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On a parametrix in some weak sense of a first order linear partial differential operator with two independent variables

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Introduction.

Let $P(x, D)$ be a partial differential operator of order m with C^{∞} coefficients in an open subset Ω of R^{n}. According to Trèves [6], a local subelliptic estimate for P near a point x_{0} of Ω is an estimate of the form

$$\|u\|_{m-1+\delta} \leqq C\|Pu\|_{0} \quad \text{for all} \quad u \in C_{0}^{\infty}(U), \tag{0.1}$$

where δ is a number such that $0<\delta \leqq 1$, C is a positive constant, and U is an open neighborhood of x_{0} in Ω.

In [6] Trèves established (0.1) for $\delta=\frac{1}{2k+1}$ when P is an operator of principal type in Ω having the property:

(0.2) Let $P_{m}(x, D)$ be the principal part of P. For any $(x_{0}, \xi^{0}) \in \Omega \times(R^{n} \backslash\{0\})$ and any complex number z such that $P_{m}(x_{0}, \xi^{0})=0$, $d_{\xi} \operatorname{Re}(zP_{m})(x_{0}, \xi^{0}) \neq 0$, the function $\operatorname{Im}(zP_{m})(x, \xi)$, restricted to the bicharacteristic strip of $\operatorname{Re}(zP_{m})(x, \xi)$ through (x_{0}, ξ^{0}), has only zeros of even order less than or equal to $2k$.

He reduced the proof of (0.1) to the estimate

$$\|u\|_{\frac{1}{2k+1}, 0}+\|D_{t} u\|_{0} \leqq C\|D_{t} u-i \beta(x, t, D_{x}) u\|_{0}, \qquad u \in C_{0}^{\infty}(U), \tag{0.3}$$

where $\|u\|_{r, s}^{2}=\iint(1+|\xi|^{2})^{r}(1+\tau^{2})^{s}|\hat{u}(\xi, \tau)|^{2} d \xi d \tau$ for every real numbers r, s, C is a positive constant, $\beta(x, t, D_{x})$ is a first order pseudo-differential operator defined in an open neighborhood of 0 in R^{n+1} having the following property (0.4), and U is an appropriate open neighborhood of 0.

(0.4) There is an open neighborhood W of 0 in R^{n} and a number $t_{0}>0$ such that, for every $(x, \xi) \in W \times(R^{n} \backslash\{0\})$ $\beta(x, t, \xi)$ as a function of t, $|t|<t_{0}$, has only zeros of even order less than or equal to $2k$.

On the other hand in [7] he proved that it is necessary for P to be hypoelliptic that it satisfies the condition:

(0.5) For every x_0, ξ^0, and z as in (0.2), the function $\mathrm{Im}\,(zP_m)$ does not vanish identically in any neighborhood of (x_0, ξ^0) on the bicharacteristic strip of $\mathrm{Re}\,(zP_m)$ through that point.

In this paper we attempt to investigate an operator of a first order with two independent variables which satisfies neither (0.2) nor (0.5), and deduce results which are analogous to but necessarily weaker than those of Trèves [6]. The operator we investigate is

$$L=\frac{\partial}{\partial t}+i\phi(x)\sigma(t)\frac{\partial}{\partial x}, \tag{0.6}$$

where $\phi(x)$ and $\sigma(t)$ are real valued functions defined in the intervals (a, b), $-\infty\leqq a<b\leqq+\infty$, and (α, β), $-\infty\leqq\alpha<0<\beta\leqq+\infty$, respectively satisfying

(0.7) $\phi\in C^\infty((a, b))$ and all derivatives of ϕ are bounded,

(0.8) $\sigma\in C^\infty((\alpha, \beta))$, $\sigma(t)\geqq 0$ in (α, β), and zeros of σ are all of finite order.

This operator has a simple form; however, near a point where ϕ vanishes the situation becomes complicated. In fact denoting by $P=\tau+i\phi(x)\sigma(t)\xi$ the symbol of $\frac{1}{i}L=\frac{1}{i}\frac{\partial}{\partial t}+\phi(x)\sigma(t)\frac{\partial}{\partial x}$, we see that $\mathrm{Im}\,P$ vanishes identically along the bicharacteristic strip of τ through $(x_0, t_0, \xi^0, 0)$ if $\phi(x_0)=0$, and so the conditions (0.2) and (0.5) with $z=\frac{1}{i}$ are violated for L. Hence the operator L is not hypoelliptic near such a point and (0.1) does not hold for L. Nevertheless, we can construct a parametrix in some weak sense for the operator L to establish some smoothness result with respect to the variable t for the solution of the equation:

$$Lu=f. \tag{0.9}$$

Also with the help of this parametrix we can show the local solvability of (0.9) which is already proved in [1] or [5] by the energy integral method.

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§ 1. Outline of the construction of a parametrix.

We consider the solution of $Lu=f$ of the form

$$u(x, t)=\frac{1}{2\pi i}\int \exp\left(i\xi\int_0^t\sigma(s)ds\right)v(x, \xi)d\xi\,, \tag{1.1}$$

where v is a function to be found.

Calculating formally, we have

$$(1.2)\qquad Lu=\frac{\sigma(t)}{2\pi}\int \exp\left(i\xi\int_0^t\sigma(s)\,ds\right)\left(\xi v(x,\,\xi)+\phi(x)\frac{\partial}{\partial x}v(x,\,\xi)\right)d\xi\,.$$

Remark that if $\sigma(t)>0$ in (α,β)

$$(1.3)\qquad g(t)=\frac{\sigma(t)}{2\pi}\int \exp\left(i\xi\int_0^t\sigma(s)\,ds\right)\left(\int \exp\left(-i\xi\int_0^{t'}\sigma(s)\,ds\right)g(t')dt'\right)d\xi\,,$$

$$\text{for every}\quad g\in C_0^\infty((\alpha,\,\beta))\,.$$

So, we can expect that when the solution of the equation

$$(1.4)\qquad \xi v(x,\,\xi)+\phi(x)\frac{\partial}{\partial x}v(x,\,\xi)=\int \exp\left(-i\xi\int_0^{t'}\sigma(s)\,ds\right)f(x,\,t')dt'$$

is substituted into the right-hand side of (1.1) $u(x,t)$ will give an approximate solution of $Lu=f$.

Thus we introduce the operator S_ξ which gives a solution of the ordinary differential equation with a real parameter ξ:

$$(1.4)'\qquad \xi v(x)+\phi(x)\frac{d}{dx}v(x)=f(x)\,,\qquad a<x<b\,.$$

Then (1.1) may be expressed as

$$(1.5)\qquad u(x,\,t)=\frac{1}{2\pi i}\int \exp\left(i\xi\int_0^t\sigma(s)\,ds\right)S_\xi\left(\int \exp\left(-i\xi\int_0^{t'}\sigma(s)\,ds\right)f(\cdot,\,t')\,dt'\right)d\xi\,.$$

If $x\in M=\{x\in(a,b)\,|\,\phi(x)=0\}$ and $\xi\neq 0$, $S_\xi f(x)$ is necessarily equal to $\frac{1}{\xi}f(x)$. In each component $I_\mu=(a_\mu,\,b_\mu)$ of $(a,\,b)\backslash M$ we define $S_\xi f(x)$ separately as follows:

$$(1.6)\qquad S_\xi f(x)=\begin{cases}\displaystyle\int_{a_\mu}^x \exp\left(\xi\int_x^y\frac{1}{\phi(s)}\,ds\right)\frac{1}{\phi(y)}f(y)dy, & x\in I_\mu \text{ and } \xi\phi(x)>0 \text{ in } I_\mu\,,\\ \displaystyle -\int_x^{b_\mu} \exp\left(\xi\int_x^y\frac{1}{\phi(s)}\,ds\right)\frac{1}{\phi(y)}f(y)dy, & x\in I_\mu \text{ and } \xi\phi(x)<0 \text{ in } I_\mu\,.\end{cases}$$

It is intended that $S_\xi f(x)$ has nice properties in the whole interval (a,b). To be correct, S_ξ is defined only for large $|\xi|$, and in (1.5) S_ξ should be replaced by $\chi(\xi)S_\xi$, where χ is a function belonging to $C^\infty(R^1)$ which is equal to 0 for small $|\xi|$, and equal to 1 for large $|\xi|$ (see (3.1.8) in § 3). Taking account of the right-hand side of (1.5), we introduce the following two operators:

$$(1.7)\qquad Tg(\xi)=\int_\alpha^\beta \exp\left(-i\xi\int_0^t\sigma(s)ds\right)g(t)dt\,,\qquad \xi\in R^1\,,\quad g\in L^1((\alpha,\,\beta))\,,$$

$$(1.8)\qquad \tilde{T}\tilde{g}(t)=\int \exp\left(i\xi\int_0^t\sigma(s)ds\right)\tilde{g}(\xi)d\xi\,,\qquad \alpha<t<\beta\,,\quad \tilde{g}\in L^1(R^1)\,.$$

To show that u defined by (1.5) is an approximate solution of $Lu=f$ with nice properties we must prove the differentiability of $S_\xi f(x)$ with respect to x and estimate the sup-norms of $S_\xi f$ and Tg. In order to obtain the L^2-estimate of u, those of $S_\xi f$, Tg, and $\tilde{T}\tilde{g}$ must be established. These properties about $S_\xi f$, Tg, and $\tilde{T}\tilde{g}$ are stated and proved in the two propositions in § 2. The technical difficulties in this paper lie mainly in the proofs of these propositions. In the proof of Proposition 2.1, by an appropriate partial integration we rewrite the right members of (1.6) in the form which is convenient to establish the desired estimates (see Lemma 2.3 and Lemma 2.4). In the proof of the L^2-estimate of $\tilde{T}\tilde{g}$ we must be careful near a point $t_0\in(\alpha,\beta)$ where $\sigma(t)$ vanishes. Noting that σ has only isolated zeros of even order, we use the change of variable $\tau=\int_{t_0}^t\sigma(s)ds$ and reduce the proof to the inequality

(1.9) $$\int|f(t)|^2\frac{1}{|t|^\alpha}dt\leqq C_\alpha\int|\hat{f}(\xi)|^2|\xi|^\alpha d\xi\,,\qquad 0<\alpha<1\,,\quad f\in C_0^\infty(R^1)\,,$$

where C_α is a positive constant depending only on α. The L^2-estimate of Tg follows from that of $\tilde{T}\tilde{g}$ since T is the dual of $\tilde{T}$. The evaluation of $|Tg(\xi)|$ is derived easily by using the change of the variable stated above and some partial integration.

§ 2. Preliminary propositions.

PROPOSITION 2.1. *Let ϕ satisfy* (0.7). *We consider the equation*

(2.1.1) $$\xi v(x)+\phi(x)\frac{d}{dx}v(x)=f(x)\,,\qquad a<x<b\,,$$

with a real parameter ξ. For every positive integer j, there exists a constant $C_j>0$, such that for $|\xi|>C_j$ we can find a linear mapping $S_\xi: C_0^{j+1}((a,b))\to C^j((a,b))$ having the following properties.

(2.1.2) $$\xi S_\xi f(x)+\phi(x)\frac{d}{dx}S_\xi f(x)=f(x)\,,\qquad a<x<b\,,$$

(2.1.3) $$\phi(x)\frac{d}{dx}S_\xi f(x)=S_\xi\Big(\phi\frac{df}{dx}\Big)(x)\,,$$

(2.1.4) *When $S_\xi f$ is considered as a function of (x,ξ), $\frac{\partial^p}{\partial x^p}S_\xi f$ is infinitely differentiable with respect to ξ in $|\xi|>C_j$ for $0\leqq p\leqq j$, and continuous in $(a,b)\times\{\xi\mid|\xi|>C_j\}$. Furthermore, for every non negative integer N the following two inequalities hold with a constant $C>0$ independent of f.*

(2.1.4.1) $$\left|\frac{\partial^N}{\partial\xi^N}\frac{\partial^p}{\partial x^p}S_\xi f(x)\right|\leqq C(1+|\xi|)^{-N-1}\sup_{a<x<b}\sum_{0\leqq l\leqq p}\left|\frac{d^l}{dx^l}f(x)\right|,$$

$$(2.1.4.2)\qquad \int_a^b \left|\frac{\partial^N}{\partial\xi^N}\frac{\partial^p}{\partial x^p}S_\xi f(x)\right|^2 dx \leqq C(1+|\xi|)^{-2N-2}\int_a^b \sum_{0\leqq l\leqq p}\left|\frac{d^l}{dx^l}f(x)\right|^2 dx$$

$$\text{for}\quad f\in C_0^{j+1}((a,b)),\qquad |\xi|>C_j,\quad \text{and}\quad 0\leqq p\leqq j.$$

PROOF. Set $M=\{x\in(a,b)\,|\,\phi(x)=0\}$ and decompose $(a,b)\backslash M$ into a disjoint union of open intervals $(a_\mu, b_\mu)_{\mu\in\Lambda}$. For every $f\in C_0^{j+1}((a,b))$ we define $S_\xi f$ by the following formula.

$$(2.1.5)\qquad S_\xi f(x)=\begin{cases} \dfrac{1}{\xi}f(x), & x\in M,\quad \xi\neq 0 \\[2ex] \displaystyle\int_{a_\mu}^x k_0(x,y,\xi)\frac{1}{\phi(y)}f(y)dy, & x\in I_\mu \text{ and } \xi\phi(x)>0 \text{ in } I_\mu, \\[2ex] \displaystyle -\int_x^{b_\mu} k_0(x,y,\xi)\frac{1}{\phi(y)}f(y)dy, & \text{otherwise}, \end{cases}$$

where $k_0(x,y,\xi)=\exp\left(\xi\int_x^y \frac{1}{\phi(s)}ds\right)$ and $I_\mu=(a_\mu, b_\mu)$. The integrals of the right-hand side of (2.1.5) are well defined for large $|\xi|$, since in the last both cases we have

$$(2.1.6)\qquad k_0(x,y,\xi)\leqq \exp\left(\int_x^y \frac{\phi'(s)}{\phi(s)}ds\right)=\left|\frac{\phi(y)}{\phi(x)}\right| \qquad \text{if}\quad |\xi|\geqq \sup_{a<x<b}|\phi'(x)|\,.$$

During the course of the proof of this proposition we shall take ξ with $|\xi|$ sufficiently large.

For every integer $m\geqq 0$, we denote by P_m the differential operator $\dfrac{d}{dx}\dfrac{1}{\xi+m\phi'}$ and we define the operators K_l, $0\leqq l\leqq j$, in the following manner:

$$(2.1.7)\qquad K_0 f(x,\xi)=\begin{cases} \displaystyle\int_{a_\mu}^x k_0(x,y,\xi)P_0 f(y,\xi)dy, & x\in I_\mu \text{ and } \xi\phi(x)>0 \text{ in } I_\mu, \\[2ex] \displaystyle -\int_x^{b_\mu} k_0(x,y,\xi)P_0 f(y,\xi)dy, & x\in I_\mu \text{ and } \xi\phi(x)<0 \text{ in } I_\mu, \end{cases}$$

for every $f\in C_0^{j+1}((a,b))$,

$$(2.1.8)\qquad K_l f(x,\xi)=\begin{cases} \displaystyle\int_{a_\mu}^x k_l(x,y,\xi)P_l\cdots P_1 f'(y,\xi)dy, & x\in I_\mu \text{ and } \xi\phi(x)>0 \text{ in } I_\mu, \\[2ex] \displaystyle -\int_x^{b_\mu} k_l(x,y,\xi)P_l\cdots P_1 f'(y,\xi)dy, & x\in I_\mu \text{ and } \xi\phi(x)<0 \text{ in } I_\mu, \end{cases}$$

for every $f\in C_0^{j+1}((a,b))$ and $1\leqq l\leqq j$, where $k_l(x,y,\xi)=\exp\left(\int_x^y \frac{\xi+l\phi'(s)}{\phi(s)}ds\right)$.

Then the next three lemmas hold.

LEMMA 2.2.

(a) *$S_\xi f\in C^{j+1}(I_\mu)$ and*

$$(2.2.1)\qquad \xi S_\xi f(x)+\phi(x)\frac{d}{dx}S_\xi f(x)=f(x) \qquad \text{in}\quad I_\mu\,.$$

(b) *For every integer l, $0 \leqq l \leqq j$, and $x \in I_\mu$*

$$\frac{1}{\phi(x)}\int_{a_\mu}^{x} k_l(x, y, \xi)f(y)dy \tag{2.2.2}$$

$$= \frac{f(x)}{\xi+(l+1)\phi'(x)} - \int_{a_\mu}^{x} k_{l+1}(x, y, \xi)P_{l+1}f(y, \xi)dy$$

if $\xi\phi(x)>0$ in I_μ,

$$\frac{-1}{\phi(x)}\int_{x}^{b_\mu} k_l(x, y, \xi)f(y)dy \tag{2.2.3}$$

$$= \frac{f(x)}{\xi+(l+1)\phi'(x)} + \int_{x}^{b_\mu} k_{l+1}(x, y, \xi)P_{l+1}f(y, \xi)dy$$

if $\xi\phi(x)<0$ in I_μ.

PROOF. (a) It is easily verified that $S_\xi f$ belongs to $C^1(I_\mu)$ and that we have

$$\frac{d}{dx}S_\xi f(x) = \frac{1}{\phi(x)}(f(x)-\xi S_\xi f(x)) \quad \text{in } I_\mu. \tag{2.2.4}$$

From these we see that $S_\xi f \in C^{j+1}(I_\mu)$ and (2.2.1) holds.

(b) We prove only (2.2.2). (2.2.3) can be proved in the similar method.

(2.2.5) the left-hand side of (2.2.2)

$$= \int_{a_\mu}^{x} k_{l+1}(x, y, \xi)\frac{1}{\phi(y)}f(y)dy$$

$$= \int_{a_\mu}^{x} \frac{\partial k_{l+1}}{\partial y}(x, y, \xi)\frac{1}{\xi+(l+1)\phi'(y)}f(y)dy$$

$$= \lim_{\varepsilon\to+0}\Big\{\Big[k_{l+1}(x, y, \xi)\frac{1}{\xi+(l+1)\phi'(y)}f(y)\Big]_{y=a_\mu+\varepsilon}^{y=x}$$

$$-\int_{a_\mu+\varepsilon}^{x} k_{l+1}(x, y, \xi)\frac{\partial}{\partial y}\Big(\frac{1}{\xi+(l+1)\phi'(y)}f(y)\Big)dy\Big\}.$$

Therefore we have only to prove that

$$\lim_{\varepsilon\to+0} k_{l+1}(x, a_\mu+\varepsilon, \xi)\frac{f(a_\mu+\varepsilon)}{\xi+(l+1)\phi'(a_\mu+\varepsilon)} = 0. \tag{2.2.6}$$

If $a_\mu=a$, this is trivial since $f=0$ near a from the hypothesis of Proposition 2.1.

When $a_\mu>a$, we have (2.2.6) by the following inequality.

$$k_{l+1}(x, y, \xi) = \exp\Big(\int_{x}^{y}\frac{\xi+(l+1)\phi'(s)}{\phi(s)}ds\Big) \tag{2.2.7}$$

$$\leqq \Big|\frac{\phi(y)}{\phi(x)}\Big|\Big|\frac{\phi(y)}{\phi(x)}\Big|^{l+1} = \Big|\frac{\phi(y)}{\phi(x)}\Big|^{l+2}$$

if $|\xi| \geqq \sup_{a<x<b}|\phi'(x)|$. Q. E. D.

LEMMA 2.3. *For every $x \in I_\mu$ we have the next four equalities.*

(c) $S_\xi f(x)=\frac{f(x)}{\xi}-K_0 f(x,\xi)$.

(d) $\frac{d}{dx}S_\xi f(x)=\frac{\xi}{\phi(x)}K_0 f(x,\xi)=\frac{f'(x)}{\xi+\phi'(x)}-K_1 f(x,\xi)$.

(e) $$\frac{d^p}{dx^p}S_\xi f(x)=\frac{1}{\phi(x)}\sum_{1\leqq l\leqq p-1}\psi_l^p(x,\xi)K_l f(x,\xi)$$
$$=\sum_{1\leqq l\leqq p-1}\psi_l^p(x,\xi)\frac{P_l\cdots P_1 f'(x,\xi)}{\xi+(l+1)\phi'(x)}-\sum_{1\leqq l\leqq p-1}\psi_l^p(x,\xi)K_{l+1}f(x,\xi),$$
$$2\leqq p\leqq j,$$

where we define $\psi_l^p(x,\xi)$, $1\leqq l\leqq p-1$, *inductively by the following formulas.*

$$\psi_1^2(x,\xi)=(\xi+\phi'(x))$$
$$\psi_1^{p+1}(x,\xi)=\frac{d^p}{dx^p}\phi(x)$$

(2.3.1) $$\psi_l^{p+1}(x,\xi)=\sum_{l\leqq j\leqq p-1}{}_pC_{j-1}\frac{d^{p+1-j}\phi}{dx^{p+1-j}}(x)\psi_{l-1}^j(x,\xi)+(\xi+p\phi'(x))\psi_{l-1}^p(x,\xi)\,,\quad 2\leqq l\leqq p-1\,,$$
$$\psi_p^{p+1}(x,\xi)=(\xi+p\phi'(x))\psi_{p-1}^p(x,\xi)\,.$$

(f) *Let* $x_0\in M'=\{x\in(a,b)\,|\,\phi'(x)=0\}$. *Then we have*

(2.3.2) $$\frac{d}{dx}\Big(\sum_{1\leqq l\leqq p-1}\psi_l^p(x,\xi)\frac{P_l\cdots P_1 f'(x,\xi)}{\xi+(l+1)\phi'(x)}\Big)=\sum_{1\leqq l\leqq p}\psi_l^{p+1}(x,\xi)\frac{P_l\cdots P_1 f'(x,\xi)}{\xi+(l+1)\phi'(x)}$$

at x_0 *if* $p\geqq 2$.

PROOF. (c) If $\xi\phi(x)>0$ in I_μ, we have

(2.3.3) $$S_\xi f(x)=\int_{a_\mu}^x k_0(x,y,\xi)\frac{1}{\phi(y)}f(y)dy$$
$$=\int_{a_\mu}^x\frac{1}{\xi}\frac{\partial}{\partial y}k_0(x,y,\xi)f(y)dy\,.$$

Hence one can prove (c) by using the same partial integration as in the Proof of Lemma 2.2 (b).

Another case where $\xi\phi(x)<0$ in I_μ can be treated in the same manner.

(d) The first equality follows from (c) above and Lemma 2.2 (2.2.1). The second follows from Lemma 2.2 (b).

(e) We shall prove (e) by induction on p. Here we remark that the second equality of (e) follows from Lemma 2.2 (b) if the first equality is proved. If $p=2$, differentiating both sides of (2.2.1), we have

(2.3.4) $$(\xi+\phi'(x))\frac{d}{dx}S_\xi f(x)+\phi(x)\frac{d^2}{dx^2}S_\xi f(x)=\frac{df}{dx}(x)\,.$$

Hence by (d)

$$\phi(x)\frac{d^2}{dx^2}S_\xi f(x)=\frac{df}{dx}(x)-(\xi+\phi'(x))\Big(\frac{f'(x)}{\xi+\phi'(x)}-K_1 f(x,\xi)\Big)\tag{2.3.5}$$
$$=(\xi+\phi'(x))K_1 f(x,\xi)\,.$$

Thus (e) is proved for $p=2$.

Next we shall prove (e) for $p+1$ under the assumption that (e) is valid for $p\geqq 2$.

Differentiating both sides of (2.2.1) p times with respect to x, we have

$$\frac{d^p f}{dx^p}(x)=\sum_{2\leqq j\leqq p-1}{}_pC_{j-1}\frac{d^{p+1-j}\phi}{dx^{p+1-j}}(x)\frac{d^j}{dx^j}S_\xi f(x)+\frac{d^p\phi}{dx^p}(x)\frac{d}{dx}S_\xi f(x)\tag{2.3.6}$$
$$+(\xi+p\phi'(x))\frac{d^p}{dx^p}S_\xi f(x)+\phi(x)\frac{d^{p+1}}{dx^{p+1}}S_\xi f(x)\,.$$

Hence by the hypothesis of the induction one can write

$$\phi\frac{d^{p+1}}{dx^{p+1}}S_\xi f\tag{2.3.7}$$
$$=\frac{d^p\phi}{dx^p}K_1 f+\sum_{\substack{2\leqq j\leqq p-1\\ 1\leqq l\leqq j-1}}{}_pC_{j-1}\frac{d^{p+1-j}\phi}{dx^{p+1-j}}\phi_l^j K_{l+1}f+(\xi+p\phi')\sum_{1\leqq l\leqq p-1}\phi_l^p K_{l+1}f$$
$$+\Big[\frac{d^p f}{dx^p}-\frac{d^p\phi}{dx^p}\frac{f'}{\xi+\phi'}-\sum_{\substack{2\leqq j\leqq p-1\\ 1\leqq l\leqq j-1}}{}_pC_{j-1}\frac{d^{p+1-j}\phi}{dx^{p+1-j}}\phi_l^j\frac{P_l\cdots P_1 f'}{\xi+(l+1)\phi'}$$
$$-(\xi+p\phi')\sum_{1\leqq l\leqq p-1}\phi_l^p\frac{P_l\cdots P_1 f'}{\xi+(l+1)\phi'}\Big]\,.$$

Thus to prove (e) for $p+1$, it suffices to show that the bracket term in the right-hand side of (2.3.7) vanishes identically, in other word we must show that

$$\frac{d^p f}{dx^p}=\frac{d^p\phi}{dx^p}\frac{f'}{\xi+\phi'}+\sum_{1\leqq l\leqq p-1}\phi_{l+1}^{p+1}\frac{P_l\cdots P_1 f'}{\xi+(l+1)\phi'}\,,\qquad p\geqq 2\,.\tag{2.3.8}$$

We prove (2.3.8) also by induction on p. When $p=2$, a straightforward computation shows that

$$\phi''\frac{f'}{\xi+\phi'}+\phi_2^3\frac{P_1 f'}{\xi+2\phi'}\tag{2.3.9}$$
$$=\phi''\frac{f'}{\xi+\phi'}+(\xi+2\phi')(\xi+\phi')\frac{1}{\xi+2\phi'}\Big(\frac{f'}{\xi+\phi'}\Big)'=f''\,.$$

Hence we have (2.3.8) for $p=2$. Assume that (2.3.8) holds for $p\geqq 2$. Then differentiating both sides of (2.3.8) one can write

$$\frac{d^{p+1}f}{dx^{p+1}}=\frac{d^{p+1}\phi}{dx^{p+1}}\frac{f'}{\xi+\phi'}+\frac{d^p\phi}{dx^p}P_1 f'\tag{2.3.10}$$
$$+\sum_{1\leqq l\leqq p-1}\frac{\partial}{\partial x}\phi_{l+1}^{p+1}\frac{P_l\cdots P_1 f'}{\xi+(l+1)\phi'}+\sum_{1\leqq l\leqq p-1}\phi_{l+1}^{p+1}P_{l+1}\cdots P_1 f'\,.$$

Therefore to prove (2.3.8) for $p+1$ we must show that

(2.3.11)
$$\frac{d^p\phi}{dx^p}P_1f'+\sum_{1\leqq l\leqq p-1}\frac{\partial}{\partial x}\psi_{l+1}^{p+1}\frac{P_l\cdots P_1f'}{\xi+(l+1)\phi'}+\sum_{1\leqq l\leqq p-1}\psi_{l+1}^{p+1}P_{l+1}\cdots P_1f'$$
$$=\sum_{1\leqq l\leqq p}\psi_{l+1}^{p+2}\frac{P_l\cdots P_1f'}{\xi+(l+1)\phi'}.$$

Since $\psi_1^{p+1}=\frac{d^p\phi}{dx^p}$ by definition (2.3.1) we see that to prove (2.3.11) it suffices to prove

(2.3.12)
$$\psi_l^{p+1}=\Big(\psi_{l+1}^{p+2}-\frac{\partial}{\partial x}\psi_{l+1}^{p+1}\Big)/(\xi+(l+1)\phi'),\qquad 1\leqq l\leqq p-1,\quad p\geqq 2.$$

This we prove by induction on p. First, when $p=2$, a direct computation shows that (2.3.12) is valid. We shall show (2.3.12) for $p\geqq 3$, assuming that (2.3.12) is already proved for the numbers less than or equal to $p-1$.

Case 1. $l=1$. By definition (2.3.1)

(2.3.13)
$$\psi_1^{p+1}=\frac{d^p\phi}{dx^p}.$$

(2.3.14)
$$\psi_2^{p+2}=\sum_{2\leqq j\leqq p}{}_{p+1}C_{j-1}\frac{d^{p+2-j}\phi}{dx^{p+2-j}}\psi_1^j+(\xi+(p+1)\phi')\psi_1^{p+1}.$$

(2.3.15)
$$\psi_2^{p+1}=\sum_{2\leqq j\leqq p-1}{}_pC_{j-1}\frac{d^{p+1-j}\phi}{dx^{p+1-j}}\psi_1^j+(\xi+p\phi')\psi_1^p.$$

Hence

(2.3.16)
$$\psi_2^{p+2}-\frac{\partial}{\partial x}\psi_2^{p+1}=\sum_{2\leqq j\leqq p-2}{}_pC_{j-1}\frac{d^{p+1-j}\phi}{dx^{p+1-j}}\Big(\psi_1^{j+1}-\frac{\partial}{\partial x}\psi_1^j\Big)$$
$$+{}_pC_0\frac{d^p\phi}{dx^p}\psi_1^2-{}_pC_{p-2}\phi''\frac{\partial}{\partial x}\psi_1^{p-1}$$
$$+(\xi+(p+1)\phi')\psi_1^{p+1}+{}_{p+1}C_{p-1}\phi''\psi_1^p$$
$$-p\phi''\psi_1^p-(\xi+p\phi')\frac{\partial}{\partial x}\psi_1^p.$$

Since by definition (2.3.1) $\psi_1^{j+1}=\frac{d}{dx}\psi_1^j=\frac{d^j\phi}{dx^j}$ for $j\geqq 2$, we can rewrite (2.3.16) as follows:

(2.3.16)′
$$\psi_2^{p+2}-\frac{\partial}{\partial x}\psi_2^{p+1}=\psi_1^{p+1}[(\xi+\phi')+(\xi+(p+1)\phi')-(\xi+p\phi')]$$
$$+\phi''\psi_1^p(-{}_pC_{p-2}+{}_{p+1}C_{p-1}-p)$$
$$=\psi_1^{p+1}(\xi+2\phi').$$

Thus (2.3.12) has been proved for $l=1$.

Case 2. $l=p-1$. As in the Case 1, one can write by (2.3.1)

(2.3.17)
$$\psi_p^{p+2}-\frac{\partial}{\partial x}\psi_p^{p+1}={}_{p+1}C_{p-1}\phi''\psi_{p-1}^p+(\xi+(p+1)\phi')\psi_{p-1}^{p+1}$$
$$-p\phi''\psi_{p-1}^p-(\xi+p\phi')\frac{\partial}{\partial x}\psi_{p-1}^p.$$

On the other hand, by the hypothesis of the induction we have

$$\psi_{p-1}^{p+1}-\frac{\partial}{\partial x}\psi_{p-1}^{p}=(\xi+(p-1)\phi')\psi_{p-2}^{p}. \tag{2.3.18}$$

Hence, keeping in mind that $\psi_{p-1}^{p+1}={}_pC_{p-2}\phi''\psi_{p-2}^{p-1}+(\xi+p\phi')\psi_{p-2}^{p}$ by (2.3.1), we obtain

$$\begin{aligned}\psi_{p}^{p+2}-\frac{\partial}{\partial x}\psi_{p}^{p+1} &=({}_{p+1}C_{p-1}-p)\phi''\psi_{p-1}^{p}+\phi'\psi_{p-1}^{p+1}+(\xi+p\phi')(\xi+(p-1)\phi')\psi_{p-2}^{p}\\ &=({}_{p+1}C_{p-1}-p)\phi''\psi_{p-1}^{p}+\phi'\psi_{p-1}^{p+1}+(\xi+(p-1)\phi')(\psi_{p-1}^{p+1}-{}_pC_{p-2}\phi''\psi_{p-2}^{p-1})\\ &=(\xi+p\phi')\psi_{p-1}^{p+1}.\end{aligned} \tag{2.3.19}$$

Thus (2.3.12) has been proved for $l=p-1$.

Case 3. $2\leqq l\leqq p-2$. By (2.3.1) one can write

$$\begin{aligned}\psi_{l+1}^{p+2}-\frac{\partial}{\partial x}\psi_{l+1}^{p+1} &={}_{p+1}C_{p-1}\phi''\psi_{l}^{p}+(\xi+(p+1)\phi')\psi_{l}^{p+1}\\ &\quad+\sum_{l+1\leqq j\leqq p-1}({}_{p+1}C_{j-1}-{}_pC_{j-1})\frac{d^{p+2-j}\phi}{dx^{p+2-j}}\psi_{l}^{j}-p\phi''\psi_{l}^{p}\\ &\quad-\sum_{l+1\leqq j\leqq p-1}{}_pC_{j-1}\frac{d^{p+1-j}\phi}{\partial x^{p+1-j}}\frac{\partial}{\partial x}\psi_{l}^{j}-(\xi+p\phi')\frac{\partial}{\partial x}\psi_{l}^{p}\\ &=\sum_{l+1\leqq j\leqq p-1}{}_pC_{j-1}\frac{d^{p+1-j}\phi}{dx^{p+1-j}}\Big(\psi_{l}^{j+1}-\frac{\partial}{\partial x}\psi_{l}^{j}\Big)\\ &\quad+{}_{p+1}C_{p-1}\phi''\psi_{l}^{p}+(\xi+(p+1)\phi')\psi_{l}^{p+1}+{}_pC_{l-1}\frac{d^{p+1-l}\phi}{dx^{p+1-l}}\psi_{l}^{l+1}\\ &\quad-{}_pC_{p-2}\phi''\psi_{l}^{p}-p\phi''\psi_{l}^{p}-(\xi+p\phi')\frac{\partial}{\partial x}\psi_{l}^{p}.\end{aligned} \tag{2.3.20}$$

On the other hand, by the hypothesis of the induction

$$\psi_{l}^{j+1}-\frac{\partial}{\partial x}\psi_{l}^{j}=(\xi+l\phi')\psi_{l-1}^{j}. \tag{2.3.21}$$

$$\psi_{l}^{p+1}-\frac{\partial}{\partial x}\psi_{l}^{p}=(\xi+l\phi')\psi_{l-1}^{p}. \tag{2.3.22}$$

Substituting (2.3.21) and (2.3.22) into (2.3.20), we have

$$\begin{aligned}\psi_{l+1}^{p+2}-\frac{\partial}{\partial x}\psi_{l+1}^{p+1} &=\sum_{l\leqq j\leqq p-1}{}_pC_{j-1}\frac{d^{p+1-j}\phi}{dx^{p+1-j}}(\xi+l\phi')\psi_{l-1}^{j}\\ &\quad-{}_pC_{l-1}\frac{d^{p+1-l}\phi}{dx^{p+1-l}}(\xi+l\phi')\psi_{l-1}^{l}+{}_{p+1}C_{p-1}\phi''\psi_{l}^{p}+(\xi+(p+1)\phi')\psi_{l}^{p+1}\end{aligned} \tag{2.3.23}$$

$$+{}_pC_{l-1}\frac{d^{p+1-l}\phi}{dx^{p+1-l}}\psi_l^{l+1}-{}_pC_{p-2}\phi''\psi_l^p-p\phi''\psi_l^p$$
$$-(\xi+p\phi')(\psi_l^{p+1}-(\xi+l\phi')\psi_{l-1}^p)\,.$$

Since $\sum_{l\leqq j\leqq p-1}{}_pC_{j-1}\frac{d^{p+1-j}\phi}{dx^{p+1-j}}\psi_{l-1}^j=\psi_l^{p+1}-(\xi+p\phi')\psi_{l-1}^p$ by (2.3.1), a direct computation shows that

$$\psi_{l+1}^{p+2}-\frac{\partial}{\partial x}\psi_{l+1}^{p+1}=(\xi+(l+1)\phi')\psi_l^{p+1}\,, \tag{2.3.24}$$

which completes the proof of (2.3.12).

(f) When $p=2$, (2.3.2) can be proved by a direct computation. Let us consider the case where $p\geqq 3$.

(2.3.25) the left-hand side of (2.3.2)

$$=\sum_{1\leqq l\leqq p-1}\frac{\partial\psi_l^p}{\partial x}\frac{P_l\cdots P_1f'}{\xi+(l+1)\phi'}+\sum_{1\leqq l\leqq p-1}\psi_l^pP_{l+1}\cdots P_1f'$$
$$=\sum_{2\leqq l\leqq p-1}\frac{\partial\psi_l^p}{\partial x}\frac{P_l\cdots P_1f'}{\xi+(l+1)\phi'}+\sum_{1\leqq l\leqq p-2}\psi_l^pP_{l+1}\cdots P_1f'$$
$$+\frac{\partial\psi_1^p}{\partial x}\frac{P_1f'}{\xi+2\phi'}+\psi_{p-1}^pP_p\cdots P_1f'\,.$$

Remarking that $\psi_{l-1}^p=\left(\psi_l^{p+1}-\frac{\partial}{\partial x}\psi_l^p\right)/(\xi+l\phi')$, $2\leqq l\leqq p-1$, by (2.3.12) and that $\phi'(x_0)=0$, we have when $x=x_0$

(2.3.26) the left-hand side of (2.3.2)

$$=\sum_{2\leqq l\leqq p-1}\psi_l^{p+1}\frac{P_l\cdots P_1f'}{\xi+(l+1)\phi'}+\psi_1^{p+1}\frac{P_1f'}{\xi+2\phi'}+\psi_{p-1}^pP_p\cdots P_1f'$$
$$=\sum_{1\leqq l\leqq p}\psi_l^{p+1}\frac{P_l\cdots P_1f'}{\xi+(l+1)\phi'}\,,$$

which completes the proof of (f). Q. E. D.

LEMMA 2.4. *For every non negative integers N and l, there exists a constant $C>0$ such that for every $f\in C_0^{l+1}((a,b))$, $x\in I_\mu$, and $|\xi|\geqq C$ we have*

$$\left|\frac{\partial^N}{\partial\xi^N}S_\xi f(x)\right|\leqq\frac{C}{(1+|\xi|)^{N+1}}\sup_{x\in I_\mu}|f(x)|\,. \tag{2.4.1}$$

$$\left|\frac{1}{\phi(x)}\frac{\partial^N}{\partial\xi^N}K_lf(x,\xi)\right|\leqq\frac{C}{(1+|\xi|)^{\max(l,1)+N+1}}\sup_{x\in I_\mu}\sum_{1\leqq k\leqq l+1}|f^{(k)}(x)|\,. \tag{2.4.2}$$

$$\int_{I_\mu}\left|\frac{\partial^N}{\partial\xi^N}S_\xi f(x)\right|^2dx\leqq\frac{C}{(1+|\xi|)^{2(N+1)}}\int_{I_\mu}|f(x)|^2dx\,. \tag{2.4.3}$$

$$\int_{I_\mu}\left|\frac{1}{\phi(x)}\frac{\partial^N}{\partial\xi^N}K_lf(x,\xi)\right|^2dx\leqq\frac{C}{(1+|\xi|)^{2(\max(l,1)+N+1)}}\int_{I_\mu}\sum_{1\leqq k\leqq l+1}|f^{(k)}(x)|^2dx\,. \tag{2.4.4}$$

PROOF. It is easy to see that if $\xi\phi(x)>0$ in I_μ and $x\geqq y$, or if $\xi\phi(x)<0$ in I_μ and $x\leqq y$, then the following inequality holds for sufficiently large $|\xi|$.

$$\left|\frac{\partial^N}{\partial\xi^N}\exp\left(\int_x^y\frac{\xi+l\phi'(s)}{\phi(s)}ds\right)\right|\leqq\frac{C}{(1+|\xi|)^N}\exp\left(\int_x^y\frac{\tilde{c}\xi}{\phi(s)}ds\right), \tag{2.4.5}$$

where C and $\tilde{c}$ are positive constants independent of ξ, x and y. On the other hand, remembering that $P_m=\frac{d}{dx}\frac{1}{\xi+m\phi'}$, we have with a constant $C>0$ which may depend on N or l,

$$\left|\frac{\partial^N}{\partial\xi^N}P_0f(x,\xi)\right|\leqq\frac{C}{(1+|\xi|)^{N+1}}|f'(x)|, \qquad a<x<b, \tag{2.4.6}$$

$$\left|\frac{\partial^N}{\partial\xi^N}P_l\cdots P_1f'(x,\xi)\right|\leqq\frac{C}{(1+|\xi|)^{N+l}}\sum_{1\leqq k\leqq l+1}|f^{(k)}(x)|, \qquad a<x<b, \tag{2.4.7}$$

for all $f\in C_0^{l+1}((a,b))$.

We now prove Lemma 2.4 only for the case where $\xi\phi(x)>0$ in I_μ. Another case where $\xi\phi(x)<0$ in I_μ can be treated in the same manner. We have by (2.4.5)

$$\left|\frac{\partial^N}{\partial\xi^N}S_\xi f(x)\right|\leqq\frac{C}{(1+|\xi|)^N}\int_{a_\mu}^x\exp\left(\int_x^y\frac{\tilde{c}\xi}{\phi(s)}ds\right)\frac{1}{|\phi(y)|}|f(y)|dy. \tag{2.4.8}$$

Hence the next inequality (2.4.9) yields (2.4.1).

$$\int_{a_\mu}^x\exp\left(\int_x^y\frac{\tilde{c}\xi}{\phi(s)}ds\right)\frac{1}{|\phi(y)|}dy\leqq\frac{1}{\tilde{c}|\xi|}. \tag{2.4.9}$$

On the other hand by the definitions (2.1.7) and (2.1.8) of K_l and by (2.4.6) and (2.4.7) we have for some constant $C>0$ independent of f,

$$\left|\frac{1}{\phi(x)}\frac{\partial^N}{\partial\xi^N}K_lf(x,\xi)\right| \tag{2.4.10}$$
$$\leqq\frac{C}{(1+|\xi|)^{\max(l,1)+N}}\int_{a_\mu}^x\exp\left(\int_x^y\frac{\tilde{c}\xi}{\phi(s)}ds\right)\frac{1}{|\phi(y)|}\sum_{1\leqq k\leqq l+1}|f^{(k)}y|dy.$$

Here we have used the relation $\frac{1}{\phi(x)}=\frac{1}{\phi(y)}\exp\left(\int_x^y\frac{\phi'(s)}{\phi(s)}ds\right)$, $x,y\in I_\mu$. Combining both (2.4.10) and (2.4.9), we obtain (2.4.2).

In view of (2.4.8) and (2.4.10), to prove (2.4.3) and (2.4.4) we have only to show the following inequality.

$$\int_{I_\mu}\left|\int_{a_\mu}^x\exp\left(\int_x^y\frac{\tilde{c}\xi}{\phi(s)}ds\right)\frac{1}{|\phi(y)|}|g(y)|dy\right|^2dx\leqq\frac{C}{(1+|\xi|)^2}\int_{I_\mu}|g(y)|^2dy, \tag{2.4.11}$$

where $g\in L^2(I_\mu)$ and C is a positive constant depending only on $\tilde{c}$.

(2.4.12) the left-hand side of (2.4.11)

$$\leqq\int_{I_\mu}\left(\int_{a_\mu}^x\exp\left(\int_x^y\frac{\tilde{c}\xi}{\phi(s)}ds\right)\frac{1}{|\phi(y)|}dy\right)$$

$$\times\Big(\int_{a_\mu}^{x}\exp\Big(\int_x^y \frac{\tilde{c}\xi}{\phi(s)}ds\Big)\frac{1}{|\phi(y)|}|g(y)|^2dy\Big)dx$$

$$\leqq \frac{1}{\tilde{c}|\xi|}\int_{I_\mu}\Big(\int_{a_\mu}^{x}\exp\Big(\int_x^y \frac{\tilde{c}\xi}{\phi(s)}ds\Big)\frac{1}{|\phi(y)|}|g(y)|^2dy\Big)dx$$

$$=\frac{1}{\tilde{c}|\xi|}\int_{I_\mu}\Big(\int_{y}^{b_\mu}\exp\Big(\int_x^y \frac{\tilde{c}\xi-\phi'(s)}{\phi(s)}ds\Big)\frac{1}{|\phi(x)|}dx\Big)|g(y)|^2dy.$$

Since $\int_x^y \frac{\tilde{c}\xi-\phi'(s)}{\phi(s)}ds \leqq \int_x^y \frac{\frac{1}{2}\tilde{c}\xi}{\phi(s)}ds$ for sufficiently large $|\xi|$, we have

$$\int_y^{b_\mu}\exp\Big(\int_x^y \frac{\tilde{c}\xi-\phi'(s)}{\phi(s)}ds\Big)\frac{1}{|\phi(x)|}dx \leqq \int_y^{b_\mu}\exp\Big(\int_x^y \frac{\frac{1}{2}\tilde{c}\xi}{\phi(x)}ds\Big)\frac{1}{|\phi(x)|}dx \leqq \frac{2}{\tilde{c}|\xi|}. \tag{2.4.13}$$

Therefore

$$\text{the left-hand side of (2.4.11)} \leqq \frac{2}{(\tilde{c}|\xi|)^2}\int_{I_\mu}|g(y)|^2dy. \tag{2.4.14}$$

This proves (2.4.11). Q. E. D.

We return to the Proof of Proposition 2.1.

(2.1.1). Since it is already proved in Lemma 2.2 that $S_\xi f\in C^{j+1}(I_\mu)$ for every $\mu\in\Lambda$, we have only to prove that $S_\xi f$ is j-times continuously differentiable at every point in $M=\{x\in(a,b)\,|\,\phi(x)=0\}$. Let x_0 be any point in M. We shall show that $S_\xi f$ is j-times continuously differentiable at x_0 and that we have

$$\frac{d^p}{dx^p}S_\xi f(x_0)=\begin{cases}\frac{1}{\xi}f(x_0), & p=0,\\[2mm] \frac{f'(x_0)}{\xi+\phi'(x_0)}, & p=1,\\[2mm] \sum_{1\leqq l\leqq p-1}\frac{\phi_l^p(x_0,\xi)}{\xi+(l+1)\phi'(x_0)}P_l\cdots P_1f'(x_0,\xi), & 2\leqq p\leqq j.\end{cases} \tag{2.1.10}$$

Case 1. x_0 is not an accumulation point of M. This means that $x_0=a_\mu$ or b_μ for some $\mu\in\Lambda$. We have by Lemma 2.3 and Lemma 2.4 (2.4.2)

$$\lim_{\substack{x\to x_0\\ x\in I_\mu}}\Big|S_\xi f(x)-\frac{1}{\xi}f(x)\Big|=0, \tag{2.1.11}$$

$$\lim_{\substack{x\to x_0\\ x\in I_\mu}}\Big|\frac{d}{dx}S_\xi f(x)-\frac{f'(x)}{\xi+\phi'(x)}\Big|=0, \tag{2.1.12}$$

$$\lim_{\substack{x\to x_0\\ x\in I_\mu}}\Big|\frac{d^p}{dx^p}S_\xi f(x)-\sum_{1\leqq l\leqq p-1}\frac{\phi_l^p(x,\xi)}{\xi+(l+1)\phi'(x)}P_l\cdots P_1f'(x,\xi)\Big|=0,\quad 2\leqq p\leqq j, \tag{2.1.13}$$

which complete the proof.

Case 2. x_0 is an accumulation point of M. By definition (2.1.5) of $S_\xi f$ and Lemma 2.3 (c)

$$S_\xi f(x)-\frac{1}{\xi}f(x)=\begin{cases} 0\,, & x\in M\,, \\ -K_0 f(x,\xi)\,, & x\in I_\mu \text{ for some } \mu\in \Lambda\,. \end{cases} \tag{2.1.14}$$

Since $\phi(x_0)=0$, we see from Lemma 2.4 (2.4.2) that

$$\lim_{x\to x_0}\Big(S_\xi f(x)-\frac{1}{\xi}f(x)\Big)=0\,. \tag{2.1.15}$$

Hence $S_\xi f$ is continuous at x_0. Furthermore since $\phi=0$ at x_0 in infinite order, we see from (2.4.2) that the right-hand side of (2.1.14) is differentiable and its derivative vanishes at x_0. This proves (2.1.10) for $p=1$. Thus we have from Case 1 above and Lemma 2.3 (d)

$$\frac{d}{dx}S_\xi f(x)-\frac{1}{\xi+\phi'(x)}f'(x)=\begin{cases} 0\,, & x\in M\,, \\ -K_1 f(x,\xi)\,, & x\in I_\mu \text{ for some } \mu\in \Lambda\,. \end{cases} \tag{2.1.16}$$

Hence by Lemma 2.4 (2.4.2) the right-hand side of (2.1.16) is continuous, differentiable, and vanishes at x_0. Using Lemma 2.3 (e), (f) and Lemma 2.4 (2.4.2) and repeating the same reasoning as above, we see that $S_\xi f$ is p times continuously differentiable at x_0 and (2.1.10) holds.

(2.1.2). Since the differentiability of $S_\xi f$ is already proved in (2.1.1), (2.1.2) follows immediately from Lemma 2.2 (2.2.1) and the definition of $S_\xi f$ in $M=\{x\in(a,b)\,|\,\phi(x)=0\}$.

(2.1.3). When $x\in M$, both sides of (2.1.3) are equal to zero by definition of $S_\xi f$. If $x\in M$ and $x\in I_\mu$ for some $\mu\in\Lambda$, Lemma 2.3 (d) shows that

$$\phi(x)\frac{d}{dx}S_\xi f(x)=\xi K_0 f(x,\xi)=S_\xi\Big(\phi\frac{d}{dx}f\Big)(x)\,, \tag{2.1.17}$$

which proves (2.1.3).

(2.1.4). The infinite differentiability of $\frac{d^p}{dx^p}S_\xi f$ with respect to ξ for large $|\xi|$ follows from Lemma 2.3, Lemma 2.4 (2.4.2) and (2.1.10). On the other hand by definition (2.3.1) of ψ_l^p

$$\Big|\frac{\partial^N}{\partial\xi^N}\psi_l^p(x,\xi)\Big|\leq C(1+|\xi|)^{l-N}\,, \tag{2.1.18}$$

where C is a positive constant independent of x and ξ. Hence the inequalities (2.1.4.1) and (2.1.4.2) follow from Lemma 2.3, Lemma 2.4, and (2.1.18). Q. E. D.

We introduce some notations. For every $g\in L^1((\alpha,\beta))$ we define $Tg(\xi)$ by

$$Tg(\xi)=\int_\alpha^\beta \exp\Big(-i\xi\int_0^t \sigma(s)ds\Big)g(t)dt\,,\qquad \xi\in R^1\,. \tag{2.5.1}$$

For $\tilde g\in L^1(R^1)$ we define $\tilde T\tilde g(t)$ by

$$\tilde{T}\tilde{g}(t)=\int \exp\Big(i\xi\int_0^t \sigma(s)ds\Big)\tilde{g}(\xi)d\xi\,, \qquad \alpha<t<\beta\,. \tag{2.5.2}$$

PROPOSITION 2.5. *Let K be any compact subset of (α, β) and k be the maximum of orders of zeros of σ in K.*

(a) *Take any $\chi\in C_0^\infty(K)$. Then for every $r\geqq 0$ there exists a constant $C>0$ such that*

$$\|\chi\tilde{T}\tilde{g}\|_r^2\leqq C\int(1+|\xi|^2)^{r+\frac{k}{2(k+1)}}|\tilde{g}(\xi)|^2d\xi\,, \qquad \tilde{g}\in L^1(R^1)\,. \tag{2.5.3}$$

(b) *For every $s\geqq\dfrac{k}{2(k+1)}$, there exists a constant $C>0$ such that for every $g\in L^1((\alpha, \beta))$ with support in K we have*

$$\int(1+|\xi|^2)^{-s}|Tg(\xi)|^2d\xi\leqq C\|g\|^2_{-s+\frac{k}{2(k+1)}}\,. \tag{2.5.4}$$

(c) *There exists a constant $C>0$ such that*

$$|Tg(\xi)|\leqq C(1+|\xi|)^{-\frac{1}{k+1}}\sup_{\alpha<t<\beta}(|g(t)|+|g'(t)|) \tag{2.5.5}$$

for every $g\in C_0^1((\alpha, \beta))$ with support in K.

PROOF. (a) We may assume that $\tilde{g}\in C_0^\infty(R^1)$. Let t_i, $1\leqq i\leqq N$, be zeros of σ in K and k_i be the order of zero of σ at t_i. We note that k_i is even since $\sigma\geqq 0$ in (α, β). For every $\phi\in C_0^\infty(R^1)$ we define ϕ_i as follows:

$$\phi_i(t)=\phi\Big(\int_{t_i}^t \sigma(s)ds\Big)\,, \qquad \alpha<t<\beta\,. \tag{2.5.6}$$

Take $\phi\in C_0^\infty(R^1)$ such that $\phi=1$ near 0 and $\operatorname{supp}\phi$ is so small that $\operatorname{supp}\phi_i$, $1\leqq i\leqq N$, are mutually disjoint and compact in (α, β).

We can write

$$\chi\tilde{T}\tilde{g}=\sum_{1\leqq i\leqq N}\phi_i\chi\tilde{T}\tilde{g}+\Big(1-\sum_{1\leqq i\leqq N}\phi_i\Big)\chi\tilde{T}\tilde{g}\,. \tag{2.5.7}$$

By the fact that $\sigma\neq 0$ near $\operatorname{supp}(1-\sum_{1\leqq i\leqq N}\phi_i)\chi$ one can easily derive

$$\Big\|\Big(1-\sum_{1\leqq i\leqq N}\phi_i\Big)\chi\tilde{T}\tilde{g}\Big\|_r^2\leqq C\int(1+|\xi|^2)^r|\tilde{g}(\xi)|^2d\xi\,, \qquad \tilde{g}\in L^1(R^1)\,, \tag{2.5.8}$$

where C is a positive constant independent of $\tilde{f}$. Hence it suffices to prove (2.5.3) for $\phi_i\chi\tilde{T}\tilde{g}$ instead of $\chi\tilde{T}\tilde{g}$. First we consider the case where $r=0$. We can write (possibly after shrinking $\operatorname{supp}\phi$)

$$\sigma(t)=\sigma_i(t)(t-t_i)^{k_i}\,, \qquad \sigma_i(t)>0\,, \quad \text{near } \operatorname{supp}\phi_i\,. \tag{2.5.9}$$

By the change of variable

$$\tau=\int_{t_i}^t \sigma(s)ds\,, \qquad \alpha<t<\beta\,, \tag{2.5.10}$$

we have

$$(2.5.11)\qquad \|\phi_i \chi \tilde{T}\tilde{g}\|_0^2$$
$$\leqq C\int|\phi(\tau)|^2\left|\int \exp i\xi\tau \exp\left(i\xi\int_0^{t_i}\sigma(s)\,ds\right)\tilde{g}(\xi)d\xi\right|^2\frac{1}{|\tau|^{\frac{k_i}{k_i+1}}}d\tau$$
$$\leqq C'\int(1+|\xi|)^{\frac{k_i}{k_i+1}}|\tilde{g}(\xi)|^2 d\xi\,,$$

where C and C' are positive constants independent of $\tilde{g}$. Here we have used the well-known inequality (cf: [2] Lemma 8.12)

$$(2.5.12)\qquad \int|f(t)|^2\frac{1}{|t|^\alpha}dt\leqq C_\alpha\int|\xi|^\alpha|\hat{f}(\xi)|^2 d\xi\,,\qquad 0<\alpha<1\,,\quad f\in C_0^\infty(R^1)\,,$$

where C_α is a positive constant depending only on α. This proves (2.5.3) for $r=0$. Differentiating $\chi\tilde{T}\tilde{g}$ less than or equal to r times, we see by (2.5.3) for $r=0$, that (2.5.3) holds also when r is any positive integer. Therefore we have (2.5.3) for all $r\geqq 0$ by the interpolation theory.

(b) This is nothing but the dual of (a).

(c) Using the same ϕ_i as in the proof of (a) we can write

$$(2.5.13)\qquad Tg=\sum_{1\leqq i\leqq N}T(\phi_i g)+T((1-\sum_{1\leqq i\leqq N}\phi_i)g)\,.$$

Since $\sigma\neq 0$ in a neighborhood of $K\cap\operatorname{supp}(1-\sum_{1\leqq i\leqq N}\phi_i)$ we have

$$(2.5.13)'\qquad |T((1-\sum_{1\leqq i\leqq N}\phi_i)g)(\xi)|\leqq C(1+|\xi|)^{-1}\sup_{\alpha<t<\beta}(|g(t)|+|g'(t)|)\,,$$

for every $g\in C_0^1((\alpha,\beta))$ with support in K, where C is a positive constant independent of f.

Next we consider about $T(\phi_i f)$. By partial integration

$$(2.5.14)\qquad T(\phi_i f)(\xi)=\int_{|t-t_i|\leqq(1+|\xi|)^{-\frac{1}{k_i+1}}}\exp\left(-i\xi\int_0^t\sigma(s)\,ds\right)\phi_i(t)g(t)dt$$
$$+\frac{1}{i\xi}\int_{|t-t_i|\geqq(1+|\xi|)^{-\frac{1}{k_i+1}}}\exp\left(-i\xi\int_0^t\sigma(s)ds\right)\frac{d}{dt}\left(\frac{\phi_i}{\sigma}g\right)(t)\,dt$$
$$+\frac{1}{i\xi}\left[\exp\left(-i\xi\int_0^t\sigma(s)\,ds\right)\frac{\phi_i(t)}{\sigma(t)}f(t)\right]_{t=t_i-(1+|\xi|)^{-\frac{1}{k_i+1}}}^{t=t_i+(1+|\xi|)^{-\frac{1}{k_i+1}}}$$

for sufficiently large $|\xi|$.

On the other hand by (2.5.9) we have with a constant $C_i>0$

$$(2.5.15)\qquad \left|\frac{1}{\sigma(t)}\right|\leqq C_i\frac{1}{|t-t_i|^{k_i}}$$

and

(2.5.16)
$$\left|\frac{d}{dt}\left(\frac{1}{\sigma(t)}\right)\right| \leqq C_i \frac{1}{|t-t_i|^{k_i+1}}$$

in a neighborhood of supp ψ_i. (2.5.14), (2.5.15) and (2.5.16) yield

(2.5.17)
$$|T(\psi_i f)(\xi)| \leqq C(1+|\xi|)^{-\frac{1}{k+1}} \sup_{\alpha<t<\beta}(|f(t)|+|f'(t)|)$$

for every $f\in C_0^1((\alpha, \beta))$ with support in K, where C is a constant >0 independent of f. Then (2.5.5) follows immediately from (2.5.13)$'$ and (2.5.17).

§ 3. Construction of a parametrix.

We introduce some notations.

$$H_{r,s}=\left\{f\in\mathcal{S}'(R_x\times R_t)\,|\,\|f\|_{r,s}^2=\iint(1+|\xi|^2)^r(1+|\tau|^2)^s|\hat{f}(\xi, \tau)|^2 d\xi d\tau<+\infty\right\},$$

$$H_{r,s}^{\mathrm{loc}}(\Omega)=\{f\in\mathcal{D}'(\Omega)\,|\,\omega f\in H_{r,s} \text{ for every } \omega\in C_0^\infty(\Omega)\},$$

$$H_{r,s}^0(\Omega)=\mathcal{E}'(\Omega)\cap H_{r,s},$$

$$H_{r,s,K}^0(\Omega)=\{f\in H_{r,s}^0(\Omega)\,|\, t\text{-projection of supp} f\subset K\Subset(\alpha, \beta)\},$$

where r, s are any real numbers, $\Omega=(a, b)\times(\alpha, \beta)$ and K is any compact set in (α, β).

THEOREM 3.1. *Denote by L the differential operator* $\frac{\partial}{\partial t}+i\phi(x)\sigma(t)\frac{\partial}{\partial x}$ *defined in* $\Omega=(a, b)\times(\alpha, \beta)$, *and assume that* ϕ *and* σ *satisfy* (0.7) *and* (0.8) *respectively. Then, for every positive integer j, there exist linear mappings* E_j, R_j *and* R_j' *such that*

(3.1.1)
$$E_j:\ H_{0,0}^0(\Omega)\longrightarrow H_{0,\delta}^{\mathrm{loc}}(\Omega),$$

(3.1.2)
$$R_j:\ H_{r,s}^0(\Omega)\longrightarrow H_{r,\tilde{s}}^{\mathrm{loc}}(\Omega),$$

(3.1.3)
$$R_j':\ H_{r,s}^0(\Omega)\longrightarrow H_{r,\tilde{s}}^{\mathrm{loc}}(\Omega),$$

for any real numbers r, s and $\tilde{s}$ *and they have the following properties.*

(3.1.4)
$$LE_jf=f+R_jf \quad in \quad \Omega, \qquad f\in H_{0,0}^0(\Omega),$$

(3.1.5)
$$E_jLf=f+R_j'f \quad in \quad \Omega \qquad for\ all \quad f\in H_{0,0}^0(\Omega)$$

such that $Lf\in H_{0,0}^0(\Omega)$.

(3.1.6) *Let K be any compact subset of* (α, β) *and k be the maximum of orders of zeros of* σ *in K. Take any* $\omega\in C_0^\infty(\Omega)$ *and denote by* l_ω *the maximum of orders of zeros of* σ *in the t-projection of* supp ω. *Let* $\delta=\frac{1}{2}\left(\frac{1}{l_\omega+1}+\frac{1}{k+1}\right)$. *Then we have with a positive constant C independent of f,*

$$\left\|\omega \frac{\partial^p}{\partial x^p} E_j f\right\|_{0,0} \leqq C\|f\|_{p,-\delta}, \qquad 0 \leqq p \leqq j, \quad f \in H^0_{p,0,K}(\Omega).$$

(3.1.7) *Let K and ω be as in* (3.1.6), *then for any real numbers $r, s, \tilde{s}$ we have with a positive constant C independent of f,*

$$\|\omega R_j f\|_{r,\tilde{s}} \leqq C\|f\|_{r,s},$$

$$\|\omega R'_j f\|_{r,\tilde{s}} \leqq C\|f\|_{r,s}, \qquad f \in H^0_{r,s,K}(\Omega).$$

PROOF. At first we define E_j, R_j and R'_j for $f \in C_0^\infty(\Omega)$. Take $\chi_j(\xi) \in C^\infty(R^1)$ such that $\chi_j(\xi)=0$ for $|\xi| \leqq 2C_j+1$ and $\chi_j(\xi)=1$ for $|\xi| \geqq 3C_j+1$, where C_j is the constant determined in Proposition 2.1. From now on in this proof we drop the subscript j for E_j, R_j, R'_j and χ_j. We define operators U and E by the following formulas.

(3.1.8) $$Uf(x, \xi) = \chi(\xi) S_\xi\Big(\int \exp\Big(-i\xi \int_0^{t'} \sigma(s)ds\Big) f(\cdot, t')dt'\Big)(x),$$

(3.1.9) $$Ef(x, t) = \frac{1}{2\pi i} \int \exp\Big(i\xi \int_0^t \sigma(s)ds\Big) Uf(x, \xi)d\xi.$$

Ef is well defined and j times continuously differentiable with respect to x by Proposition 2.1 (2.1.4) and Proposition 2.5 (2.5.5). Furthermore one can write

(3.1.10) $$\frac{\partial^p}{\partial x^p} Ef(x, t) = \frac{1}{2\pi i} \int \exp\Big(i\xi \int_0^t \sigma(s)ds\Big) \frac{\partial^p}{\partial x^p} Uf(x, \xi)d\xi, \qquad 0 \leqq p \leqq j.$$

We prove (3.1.6) for $f \in C^\infty_{0,K}(\Omega) = \{\phi \in C_0^\infty(\Omega) \mid t\text{-projection of } \operatorname{supp}\phi \subset K\}$, where K is a compact set in (α, β). Take the functions $\psi_1, \psi_2 \in C_0^\infty((\alpha, \beta))$ such that ψ_1 (resp. ψ_2) is equal to 1 in a neighborhood of the t-projection of $\operatorname{supp}\omega$ (resp. K) and the maximum of orders of zeros of σ in $\operatorname{supp}\psi_1$ (resp. $\operatorname{supp}\psi_2$) is equal to l_ω (resp. k). Then, applying Proposition 2.5 (2.5.3), Proposition 2.1 (2.1.4.2) and Proposition 2.5 (2.5.4) successively to (3.1.10) we have

$$\begin{aligned}
&\left\|\omega \frac{\partial^p}{\partial x^p} Ef\right\|^2_{0,0} \\
&= \iint \left|\psi_1(t)\omega(x, t) \frac{\partial^p}{\partial x^p} E(\psi_2 f)(x, t)\right|^2 dxdt \\
&\leqq C_1 \iint |\psi_1(t)|^2 \left|\int \exp\Big(i\xi \int_0^t \sigma(s)ds\Big) \frac{\partial^p}{\partial x^p} U(\psi_2 f)(x, \xi)d\xi\right|^2 dxdt \\
&\leqq C_2 \iint (1+|\xi|^2)^{\frac{l_\omega}{2(l_\omega+1)}} \left|\frac{\partial^p}{\partial x^p} U(\psi_2 f)(x, \xi)\right|^2 dxd\xi \\
&= C_2 \iint (1+|\xi|^2)^{\frac{l_\omega}{2(l_\omega+1)}} \left|\chi(\xi) \frac{\partial^p}{\partial x^p} S_\xi\Big(\int \exp\Big(-i\xi \int_0^{t'} \sigma(s)ds\Big)\psi_2(t') f(\cdot, t')dt'\right|^2 dxd\xi \\
&\leqq C_3 \sum_{0 \leqq l \leqq p} \iint (1+|\xi|^2)^{\frac{l_\omega}{2(l_\omega+1)}-1} \left|\int \exp\Big(-i\xi \int_0^{t'} \sigma(s)ds\Big)\psi_2(t') \frac{\partial^l}{\partial x^l} f(x, t')dt'\right|^2 dxd\xi
\end{aligned}$$

$$\leqq C_4\|\psi_2 f\|^2_{p,\frac{l_\omega}{2(l_\omega+1)}-1+\frac{k}{2(k+1)}}$$

$$\leqq C_5\|f\|^2_{p,-\delta},$$

where C_i, $1\leqq i\leqq 5$, is a positive constant independent of $f\in C^\infty_{0,K}(\Omega)$. Thus (3.1.6) has been proved for $f\in C^\infty_{0,K}(\Omega)$.

On the other hand if $f\in C^\infty_0(\Omega)$ and $f=0$ in $(a,b)\times\tilde{M}'$, where $\tilde{M}'$ is an open set including $\tilde{M}=\{t\in(\alpha,\beta)\,|\,\sigma(t)=0\}$, we have, by Proposition 2.1 (2.1.4.1) for any positive integer N

$$(3.1.11)\qquad |Uf(x,\xi)|\leqq\frac{C}{(1+|\xi|)}\sup_{a<x<b}\left|\int\exp\left(-i\xi\int_0^{t'}\sigma(s)\,ds\right)f(x,t')\,dt'\right|$$

$$\leqq\frac{C_{N,f}}{(1+|\xi|)^N},$$

where $C_{N,f}$ is a positive constant dependent on f and N but independent of x, ξ. Hence Ef is continuously differentiable with respect to t also and the order of integration and differentiation can be interchanged. By Proposition 2.1 (2.1.2) and the Fourier inversion formula we can write for f as is stated above

$$(3.1.12)\qquad LEf(x,t)$$

$$=\frac{\sigma(t)}{2\pi}\int\exp\left(i\xi\int_0^t\sigma(s)\,ds\right)\left(\xi Uf(x,\xi)+\phi(x)\frac{\partial}{\partial x}Uf(x,\xi)\right)d\xi$$

$$=\frac{\sigma(t)}{2\pi}\int\exp\left(i\xi\int_0^t\sigma(s)\,ds\right)\left(\int\exp\left(-i\xi\int_0^{t'}\sigma(s)ds\right)f(x,t')dt'\right)d\xi$$

$$+\frac{\sigma(t)}{2\pi}\int\exp\left(i\xi\int_0^t\sigma(s)\,ds\right)(\chi(\xi)-1)$$

$$\times\left(\int\exp\left(-i\xi\int_0^{t'}\sigma(s)\,ds\right)f(x,t')\,dt'\right)d\xi$$

$$=f(x,t)+\frac{\sigma(t)}{2\pi}\iint\exp\left(i\xi\int_{t'}^t\sigma(s)\,ds\right)(\chi(\xi)-1)f(x,t')dt'd\xi\,.$$

For general $f\in C^\infty_0(\Omega)$ approximating it in L^2-norm by functions as above with supports contained in a common compact set K in Ω, we see, by (3.1.6) for $f\in C^\infty_{0,K}(\Omega)$ and $p=0$, that (3.12) also holds for $f\in C^\infty_0(\Omega)$.

Here we define R as follows:

$$(3.1.13)\qquad Rf(x,t)$$

$$=\frac{\sigma(t)}{2\pi}\iint\exp\left(i\xi\int_{t'}^t\sigma(s)ds\right)(\chi(\xi)-1)f(x,t')dt'd\xi\,,\qquad f\in C^\infty_0(\Omega)\,.$$

Then we have (3.1.4) for every $f\in C^\infty_0(\Omega)$ and the first inequality in (3.1.7) is immediate.

Next we shall prove (3.1.5) for $f\in C_0^\infty(\Omega)$. Integrating partially with respect to t', we obtain from Proposition 2.1 (2.1.2) and Proposition 2.1 (2.1.3)

(3.1.14)
$$\begin{aligned} &ULf(x,\xi)\\ &=i\chi(\xi)\Big\{\xi S_\xi\Big(\int\exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\sigma(t')f(\cdot,t')dt'\Big)(x)\\ &\qquad+\phi(x)\frac{\partial}{\partial x}S_\xi\Big(\int\exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\sigma(t')f(\cdot,t')dt'\Big)(x)\Big\}\\ &=i\chi(\xi)\int\exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\sigma(t')f(x,t')dt'.\end{aligned}$$

Hence

(3.1.15)
$$\begin{aligned} ELf(x,t)=\frac{1}{2\pi}&\int\exp\Big(i\xi\int_0^{t}\sigma(s)ds\Big)\\ &\times\chi(\xi)\Big(\int\exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\sigma(t')f(x,t')dt'\Big)d\xi.\end{aligned}$$

Note that we have by Proposition 2.5 (2.5.5)

(3.1.16)
$$\begin{aligned} &\Big|\int\exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\sigma(t')f(x,t')dt'\Big|\\ &=\Big|\frac{1}{i\xi}\int\exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\frac{\partial f}{\partial t'}(x,t')dt'\Big|\\ &\leqq\frac{C}{(1+|\xi|)^{1+\frac{1}{k+1}}}\sup_{\substack{a<x<b\\ \alpha<t<\beta}}\Big(\Big|\frac{\partial f}{\partial t}(x,t)\Big|+\Big|\frac{\partial^2 f}{\partial t^2}(x,t)\Big|\Big)\\ &\qquad\qquad\text{for large } |\xi|,\quad f\in C_{0,K}^\infty(\Omega),\end{aligned}$$

where K is a compact subset in (α,β), k is the maximum of orders of zeros of σ in K, and C is a positive constant independent of f. By the assumption (0.8) on σ, the change of variable $\tau=\int_0^t\sigma(s)ds$ is a homeomorphism between (α,β) and (α',β'), where $\alpha'=\lim_{t\to\alpha+0}\int_0^t\sigma(s)ds$ and $\beta'=\lim_{t\to\beta-0}\int_0^t\sigma(s)ds$. Also by (0.8), the zeros of σ in (α,β) are all isolated. Hence we have

$$\int\exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\sigma(t')f(x,t')dt'=\int\exp(-i\xi\tau')\tilde f(x,\tau')d\tau',\qquad f\in C_{0,K}^\infty(\Omega),$$

where $\tau'=\int_0^{t'}\sigma(s)ds$ and $\tilde f(x,\tau')=f(x,t')$. For every fixed $x\in(a,b)$, $\tilde f(x,\tau)$ is a continuous function of τ with compact support. Therefore by (3.1.16) and the Fourier inversion formula we get

$$\begin{aligned} f(x,t)&=\tilde f(x,\tau)\\ &=\frac{1}{2\pi}\int\exp(i\xi\tau)\Big(\int\exp(-i\xi\tau')\tilde f(x,\tau')d\tau'\Big)d\xi\end{aligned}$$

$$=\frac{1}{2\pi}\int \exp\Big(i\xi\int_0^t \sigma(s)ds\Big)\Big(\int \exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\sigma(t')f(x, t')dt'\Big)d\xi .$$

Hence we have by (3.1.15)

(3.1.17) $$ELf(x, t)=f(x, t) +\frac{1}{2\pi}\iint \exp\Big(i\xi\int_{t'}^t \sigma(s)ds\Big)(\chi(\xi)-1)\sigma(t')f(x, t')dt'd\xi , \qquad f\in C_{0,K}^\infty(\Omega).$$

Now define $R'f$ by

(3.1.18) $$R'f(x, t)=\frac{1}{2\pi}\iint \exp\Big(i\xi\int_{t'}^t \sigma(s)ds\Big)(\chi(\xi)-1)\sigma(t')f(x, t')dt'd\xi .$$

Then

(3.1.19) $$ELf=f+R'f \qquad \text{for every} \quad f\in C_0^\infty(\Omega).$$

The second inequality in (3.1.7) is immediate from (3.1.18).

Finally we shall extend E, R and R'. Take $\rho\in C_0^\infty(R_x\times R_t)$ such that $\iint \rho dxdt=1$ and set $\rho_\varepsilon(x, t)=\frac{1}{\varepsilon^2}\rho\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\Big)$. For every $f\in H_{0,0}^0(\Omega)$ we define $Ef\in\mathcal{D}'(\Omega)$ in the following manner.

(3.1.20) $$\langle Ef, g\rangle=\lim_{\varepsilon\to 0}\langle E(\rho_\varepsilon * f), g\rangle , \qquad g\in C_0^\infty(\Omega).$$

And for $f\in H_{r,s}^0(\Omega)$, where r and s are any real numbers, we define $Rf\in\mathcal{D}'(\Omega)$ and $R'f\in\mathcal{D}'(\Omega)$ as follows:

(3.1.21) $$\langle Rf, g\rangle=\lim_{\varepsilon\to 0}\langle R(\rho_\varepsilon * f), g\rangle , \qquad g\in C_0^\infty(\Omega).$$

(3.1.22) $$\langle R'f, g\rangle=\lim_{\varepsilon\to 0}\langle R'(\rho_\varepsilon * f), g\rangle , \qquad g\in C_0^\infty(\Omega).$$

Since Theorem 3.1 is already proved for $f\in C_0^\infty(\Omega)$, these definitions are well defined and it is not difficult to verify that Theorem 3.1 holds for extended E, R and R'. Q. E. D.

REMARK. Let (x_0, t_0) be any point in Ω. By (3.1.7) we have, for sufficiently small neighborhood Ω_0 of (x_0, t_0)

(3.1.23) $$\iint_{\Omega_0}|R_jf(x, t)|^2dxdt\leqq\frac{1}{2}\iint_{\Omega_0}|f(x, t)|^2dxdt , \qquad f\in H_{0,0}^0(\Omega_0).$$

Combining this with (3.1.4) we see that L is locally solvable at (x_0, t_0).

§ 4. L^2-estimate.

LEMMA 4.1. *For every positive integer j, let E_j be the parametrix constructed in Theorem 3.1. Let p be any positive integer and assume that $f\in H_{0,0}^0(\Omega)$ and $\Big(\phi\frac{\partial}{\partial x}\Big)^k f\in H_{0,0}^0(\Omega)$ for $1\leqq k\leqq p$. Then we can write*

(4.1.1) $$\frac{\partial^p}{\partial t^p}E_j f=\sum_{1\leq k\leq p}\sigma_{p,k}E_j\Big(\Big(\phi\frac{\partial}{\partial x}\Big)^k f\Big)+\sum_{0\leq l+m\leq p-1}\sigma_{p,l,m}\frac{\partial^l}{\partial t^l}\Big(\phi\frac{\partial}{\partial x}\Big)^m(f+R_j f),$$

where $\sigma_{p,k}, \sigma_{p,l,m}\in C^\infty((\alpha,\beta))$ *are appropriate functions of* t *independent of* f.

PROOF. At first we shall prove the next two relations.

(4.1.2) $$\Big(\phi\frac{\partial}{\partial x}\Big)^k E_j f=E_j\Big(\Big(\phi\frac{\partial}{\partial x}\Big)^k f\Big),\qquad 1\leqq k\leqq p,$$

(4.1.3) $$\Big(\phi\frac{\partial}{\partial x}\Big)^k R_j f=R_j\Big(\Big(\phi\frac{\partial}{\partial x}\Big)^k f\Big),\qquad 1\leqq k\leqq p.$$

Clearly it suffices to prove these for $k=1$. By Proposition 2.1 (2.1.3), (3.1.10) and (3.1.8) we have

(4.1.2)′ $$\Big(\phi\frac{\partial}{\partial x}\Big)E_j f=E_j\Big(\Big(\phi\frac{\partial}{\partial x}\Big)f\Big)\qquad \text{for every}\quad f\in C_0^\infty(\Omega).$$

As for R_j, by its definition (3.1.13),

(4.1.3)′ $$\Big(\phi\frac{\partial}{\partial x}\Big)R_j f=R_j\Big(\Big(\phi\frac{\partial}{\partial x}\Big)f\Big)\qquad \text{for every}\quad f\in C_0^\infty(\Omega).$$

Let ρ_ε be the function selected for the extension of E, R and R' in the proof of Theorem 3.1. Then

(4.1.4) $$\lim_{\varepsilon\to 0} f*\rho_\varepsilon=f\qquad \text{in}\quad H_{0,0}$$

and

(4.1.5) $$\lim_{\varepsilon\to 0}\Big[\phi\Big(\frac{\partial f}{\partial x}*\rho_\varepsilon\Big)-\Big(\phi\frac{\partial f}{\partial x}\Big)*\rho_\varepsilon\Big]=0\qquad \text{in}\quad H_{0,0}.$$

Since the supports of the left-hand sides of (4.1.4) and (4.1.5) are contained in a common compact set in Ω if ε is sufficiently small, (4.1.2) and (4.1.3) follow from (4.1.2)′, (4.1.3)′, (4.1.4), (4.1.5), and Theorem 3.1 (3.1.6), (3.1.7). The proof of (4.1.1) is by induction on p. When $p=1$ (4.1.1) is immediate from Theorem 3.1 (3.1.4) and (4.1.2). Assuming that (4.1.1) holds for $p\geqq 1$ we shall prove (4.1.1) for $p+1$. Differentiating both sides of (4.1.1) with respect to t, we have

(4.1.6) $$\frac{\partial^{p+1}}{\partial t^{p+1}}E_j f=\sum_{1\leq k\leq p}\frac{d\sigma_{p,k}}{dt}E_j\Big(\Big(\phi\frac{\partial}{\partial x}\Big)^k f\Big)+\sum_{1\leq k\leq p}\sigma_{p,k}\frac{\partial}{\partial t}E_j\Big(\Big(\phi\frac{\partial}{\partial x}\Big)^k f\Big)+\frac{\partial}{\partial t}\Big(\sum_{0\leq l+m\leq p-1}\sigma_{p,l,m}\frac{\partial^l}{\partial t^l}\Big(\phi\frac{\partial}{\partial x}\Big)^m(f+R_j f)\Big).$$

On the other hand by Theorem 3.1 (3.1.4), (4.1.2) and (4.1.3) we can write

$$\frac{\partial}{\partial t}E_j\Big(\Big(\phi\frac{\partial}{\partial x}\Big)^k f\Big)=-i\sigma E_j\Big(\Big(\phi\frac{\partial}{\partial x}\Big)^{k+1} f\Big)+\Big(\phi\frac{\partial}{\partial x}\Big)^k f+\Big(\phi\frac{\partial}{\partial x}\Big)^k R_j f. \tag{4.1.7}$$

From (4.1.6) and (4.1.7) we obtain (4.1.1) for $p+1$. Q. E. D.

LEMMA 4.2. *For every positive integer j, let E_j be the parametrix constructed in Theorem 3.1. Take any functions ω, $\tilde{\omega}\in C_0^\infty(\Omega)$ such that $\tilde{\omega}=0$ near* supp ω, *and fix any integer p such that $0\leqq p\leqq j$. Then $\omega E_j(\tilde{\omega}f)\in H_{p,q}$ for any positive integer q if $f\in H_{p,0}^{\mathrm{loc}}(\Omega)$, and we have with a constant $C>0$ independent of f*

$$\|\omega E_j(\tilde{\omega}f)\|_{p,q}\leqq C\|\tilde{\omega}f\|_{p,0}, \qquad f\in H_{p,0}^{\mathrm{loc}}(\Omega). \tag{4.2.1}$$

PROOF. We may assume that $f\in C_0^\infty(\Omega)$. By the continuity of S_ξ in the sup-norm in x (cf. Proposition 2.1 (2.1.4.1)) we can write for large $|\xi|$

$$S_\xi\Big(\int \exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\tilde{\omega}f(\cdot,t')dt'\Big)(x)=\int \exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)S_\xi(\tilde{\omega}f)(x,t')dt'. \tag{4.2.2}$$

Hence we have for $0\leqq p'\leqq p$

$$\begin{aligned}\frac{\partial^{p'}}{\partial x^{p'}}(\omega E_j(\tilde{\omega}f))(x,t)&=\sum_{0\leqq r\leqq p'}\omega_{p',r}(x,t)\int \exp\Big(i\xi\int_0^t\sigma(s)ds\Big)\\&\quad\times\Big(\int\exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\frac{\partial^r}{\partial x^r}\{\chi_j(\xi)S_\xi(\tilde{\omega}f)\}(x,t')dt'\Big)d\xi,\end{aligned} \tag{4.2.3}$$

where $\omega_{p',r}(x,t)=\frac{{}_{p'}C_{p'-r}}{2\pi i}\frac{\partial^{p'-r}\omega}{\partial x^{p'-r}}(x,t)$. Let $\delta=\mathrm{dist}\,(\mathrm{supp}\,\omega,\ \mathrm{supp}\,\tilde{\omega})$ and take $\rho(t)\in C^\infty(R^1)$ such that $\rho(t)=0$, $|t|\leqq\frac{\delta}{4}$, and $\rho(t)=1$, $|t|\geqq\frac{\delta}{2}$. We now set

$$F_{1,r}(x,t,\xi)=\int\exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)(1-\rho(t-t'))\frac{\partial^r}{\partial x^r}\{\chi_j(\xi)S_\xi(\tilde{\omega}f)\}(x,t')dt', \tag{4.2.4}$$

$$F_{2,r}(x,t,\xi)=\int\exp\Big(-i\xi\int_0^{t'}\sigma(s)ds\Big)\rho(t-t')\frac{\partial^r}{\partial x^r}\{\chi_j(\xi)S_\xi(\tilde{\omega}f)\}(x,t')dt'. \tag{4.2.5}$$

Then (4.2.3) can be rewritten as follows:

$$\begin{aligned}\frac{\partial^{p'}}{\partial x^{p'}}(\omega E_j(\tilde{\omega}f))(x,t)&=\sum_{0\leqq r\leqq p'}\omega_{p',r}(x,t)\int\exp\Big(i\xi\int_0^t\sigma(s)ds\Big)F_{1,r}(x,t,\xi)d\xi\\&\quad+\sum_{0\leqq r\leqq p'}\omega_{p',r}(x,t)\int\exp\Big(i\xi\int_0^t\sigma(s)ds\Big)F_{2,r}(x,t,\xi)d\xi.\end{aligned} \tag{4.2.6}$$

At first we shall estimate the $H_{0,q}$ norm of the first term of the right-hand side of (4.2.6). Since $(x, t')\notin \operatorname{supp}\tilde{\omega}$ if $(x, t)\in \operatorname{supp}\omega_{p',r}(\subset \operatorname{supp}\omega)$ and if $t-t'\in \operatorname{supp}(1-\rho)$, we have by (2.1.10)

$$\omega_{p',r}(x, t)(1-\rho(t-t'))\frac{\partial^r}{\partial x^r}\{\chi_j(\xi)S_\xi(\tilde{\omega}f)\}(x, t')=0 \quad \text{if} \quad x\in M. \tag{4.2.7}$$

Let $x\notin M$. We see that $|x-y|\geqq\frac{\delta}{2}$ if $(x, t)\in \operatorname{supp}\omega_{p',r}(\subset \operatorname{supp}\omega)$, $t-t'\in \operatorname{supp}(1-\rho)$ and if $(y, t')\in \operatorname{supp}\tilde{\omega}$. Hence, then, for any non negative integer α

$$\begin{aligned}\frac{1}{|\phi(x)|}\exp\Big(\int_x^y \frac{\xi+\alpha\phi'(s)}{\phi(s)}ds\Big) \\ \leqq \exp(-c_1'|\xi|)\exp\Big(\int_x^y\frac{c_2'\xi}{\phi(s)}ds\Big)\frac{1}{|\phi(y)|}\end{aligned} \tag{4.2.8}$$

for sufficiently large $|\xi|$, either $a_\mu<y\leqq x<b_\mu$ and $\xi\phi(x)>0$ in I_μ or $a_\mu<x\leqq y<b_\mu$ and $\xi\phi(x)<0$ in I_μ, where c_1' and c_2' are positive constants independent of x, y and ξ. Therefore by Lemma 2.3, (2.4.6) and (2.4.7) we have for any non negative integer q'

$$\begin{aligned}&\Big|\frac{\partial^{q'}}{\partial t^{q'}}\Big(\omega_{p',r}(x, t)(1-\rho(t-t'))\frac{\partial^r}{\partial x^r}\{\chi_j(\xi)S_\xi(\tilde{\omega}f)\}(x, t')\Big)\Big| \\ &\leqq\begin{cases}0 & x\in M, \\ C(x, t)\chi_j(\xi)\exp(-c_1''|\xi|) \\ \quad\times\int_{a_\mu}^x \exp\Big(\int_x^y\frac{c_2\xi}{\phi(s)}ds\Big)\frac{1}{|\phi(y)|}\sum_{0\leqq\alpha\leqq r}\Big|\frac{\partial^\alpha}{\partial y^\alpha}(\tilde{\omega}f)(y, t')\Big|dy, \\ \qquad x\in I_\mu \text{ for some } \mu\in\Lambda,\ \xi\phi(x)>0 \text{ in } I_\mu, \\ C(x, t)\chi_j(\xi)\exp(-c_1''|\xi|) \\ \quad\times\int_x^{b_\mu}\exp\Big(\int_x^y\frac{c_2''\xi}{\phi(s)}ds\Big)\frac{1}{|\phi(y)|}\sum_{0\leqq\alpha\leqq r}\Big|\frac{\partial^\alpha}{\partial y^\alpha}(\tilde{\omega}f)(y, t')\Big|dy, \\ \qquad x\in I_\mu \text{ for some } \mu\in\Lambda,\ \xi\phi(x)<0 \text{ in } I_\mu,\end{cases}\end{aligned} \tag{4.2.9}$$

where $C(x, t)\geqq 0$ is an appropriate function belonging to $C_0^0(\Omega)$ and independent of f, and c_1'', c_2'' are positive constant independent of f and x, t, ξ, y, t'. From (4.2.4) and (4.2.9) the next inequality is immediate.

$$\Big|\frac{\partial^{q'}}{\partial t^{q'}}(\omega_{p',r}(x, t)F_{1,r}(x, t, \xi))\Big| \tag{4.2.10}$$

$$
\leqq\begin{cases}
0 & x\in M, \\
C(x,t)\chi_j(\xi)\exp(-c_1|\xi|) & \\
\quad\times\displaystyle\iint_{a_\mu}^{x}\exp\Big(\int_x^y\frac{c_2\xi}{\phi(s)}ds\Big)\frac{1}{|\phi(y)|}\sum_{0\leqq\alpha\leqq r}\Big|\frac{\partial^\alpha}{\partial y^\alpha}(\tilde{\omega}f)(y,t')\Big|dydt', & \\
& x\in I_\mu \text{ for some } \mu\in\Lambda,\ \xi\phi(x)>0 \text{ in } I_\mu, \\
C(x,t)\chi_j(\xi)\exp(-c_1|\xi|) & \\
\quad\times\displaystyle\iint_{x}^{b_\mu}\exp\Big(\int_x^y\frac{c_2\xi}{\phi(s)}ds\Big)\frac{1}{|\phi(y)|}\sum_{0\leqq\alpha\leqq r}\Big|\frac{\partial^\alpha}{\partial y^\alpha}(\tilde{\omega}f)(y,t')\Big|dydt', & \\
& x\in I_\mu \text{ for some } \mu\in\Lambda,\ \xi\phi(x)<0 \text{ in } I_\mu,
\end{cases}
$$

where c_1, c_2 are positive constants independent of f and x, t, ξ, y, t'. Now by (2.4.11) it is not difficult to show that

(4.2.11) the $H_{0,q}$ norm of the first term of the right-hand side of (4.2.6) $\leqq C\|\tilde{\omega}f\|_{p,0}$.

Next we shall estimate the $H_{0,q}$ norm of the second term of the right-hand side of (4.2.6). Take a function $\psi(\xi)\in C_0^\infty(R^1)$ such that $\psi(0)=1$. Then

$$
\begin{aligned}
(4.2.12)\quad & \omega_{p',r}(x,t)\int\exp\Big(i\xi\int_0^t\sigma(s)ds\Big)F_{2,r}(x,t,\xi)d\xi \\
&=\lim_{\varepsilon\to0}\omega_{p',r}(x,t)\iint\exp\Big(i\xi\int_{t'}^t\sigma(s)ds\Big) \\
&\qquad\times\rho(t-t')\psi(\varepsilon\xi)\frac{\partial^r}{\partial x^r}\{\chi_j(\xi)S_\xi(\tilde{\omega}f)\}(x,t')dt'd\xi.
\end{aligned}
$$

Since $|t-t'|\geqq\frac{\delta}{4}$ when $t-t'\in\operatorname{supp}\rho$,

(4.2.13) $\Big|\int_{t'}^t\sigma(s)ds\Big|\geqq C>0$ for some positive constant C if $(x,t)\in\operatorname{supp}\omega_{p',r}$ and $(x,t')\in\operatorname{supp}\frac{\partial^r}{\partial x^r}S_\xi(\omega f)$.

By partial integration with respect to ξ we have, for large positive integer N,

(4.2.14) the right-hand side of (4.2.12)

$$
\begin{aligned}
&=\omega_{p',r}(x,t)\iint\exp\Big(i\xi\int_{t'}^t\sigma(s)ds\Big) \\
&\qquad\times\frac{(-1)^N}{\Big(i\int_{t'}^t\sigma(s)ds\Big)^N}\rho(t-t')G_{r,N}(x,t',\xi)dt'd\xi,
\end{aligned}
$$

where $G_{r,N}(x,t',\xi)=\frac{\partial^N}{\partial\xi^N}\frac{\partial^r}{\partial x^r}\{\chi_j(\xi)S_\xi(\tilde{\omega}f)\}(x,t')$. If we apply Proposition 2.1 (2.1.4.2) to (4.2.14) it is not difficult to see that

(4.2.15) the $H_{0,q}$ norm of the second term of the right-hand side of (4.2.6) $\leqq C\|\tilde{\omega}f\|_{p',0}$.

Combining (4.2.6), (4.2.11) and (4.2.15), we obtain (4.2.1). Q. E. D.

THEOREM 4.3. *Let I, J and N be any non negative integers.*

(a) *Assume that u, $\left(\phi\frac{\partial}{\partial x}\right)^k(Lu)$, and $\frac{\partial^l}{\partial t^l}\left(\phi\frac{\partial}{\partial x}\right)^m(Lu)\in H^{\text{loc}}_{I,0}(\Omega)$ for $0\leqq k\leqq J$ and $0\leqq l+m\leqq J-1$. Then $u\in H^{\text{loc}}_{I,J}(\Omega)$.*

(b) *Under the same assumptions as in* (a), *take any two functions $\omega, \tilde{\omega}\in C_0^\infty(\Omega)$ such that $\tilde{\omega}=1$ near* $\operatorname{supp}\omega$, *and let l_ω and $l_{\tilde{\omega}}$ be the maximum of orders of zeros of σ in the t-projections of* $\operatorname{supp}\omega$ *and* $\operatorname{supp}\tilde{\omega}$ *respectively.*

Then we have with a constant $C>0$ independent of u

(4.3.1)
$$\|\omega u\|_{I,J}\leqq C\Big(\sum_{0\leqq k\leqq J}\Big\|\Big(\phi\frac{\partial}{\partial x}\Big)^k(\tilde{\omega}f)\Big\|_{I,-\delta} + \sum_{0\leqq l+m\leqq J-1}\Big\|\Big(\phi\frac{\partial}{\partial x}\Big)^m(\tilde{\omega}f)\Big\|_{I,l} + \|(L\tilde{\omega})u\|_{I,0}+\|\tilde{\omega}u\|_{I,-N}\Big)$$

where $f=Lu$ and $\delta=\frac{1}{2}\left(\frac{1}{l_\omega+1}+\frac{1}{l_{\tilde{\omega}}+1}\right)$.

PROOF. We shall apply Theorem 3.1, Proposition 4.1 and Proposition 4.2 with $j=I+1$. We have

(4.3.2)
$$L(\tilde{\omega}u)=\tilde{\omega}Lu+(L\tilde{\omega})u=\tilde{\omega}f+(L\tilde{\omega})u\,.$$

By the hypothesis of Theorem 4.3, both $\tilde{\omega}f$ and $(L\tilde{\omega})u$ belong to $H^0_{0,0}(\Omega)$. Hence $L(\tilde{\omega}u)\in H^0_{0,0}(\Omega)$ and we have, by Theorem 3.1 (3.1.5),

(4.3.3)
$$E_{I+1}L(\tilde{\omega}u)=\tilde{\omega}u+R'_{I+1}(\tilde{\omega}u)=E_{I+1}(\tilde{\omega}f)+E_{I+1}((L\tilde{\omega})u)\,.$$

Since $\tilde{\omega}=1$ near $\operatorname{supp}\omega$ we can write

(4.3.4)
$$\omega u=\omega E_{I+1}(\tilde{\omega}f)+\omega E_{I+1}((L\tilde{\omega})u)-\omega R'_{I+1}(\tilde{\omega}u)\,.$$

Let i, j be any non negative integers such that $0\leqq i\leqq I$ and $0\leqq j\leqq J$. By Leibniz' formula, we have

(4.3.5)
$$\frac{\partial^{i+j}}{\partial x^i\partial t^j}(\omega u)=\sum_{\substack{0\leqq p\leqq i\\0\leqq q\leqq j}}\omega_{i,j,p,q}\frac{\partial^{p+q}}{\partial x^p\partial t^q}E_{I+1}(\tilde{\omega}f) + \frac{\partial^{i+j}}{\partial x^i\partial t^j}(\omega E_{I+1}((L\tilde{\omega})u))-\frac{\partial^{i+j}}{\partial x^i\partial t^j}(\omega R'_{I+1}(\tilde{\omega}u))\,,$$

where $\omega_{i,j,p,q}={}_iC_{i-p}\,{}_jC_{j-q}\frac{\partial^{i+j-p-q}}{\partial x^{i-p}\partial t^{j-q}}\omega$.

By the hypothesis of Theorem 4.3 about $f=Lu$, we can apply Lemma 4.1 to $\frac{\partial^q}{\partial t^q}E_{I+1}(\tilde{\omega}f)$, $q\geqq 1$. Hence we can rewrite (4.3.5) as follows:

(4.3.5)′
$$\frac{\partial^{i+j}}{\partial x^i \partial t^j}(\omega u)=\sum_{0\leq p\leq i}\omega_{i,j,p,0}\frac{\partial^p}{\partial x^p}E_{I+1}(\tilde{\omega}f)$$
$$+\sum_{\substack{0\leq p\leq i\\ 1\leq k\leq q\leq j}}\omega_{i,j,p,q}\sigma_{p,k}\frac{\partial^p}{\partial x^p}E_{I+1}\Big(\Big(\phi\frac{\partial}{\partial x}\Big)^k(\tilde{\omega}f)\Big)$$
$$+\sum_{\substack{0\leq p\leq i\\ 0\leq l+m\leq q-1\leq j-1}}\omega_{i,j,p,q}\sigma_{p,l,m}\frac{\partial^p}{\partial x^p}\frac{\partial^l}{\partial t^l}\Big(\phi\frac{\partial}{\partial x}\Big)^m\big(\tilde{\omega}f+R_{I+1}(\tilde{\omega}f)\big)$$
$$+\frac{\partial^{i+j}}{\partial x^i\partial t^j}(\omega E_{I+1}((L\tilde{\omega})u))-\frac{\partial^{i+j}}{\partial x^i\partial t^j}(\omega R'_{I+1}(\tilde{\omega}u)).$$

Since $L\tilde{\omega}=0$ near supp ω, we may apply Lemma 4.2 to the fourth term in the right-hand side of (4.3.5)′. Next we apply Theorem 3.1 (3.1.6) and (3.1.7) to the terms in the right-hand side of (4.3.5)′ involving E_{I+1}, R_{I+1} and R'_{I+1}. Then we have with a constant $C>0$ independent of u, f

(4.3.6)
$$\iint\Big|\frac{\partial^{i+j}}{\partial x^i\partial t^j}(\omega u)\Big|^2 dxdt$$
$$\leq C\Big(\sum_{0\leq k\leq j}\Big\|\Big(\phi\frac{\partial}{\partial x}\Big)^k(\tilde{\omega}f)\Big\|^2_{i,-\delta}$$
$$+\sum_{0\leq l+m\leq j-1}\Big\|\Big(\phi\frac{\partial}{\partial x}\Big)^m(\tilde{\omega}f)\Big\|^2_{i,l}+\|(L\tilde{\omega})u\|^2_{\bar{i},0}+\|\tilde{\omega}u\|^2_{\bar{i},-N}\Big).$$

Hence $u\in H^{\mathrm{loc}}_{I,J}(\Omega)$ and (4.3.1) is immediate from (4.3.6). Q. E. D.

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