J. Math. Soc. Japan
Vol. 29, No. 1, 1977

Freeness of the group $\langle a^n, b^n \rangle$ for some integer n, $a, b \in SL(2, C)$

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(Received May 28, 1975)
(Revised June 30, 1976)

Suppose that $\langle a, b\rangle$ is an irreducible subgroup of the real special linear group $SL(2, R)$ with $\operatorname{tr} a=\alpha$, $\operatorname{tr} b=\beta$, $\operatorname{tr} ab=\gamma$. Let $\alpha\geqq 2$, $\beta\geqq 2$. Purzitsky [7] and Rosenberger [9] proved that $\gamma\geqq\alpha\beta+2$ or $\gamma\leqq -2$ are the necessary and sufficient conditions for $\langle a, b\rangle$ to be the discrete free product of cyclic group $\langle a\rangle$ and $\langle b\rangle$. For $\alpha, \beta\in C$, suppose that $|\alpha|\geqq 2|\beta|\geqq 2$ and $\langle a, b\rangle$ is not a free product of $\langle a\rangle$ and $\langle b\rangle$. What can we say about the freeness of the groups $\langle a^n, b^n\rangle$ for some integer n? In the present paper we shall discuss this question.

It was shown (cf. [5] Theorem 3.5) that if $\operatorname{tr} a=2=\operatorname{tr} b$ then a, b can be reduced simultaneously into the form:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \gamma-2 \\ 0 & 1 \end{pmatrix} \qquad \gamma\neq 2$$

respectively. In this case, a positive integer n can be chosen such that

$$|n(\gamma-2)|\geqq 4$$

so that $\langle a^n, b\rangle$ is free by a result of Chang, Jennings and Ree [1], even though $\langle a, b\rangle$ need not be free. Consequently $\langle a^n, b^n\rangle$ is free for some integer sufficiently large. However if the traces of both a and b are not equal to 2, then it is not so obvious that we conclude about the freeness of $\langle a^n, b^n\rangle$. We shall show that if $|\alpha|>2$, $|\beta|>2$, and $\langle a, b\rangle$ is irreducible then there always exists an integer n such that $\langle a^n, b^n\rangle$ becomes a free group. We shall prove that if the trace of one of the a and b is 2 while that of the other is >2, and a, b are non-trivial elements in $SL(2, R)$, then $\langle a^n, b^n\rangle$ is free for sufficiently large n. Throughout this paper, R and C stand for the sets of real and complex numbers respectively. I denotes the 2×2 identity matrix. Explanation for other concepts can be found in Dixon [2] or Wehrfritz [10].

Before we prove our main theorems, we mention some of the results used to prove them.

1. PING PONG LEMMA OF MACBEATH [4]. *Let A and B be groups of permutations of a set Ω and let G be the group generated by A and B together.*

Suppose that Ω contains two disjoint non-empty sets Γ and Δ such that each non-trivial element of A maps Γ into Δ and each non-trivial element of B maps Δ into Γ. Then either G is the free product of its subgroups A and B or else both A and B have order 2 and G is a dihedral group.

For another proof of Macbeath's lemma see Lyndon and Ullman [3].

2. LEMMA. *Let $\langle a, b\rangle$ be an irreducible subgroup of $SL(2, C)$ with* $\mathrm{tr}\, a=2$, $\mathrm{tr}\, b=\beta$, $\mathrm{tr}\, ab=\gamma$. *Then a and b can be brought, by conjugation, simultaneously into the form*

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \gamma-\beta \\ -1(\gamma-\beta) & \beta \end{pmatrix}.$$

PROOF. Follows from the Proof of Theorem 3.1 [5].

3. LEMMA (Theorem 3.6 [5]). *Let $G=\langle a, b\rangle$ be an irreducible subgroup of $SL(2, C)$, λ a characteristic root of a and μ a characteristic root of b. Then a and b can be brought, by conjugation, simultaneously into the form:*

$$\begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \quad \begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}$$

where

$$|\lambda| \geqq |\lambda^{-1}|, \qquad |\mu| \geqq |\mu^{-1}|.$$

Now we prove the main result of this paper.

4. THEOREM. *For an irreducible subgroup $\langle a, b\rangle$ of $SL(2, C)$ with* $\mathrm{tr}\, a=\alpha$, $\mathrm{tr}\, b=\beta$, $\mathrm{tr}\, ab=\gamma$, *let $|\alpha|>2$, $|\beta|>2$. Then $\langle a^n, b^n\rangle$ is free for some sufficiently large n.*

PROOF. Step I. By Lemma 3, we can take a and b as

$$a=\begin{pmatrix} \lambda & 0 \\ \xi & \lambda^{-1} \end{pmatrix}, \qquad b=\begin{pmatrix} \mu & \eta \\ 0 & \mu^{-1} \end{pmatrix}.$$

Let $\boldsymbol{a}, \boldsymbol{b}$ denote the induced projective transformations of the projective line $C\cup\{\infty\}$. Then

$$\boldsymbol{a}(z)=\frac{\lambda z}{\xi Z+\lambda^{-1}}, \qquad \boldsymbol{b}(z)=\frac{\mu z+\eta}{\mu^{-1}}, \qquad z\in C\cup\{\infty\},$$

$$\boldsymbol{a}^n(z)=\frac{\lambda^n z}{\xi'(\lambda^n-\lambda^{-n})z+\lambda^{-n}}, \qquad \boldsymbol{b}^n(z)=\frac{\mu^n z+\eta'(\mu^n-\mu^{-n})}{\mu^{-n}}$$

where

$$\xi'=\xi/(\lambda-\lambda^{-1}), \qquad \eta'=\eta/(\mu-\mu^{-1}).$$

Step 2. If $z\neq 0$, then

$$\boldsymbol{a}^n(z)=\frac{\lambda}{\xi'(\lambda^n-\lambda^{-n})z+\lambda^{-n}} \longrightarrow 1/\xi' \quad \text{as} \quad n\to\infty$$

and if $z\neq 1/\xi'$ and $z\neq\infty$

$$\boldsymbol{a}^{-n}(z)=\frac{\lambda^{-n}z}{\xi'(\lambda^{-n}-\lambda^{n})z+\lambda^{n}}\longrightarrow 0 \qquad \text{as} \quad n\rightarrow\infty .$$

Similarly, if $z\neq 0$, then

$$\boldsymbol{b}^{n}(z)=\frac{\mu^{n}z+\eta'(\mu^{n}-\mu^{-n})}{\mu^{-n}}\longrightarrow\infty \qquad \text{as} \quad n\rightarrow\infty$$

and if $z\neq-\eta'$ and $z\neq\infty$,

$$\boldsymbol{b}^{-n}(z)=\frac{\mu^{-n}z+\eta'(\mu^{-n}-\mu^{n})}{\mu^{n}}\longrightarrow\eta' \qquad \text{as} \quad n\rightarrow\infty .$$

Step 3. Choose disjoint open sets A_1, A_2, B_1, B_2 of $C^*=C\cup\{\infty\}$ such that $0\in A_1$, $1/\xi'\in A_2$, $\infty\in B_1$, $-\eta'\in B_2$. Such sets always exist, for example, for suitably small $\varepsilon>0$, one can take:

$$A_1=\{z\in C^*:\ |z|<\varepsilon\}\ , \qquad A_2=\{z\in C^*:\ |z-1/\xi'|<\varepsilon\}\ ,$$
$$B_1=\{z\in C^*:\ |1/z|<\varepsilon\}\ , \qquad B_2=\{z\in C^*:\ |z+\eta'|<\varepsilon\}$$

where $1/\xi'\neq-\eta'$ for otherwise $\xi\eta=-(\lambda-\lambda^{-1})(\mu-\mu^{-1})$ which is a condition of reducibility of $\langle a, b\rangle$ by 3.8, [5]. Put

$$A=A_1\cup A_2\,, \qquad B=B_1\cup B_2\,.$$

Step 4. For each $z\in B$, $z\neq\infty$

$$\boldsymbol{a}^{n}(z)\longrightarrow 1/\xi'\in A \quad \text{and} \quad \boldsymbol{a}^{-n}(z)\longrightarrow 0\in A \qquad \text{as} \quad n\rightarrow\infty$$

and $\boldsymbol{a}^{n}(\infty)=\boldsymbol{a}^{-n}(\infty)=1/\xi'$. Hence, for sufficiently large n,

$$\boldsymbol{a}^{\pm n}(B)\subseteq A\,.$$

Likewise, for all $z\in A$, $z\neq 0$

$$\boldsymbol{b}^{n}(z)\longrightarrow\infty\in B \quad \text{and} \quad \boldsymbol{B}^{-n}(z)\longrightarrow-\eta'\in B \qquad \text{as} \quad n\rightarrow\infty$$

and $\boldsymbol{b}^{n}(0)=\infty$, $\boldsymbol{b}^{-n}(0)=-\eta'$. Hence, for sufficiently large n,

$$\boldsymbol{b}^{\pm n}(A)\subseteq B\,.$$

By the Ping Pong Lemma 1 of Macbeath $\langle a^n, b^n\rangle$ is a free group on two generators a^n and b^n for sufficiently large n, as required.

As has already been mentioned if $\operatorname{tr} a=\alpha=2$, $\operatorname{tr} b=\beta=2$ and $\langle a, b\rangle$ is an irreducible subgroup then $\langle a^n, b^n\rangle$ is free for some positive integer n. In the next theorem, we show that a similar statement is valid if the trace of at least one of the matrices a or b is 2 while that of the other >2 and $a, b\in SL(2, R)$.

5. THEOREM. *Let* $\langle a, b\rangle$ *be an irreducible subgroup of* $SL(2, R)$ *such that the trace of at least one of the matrices* a, b, *say of* a, *is* 2 *and* $\operatorname{tr} b>2$. *Then* $\langle a^n, b^n\rangle$ *is free for some positive integer* n.

PROOF. If the trace of at least one of the matrices a or b, say of a, is 2 then by Lemma 2 we can conjugate a and b and transform them simultaneously into the form

$$a_1=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \qquad b_1=\begin{pmatrix}0 & \gamma-\beta\\ -1/(\gamma-\beta) & \beta\end{pmatrix}$$

where $\beta=\operatorname{tr} b$, $\gamma=\operatorname{tr} ab$ and $\gamma\neq\beta$ in the case when $\langle a, b\rangle$ is irreducible. Clearly, $\langle a^n, b^n\rangle$ is free if and only if $\langle a_1^n, b_1^n\rangle$ is free. We have the following two cases to discuss

(i) $\gamma-\beta=x>0$. In this case

$$a_1^n b_1=\begin{pmatrix}0 & x\\ -1/x & nx+\beta\end{pmatrix}.$$

Hence, $\langle a_1^n, b_1\rangle$ is, by the theorem of Purzitsky and Rosenberger [7, 9] free provided that

$$nx+\beta\geqq\beta\times 2+2$$

that is, if

$$nx\geqq\beta+2. \tag{1}$$

Since $x=\gamma-\beta>0$, and β is a fixed real number >2, a positive integer n can be chosen such that the inequality (1) is satisfied.

Hence, $\langle a_1^n, b_1\rangle$ and consequently, $\langle a^n, b^n\rangle$ is free for some positive integer n.

(ii) $\gamma-\beta=-x<0$ so that $x>0$. In this case, we consider the group $\langle a_1^n, b_1^{-1}\rangle$. Since

$$b_1^{-1}=\begin{pmatrix}\beta & x\\ -1/x & 0\end{pmatrix}=b'$$

we have

$$a_1^n\cdot b'=\begin{pmatrix}\beta & x\\ n\beta-1/x & nx\end{pmatrix}$$

so $\langle a_1^n, b'\rangle$ is, by the theorem of Purzitsky and Rosenberger [7, 9], free if

$$nx+\beta\geqq 2\beta+2$$

that is if,

$$nx\geqq\beta+2. \tag{2}$$

As before, since $x>0$ and β are fixed real numbers >2, an n can be chosen such that the inequality (2) is satisfied. Hence $\langle a_1^n, b'\rangle$ and so also $\langle a^n, b^n\rangle$ is free for some positive integer n. This completes the proof of the theorem.

ACKNOWLEDGEMENT. The author is thankful to Professor J. D. Dixon for his help in the preparation of this paper which forms a part of author's Ph. D. Thesis [6]. Financial help from Carleton University, Ottawa, is gratefully acknowledged.

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