tau(h)x_h(g)d\mu(h) \qquad \text{where} \quad x_h(g)=x(g-h)\,.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_\tau(g)-x(g)\|&=\left\|\int_G p_\tau(g)(x_h(g)-x(g))d\mu(h)\right\|\\ &\leqq \sup_{h\in U}\|x_h(g)-x(g)\|+\sup_{h\notin U}|p_\tau(h)|\cdot 2\|x\|\end{aligned}$$

and

$$\|x-x_\tau\|\leqq \sup_{h\in U}\|x-x_h\|+2\|x\|\sup_{h\in U}|p_\tau(h)|$$

for every neighborhood U of the identity. Now the map $h\to x_h$ is a continuous map of G into $C(G, M)=B$ so that the first term will be small for an appropriate U, where as the second term will become smaller for that U by the property (iii) of (4.2).

LEMMA 4.4. *If x is an analytic element of B, then $\hat{x}(\gamma)=0$ for every $\gamma<0$.*

PROOF. For every $\varphi\in M^*$, $L_\varphi(x)\in A(\alpha)$ by Lemma 4.1. Hence,

$$\int_G L_\varphi(x)(g)\langle g, \gamma\rangle d\mu=0 \qquad \text{for every} \quad \gamma<0\,.$$

It follows that

$$\begin{aligned}\langle\hat{x}(\gamma), \varphi\rangle&=\int_G\langle x(g), \varphi\rangle\langle g, \gamma\rangle d\mu\\ &=\int_G L_\varphi(x)(g)\langle g, \gamma\rangle d\mu=0 \qquad \text{for every} \quad \varphi\in M^*\,.\end{aligned}$$

Hence, $\hat{x}(\gamma)=0$.

Let R_μ be the right Fubini mapping of B to M associated to the state μ. Then, if we identify M with the subalgebra $1\otimes M$ in B, R_μ is a faithful projection of norm one of B to M because μ is a faithful state on $C(G)$. Denote this projection by ε_μ.

THEOREM 4.5. *$A(\tilde{\alpha})=A(G)\underset{\lambda}{\otimes}M$ and it is a maximal C^*-subdiagonal algebra with respect to ε_μ with the diagonal M.*

PROOF. It is apparent that $A(G)\underset{\lambda}{\otimes}M\subset A(\tilde{\alpha})$. Let x be an analytic element of B, then by Lemma 4.3 and 4.4 x is approximated uniformly by the elements of $A(G)\underset{\lambda}{\otimes}M$. Hence $A(\tilde{\alpha})=A(G)\underset{\lambda}{\otimes}M$, which implies that ε_μ is multiplicative on $A(\tilde{\alpha})$ because μ is multiplicative on $A(G)$. Finally, since $A(\tilde{\alpha})$ is a C^*-subdiagonal algebra defined by a flow, by Theorem 3.1 the same argument as in Theorem 2.4 shows that $A(\tilde{\alpha})$ is the maximal C^*-subdiagonal algebra for ε_μ.

It is worth noticing that if we denote by $A(G)_0$ the kernel of the multiplicative state μ in $A(G)$ then the kernel of ε_μ in $A(\tilde{\alpha})$ coincides with $A(\alpha)_0\underset{\lambda}{\otimes}M$ $=\{x\in A(\tilde{\alpha})\,|\,\hat{x}(e)=0\}$, the uniform closure of the set of elements of B with positive spectrums. The authors do not know whether or not this latter thing holds for an arbitrary C^*-subdiagonal algebra.

Now let us consider $C(G)$ as a C^*-algebra of bounded (multiplication) operators on the Hilbert space $L^2(G)$. Let $\mathcal{M}=L^\infty(G)\overline{\otimes}M$ be the tensor product of $L^\infty(G)$ and M on $L^2(G)\underset{\sigma}{\otimes}H$ as von Neumann algebras. The flow $\tilde{\alpha}$ of B and the projection ε_μ can be naturally extended to $\mathcal{M}$ as a (σ-weakly continuous) flow $\tilde{\alpha}$ (use the same letter) and a faithful normal projection ε_μ of $\mathcal{M}$ to M. The predual of $\mathcal{M}$ is expressed as the tensor product $L^1(G)\underset{\sigma}{\otimes}M_*$ with the greatest cross norm γ. Let $H'(G)$ be the Hardy class associated to $A(G)$. Then, since the predual $\mathcal{M}_*$ is identified with $L^1(G, M_*)$, the product space $H^1(G)\underset{\gamma}{\otimes}M_*$ may be viewed as a closed subspace of $\mathcal{M}_*$. Thus, we have

THEOREM 4.6. *In the von Neumann algebra $\mathcal{M}$; the algebra $H^\infty(\tilde{\alpha})$ is the σ-weak closure of $A(\tilde{\alpha})$ and the maximal subdiagonal algebra with respect to ε_μ. The space of analytic functionals $H^1(\tilde{\alpha})$ is the tensor product $H^1(G)\underset{\gamma}{\otimes}M_*$.*

The first assertion corresponds to Theorem 5.3.2 in [2]. The result means that $H^\infty(\tilde{\alpha})$ is the σ-weak closure of the finite combinations, $\{\sum_{\gamma\geqq 0}\langle\cdot,\gamma\rangle a_\gamma\,|\,a_\gamma\in M\}$.

PROOF. The first assertion is an easy consequence of Proposition 3.3. For the second assertion one can easily see that the algebraic tensor product of the Hardy class $H^1(G)$ and M_* is contained in $H^1(\tilde{\alpha})$. Let φ be an element of $H^1(\tilde{\alpha})$ and regard it as an M_*-valued integrable function $\varphi(g)$ of G. Take an element a of M and consider the left Fubini mapping L_a associated to the functional a on M_*. Then, the same arguments as in Lemma 4.2 show that

$La(\varphi)\in H^1(G)$.

Hence,

$$\int_G L_a(\varphi)(g)\langle g,\gamma\rangle d\mu=0 \qquad \text{for every} \quad \gamma<0.$$

Now consider " the negative coefficients of the Fourier transform $\hat{\varphi}$ " of φ which is defined by $\hat{\varphi}(\gamma)=\int_G \varphi(g)\overline{\langle g,\gamma\rangle}d\mu$. We have;

$$\begin{aligned}\langle a,\hat{\varphi}(\gamma)\rangle &= \int_G \langle a,\varphi(g)\rangle\overline{\langle g,\gamma\rangle}d\mu \\ &= \int_G L_a(\varphi)(g)\langle g,\gamma\rangle d\mu=0\end{aligned}$$

for every $\gamma<0$ and every $a\in M$. Thus, $\hat{\varphi}(\gamma)=0$ for $\gamma<0$. Put $\varphi_\tau(g)=\sum_{\gamma\in\Gamma}\lambda_\tau(g)\langle g,\gamma\rangle\hat{\varphi}(\gamma)$ for the approximate identity $p_\tau(g)$. As in the proof of Lemma 4.3 we have

$$\varphi_\tau(g)=\int_G p_\tau(h)\varphi(g-h)d\mu=\int_G p_\tau(h)\varphi_h(g)d\mu(h).$$

Hence,

$$\begin{aligned}\|\varphi_\tau-\varphi\| &= \int_G \|\varphi_\gamma(g)-\varphi(g)\|d\mu \\ &\leqq \int_G\int_G \|p_\gamma(h)(\varphi_h(g)-\varphi(g))\|d\mu(h)d\mu(g) \\ &\leqq \sup_{h\in U}\|\varphi_h-\varphi\|+2\|\varphi\|\sup_{h\in U}|p_\tau(h)|\end{aligned}$$

for every neighborhood U of the identity. Since the map $h\to\varphi_h$ is a continuous map of G into $L^1(G, M_*)$, the same reason as in the proof of Lemma 4.3 says that φ_τ converges to φ in norm. Therefore, φ belongs to $H^1(G)\otimes M_*$. This completes the proof.

It may be worth noticing that as the flow α extends to $L^2(G)$ as a flow of unitary operators we can define a flow $\hat{\alpha}$ in $L^2(G)\underset{\sigma}{\otimes}H$ and that if we consider the noncommutative Hardy class $H^2(\hat{\alpha})$ then $H^2(\hat{\alpha})$ coincides with the tensor product $H^2(G)\underset{\sigma}{\otimes}H$.

Let N be a von Neumann subalgebra of M and let $\gamma\to u_\gamma\in M$ be a unitary representation of Γ into M such that the mapping $x\to\sigma_\gamma(x)=u_\gamma x u_\gamma^*$ defines a group of $*$-automorphisms of N. For the moment we need not specify the action of σ_γ of finite combinations $\{\sum_{\gamma\in\Gamma}\langle\cdot,\gamma\rangle a_\gamma u_\gamma\,|\,a_\gamma\in N\}$ forms a $*$-algebra in $\mathcal{M}$. Let B_1 be its uniform closure and $\mathcal{M}_1$ be the σ-weak closure of B_1. One easily sees that B_1 is an α-invariant C^*-algebra. Put $A_1(\tilde{\alpha})=A(\hat{\varphi})\cap B_1$, the algebra of analytic elements of B_1 for the restricted flow $\tilde{\alpha}$. We denote by $H_1^\infty(\tilde{\alpha})$ the algebra of analytic elements in $\mathcal{M}_1$.

THEOREM 4.7. *$A_1(\tilde{\alpha})$ is the norm closure of the set* $\{\sum_{\gamma\geqq 0}\langle\cdot,\gamma\rangle a_\gamma u_\gamma \mid a_\gamma\in N\}$ *and it is the maximal C^*-subdiagonal algebra of B_1 with respect to the projection $\varepsilon_\mu | B_1$ with the diagonal N. Furthermore, $H_1^\infty(\tilde{\alpha})$ is the σ-weak closure $A_1(\tilde{\alpha})$, which is the maximal subdiagonal of $\mathcal{M}_1$ for the projection $\varepsilon_\mu | \mathcal{M}_1$ with the diagonal N.*

PROOF. Note first that ε_μ maps $\mathcal{M}_1$ to N. In fact, we have, for an element $x=\sum_{\gamma\in\Gamma}\langle\cdot,\gamma\rangle a_\gamma u_\gamma$,

$$\varepsilon_\mu(x)=\sum_{\gamma\in\Gamma}\varepsilon_\mu(\langle\cdot,\gamma\rangle a_\gamma u_\gamma)=\sum_{\gamma\in\Gamma}a_\gamma u_\gamma\int_G\langle g,\gamma\rangle d\mu=a_e\,,$$

and hence $\varepsilon_\mu(\mathcal{M}_1)\subset N$. It follows that $A_1(\tilde{\alpha})$ is a C^*-subdiagonal algebra with the diagonal N, and so because it is defined by a flow it is the maximal C^*-subdiagonal algebra by Theorem 3.1. Take an element x of $A_1(\tilde{\alpha})$. By Theorem 4.5, x is approximated in norm by $\{x_\tau\}$ where $x_\tau=\sum_{\gamma\geqq 0}\lambda_\tau(\gamma)\langle\cdot,\gamma\rangle x(\gamma)$. For $\gamma\in\Gamma$, we have

$$\begin{aligned}\langle\cdot,\gamma\rangle\hat{x}(\gamma)&=\langle\cdot,\gamma\rangle\int_G\overline{\langle g,\gamma\rangle}x(g)d\mu\\&=\langle\cdot,\gamma\rangle\Big(\int_G\overline{\langle g,\gamma\rangle}x(g)u_\gamma^*d\mu\Big)u_\gamma\\&=\langle\cdot,\gamma\rangle\varepsilon_\mu(\langle\cdot,-\gamma\rangle xu_{-\gamma})u_\gamma\,.\end{aligned}$$

Here one may easily see that $\varepsilon_\mu(\langle\cdot,-\gamma\rangle xu_{-\gamma})$ belongs to N and we get the first half of the theorem. The rest of the theorem is an easy consequence of the fact that the diagonal of $H_1^\infty(\tilde{\alpha})$ is also N.

We notice that, as in the case of $A(\tilde{\alpha})$, the kernel of ε in $A_1(\tilde{\alpha})$ coincides with the norm closure of those elements of B_1 with positive spectrums, i. e. the norm closure of $\{\sum_{\gamma>0}\langle\cdot,\gamma\rangle a_\gamma u_\gamma \mid a_\gamma\in N\}$.

Now let us assume that Γ is countable such as the additive group of rational numbers. As it was described in Arveson [3] the above von Neumann algebra $\mathcal{M}_1$ is shown to be spatially isomorphic to the discrete crossed product $N\times\Gamma$ of N by Γ with the automorphism group $\{\sigma_\gamma\}$. We sketch these arguments in the following in terms of tensor products with some additional informations for C^*-subdiagonal algebras.

Let $\mathcal{F}$ be the usual Fourier transform of $L^2(G)$ onto $l_2(\Gamma)$ and put $U=\mathcal{F}\otimes 1$. Then, U is an isometry between the Hilbert space $L^2(G)\underset{\sigma}{\otimes}H$ and $l_2(\Gamma)\underset{\sigma}{\otimes}H$ such that $U^*(l_\gamma\otimes x)U=\langle\cdot,\gamma\rangle x$ for $x\in B(H)$, where l_γ means the left regular representation of γ. Therefore, the isometry U induces an isomorphism between the C^*-algebra B_1 and the C^*-algebra $C^*(N,\Gamma)$ on $l_2(\Gamma)\underset{\sigma}{\otimes}H$ generated by those elements $\{l_\gamma\otimes au_\gamma \mid a\in N,\ \gamma\in\Gamma\}$. The latter is nothing but the reduced C^*-crossed product of N by Γ, and as Γ is amenable it coincides with the C^*-

crossed product $C^*(N, \Gamma)$ (cf. [**30**]). Thus, the von Neumann algebra $\mathcal{M}_1$ is also isomorphic to the discrete crossed product $N \otimes \Gamma$, in which the subdiagonal algebra of the σ-weak closure of the set $\{\sum_{\gamma \geqq 0} l_\gamma \otimes a u_\gamma | a \in N\}$ is the image of subdiagonal algebra $H_1^\infty(\tilde{\alpha})$. Furthermore, in some case one sees that B_1 is also $*$-isomorphic to the covariant algebra $C^*(N, u_\gamma | \gamma \in \Gamma)$ by [**30**; Theorem 5.2]. Thus, in this sense the C^*-subdiagonal algebra $A(\beta)$ in Arveson [**3**] is seen to be the transform of the algebra $A_1(\tilde{\alpha})$ of B_1 with $B(\beta)$ as the image of B_1 where $N = L^\infty[0,1]$ on the space $H = L^2[0,1]$, Γ=the group of integers and β is an ergodic automorphism of N preserving the Lebesgue measure, so that some of the results in [**3**] are understood as natural consequences of our arguments here for C^*-subdiagonal algebras. It is to be noticed that in this case the algebra should be a C^*-subdiagonal and not a σ-weakly closed subdiagonal algebra because the σ-weak closure of the algebra $A(\beta)$ is generally too big to settle down the problem.

Quite recently, Loebl and Muhly [**16**] has pointed out that in some cases the algebra $H^\infty(\alpha)$ happens to be a reductive algebra. The authors believe that an investigation for $H^\infty(\alpha)$ from the point of view of invariant subspaces (both simply and doubly) will be fruitful.

Additions. After this paper was written the authors have found the notice; R. I. Loebl and P. S. Muhly, Flows on von Neumann algebras, Noticies of American Mathematical Society, 21 (1974), 74T-B176, in which our Theorem 2.4 has been announced. The authors suspect that some of other results here may overlap with theirs in their forthcoming paper, though all works in the present paper have been investigated independently with theirs.

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