J. Math. Soc. Japan
Vol. 28, No. 3, 1976

Complex structures on $S^1 \times S^5$

By Masahide KATO

(Received July 16, 1975)

In this paper we study the structure of a compact complex manifold X of dimension 3 of which the 1st Betti number is equal to 1 and the 2nd Betti number vanishes. This manifold X has at most two algebraically independent meromorphic functions. Here we restrict ourselves to the case where X has exactly two algebraically independent meromorphic functions. Then X has an algebraic net of elliptic curves. We assume furthermore that this net has no base points, in other words, there exists a holomorphic mapping f of X onto a projective algebraic (non-singular) surface whose general fibres are connected non-singular elliptic curves. Finally we assume that f is equi-dimensional. Under these assumptions we prove the following:

(1) *There exists an infinite cyclic unramified covering manifold W of X such that $W \cup \{$one point$\}$ is holomorphically isomorphic to an affine variety which admits an algebraic $\boldsymbol{C}^*$-action* (Theorem 3).

(2) *Let X_t be any small deformation of X and W_t the deformation of W corresponding to X_t. Then, attaching one point 0_t to each W_t, we can construct a complex analytic family of complex spaces $\bigcup_t (W_t \cup \{0_t\})$ such that, for each t, $W_t \cup \{0_t\}$ is holomorphically isomorphic to an affine variety* (Theorem 4).

(3) *X_t is holomorphically isomorphic to a submanifold of $\boldsymbol{C}^{n_t} - \{0\}/\langle \tilde{g}_t \rangle$, where $\tilde{g}_t$ is a contracting holomorphic automorphism of the n_t-dimensional affine space $\boldsymbol{C}^{n_t}$ which fixes the origin* (Theorem 5).

In proving (1)—(3), we use the following fact:

A complex space[1] *which admits a contracting holomorphic automorphism is holomorphically isomorphic to an affine algebraic set* (Theorem 1)[2].

As corollaries to (1)—(3), we obtain some results concerning about certain complex structures on $S^1 \times S^5$. In connection with our investigation, we also have some results on elliptic surfaces of which the 1st Betti numbers are odd (Theorems 6, 7).

Some of the results of this paper were announced in [**3**, **4**].

The author would like to express his hearty thanks to Professor K. Kodaira, Professor S. Iitaka and Dr. K. Akao for their valuable advices.

1) By a *complex space*, we mean a reduced Hausdorff complex space.

2) See footnote 3).

§ 1. Preliminaries I.

Throughout this paper except in § 4, we use the following notation:
For a topological space M,

$\pi_1(M)$ = the fundamental group of M,

$b_i(M)$ = the i-th Betti number of M,

$\chi(M) = \Sigma(-1)^i b_i(M)$ = the Euler number of M.

For a compact complex manifold Y of dimension m,

$a(Y)$ = the algebraic dimension of Y

= the transcendental degree over $\boldsymbol{C}$ of the field of all meromorphic functions on Y,

$\mathcal{O}_Y$ = the sheaf of germs of holomorphic functions on Y,

$\mathcal{O}_Y(L)$ = the sheaf of germs of holomorphic sections of a line bundle L on Y,

$q(Y) = \dim H^1(Y, \mathcal{O}_Y)$,

$p_g(Y) = \dim H^m(Y, \mathcal{O}_Y)$,

$c_i[Y]$ = the i-th Chern class of Y,

$c(L)$ = the Chern class of a line bundle L on Y.

For a compact complex manifold X of dimension 3,

$p(X) = \dim H^2(X, \mathcal{O}_X)$,

$T(X) = (1/24)c_1[X] \cdot c_2[X]$ = the Todd genus of X.

We consider the fibre space (X, V, f) with the projection $f: X \to V$ which satisfies the following three conditions;

(i) *X is a compact complex manifold of dimension* 3,

(ii) *V is a non-singular projective algebraic manifold of dimension* 2,

(iii) *f is an equi-dimensional holomorphic surjective mapping whose general fibres are connected non-singular elliptic curves.*

We list up the additional conditions which will be occasionally imposed on (X, V, f).

(a) $b_1(X)=1$ *and* $b_2(X)=0$.

(b) *There exists a finite set A of points a_j $(j=1, 2, \cdots, \rho)$ on V such that, for each $v \in V-A$, $f^{-1}(v)$ is a (non-singular) elliptic curve.*

(c) $q(X) \geqq q(V)+1$.

(d) $T(X)=0$.

(e) $q(V)=p_g(V)=0$.

(f) $\chi(X)\leqq 0$.

If (X, V, f) satisfies (a), we denote it by $(X, V, f)_a$. In § 1, we study (X, V, f). In §§ 2-6 except in § 4, we study $(X, V, f)_a$.

Assume that (X, V, f) satisfies (b). We denote by F_j the fibre of f over $a_j \in A$. It is well-known that under the condition (b) general fibres of f are isomorphic to each other. Define proper analytic subsets Σ', Σ'' of X as follows;

$$\Sigma' = \{x \in X : \operatorname{rank} df(x) \leqq 1\} ,$$

$$\Sigma'' = \{x \in X : \operatorname{rank} df(x) = 0\} .$$

Since f is equi-dimensional, we have $\dim \Sigma'' \leqq 1$. By (b), $f(\Sigma'')$ is a finite set of points on V. Let Σ_i $(i=1, 2, \cdots, r)$ be all the irreducible 2-dimensional components of Σ' and Σ the union of Σ_i's. The sets $C_i=f(\Sigma_i)$ are irreducible curves on V. For a general point c on C_i, $F=f^{-1}(c)$ is a non-singular elliptic curve. Let p be any point of F. Choose a system of local coordinates $(\tilde{U}, (x, y, z))$ on X with center p such that $\Sigma_i \cap \tilde{U}=\{y=0\}$, $F \cap \tilde{U}=\{x=y=0\}$ and such that $\tilde{U}$, $\Sigma_i \cap \tilde{U}$ and $F \cap \tilde{U}$ are simply connected. Let $(U, (u, v))$ be a system of local coordinates on V with center $c=f(p)$ such that $C_i \cap U=\{v=0\}$ and $C_j \cap U=\emptyset$ for every $j \neq i$. Let the holomorphic mapping f be

$$\begin{cases} u = f_1(x, y, z) , \\ v = f_2(x, y, z) . \end{cases}$$

There exists a certain positive integer m such that

$$v = f_2(x, y, z) = y^m g_2(x, y, z) ,$$

where g_2 is an everywhere non-vanishing holomorphic function on U. Then the Jacobian matrix of f is

$$df_{|\Sigma_i \cap \tilde{U}} = \begin{cases} \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ 0 & g_2 & 0 \end{pmatrix}_{y=0} , & \text{if } m=1 , \\ \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ 0 & 0 & 0 \end{pmatrix}_{y=0} , & \text{if } m \geqq 2 . \end{cases}$$

On the other hand, $u_{|C_i \cap U}=f_1(x, 0, z)=x^n g_1(x, z)$ $(n \geqq 1)$. Since F is non-singular, we can assume that $g_1 \neq 0$ on $\Sigma_i \cap \tilde{U}$, i.e., $g_1 \neq 0$ on $\tilde{U}$, by choosing $\tilde{U}$ to be sufficiently small. Moreover the equality $n=1$ holds, since $c=f(p)$ is a general point of C_i. If $m=1$, then

$$df_{|\Sigma_i\cap\tilde{U}}=\begin{pmatrix} g_1+x\dfrac{\partial g_1}{\partial x} & * & * \\ 0 & g_2 & 0\end{pmatrix}_{y=0}$$

has rank 2 in a neighborhood of p. This contradicts the definition of Σ_i. Thus we obtain

$$f:\begin{cases} u=f_1(x, y, z)=x\cdot g_1(x, z)+y\cdot h(x, y, z)\,, \\ v=f_2(x, y, z)=y^m g_2(x, y, z) \qquad (m\geqq 2)\,, \end{cases}$$

where g_1, g_2 and h are holomorphic functions on $\tilde{U}$ and moreover g_1 and g_2 are everywhere non-vanishing on $\tilde{U}$. Letting a new system of local coordinates (x', y', z') on X with center p be

$$\begin{cases} x'=x\cdot g_1(x, z)+y\cdot h(x, y, z)\,, \\ y'=y(g_2(x, y, z))^{1/m}\,, \\ z'=z\,, \end{cases}$$

we have

$$f:\begin{cases} u=x'\,, \\ v=y'^m\,. \end{cases} \tag{0}$$

Note that m is constant in a small neighborhood of c in C_i.

Let D and $\tilde{D}$ be open subsets of U and $\boldsymbol{C}^2$ respectively defined by

$$D=\{(u, v)\in U: |u|<\varepsilon,\ |v|<\varepsilon^m\}\,,$$

$$\tilde{D}=\{(\sigma, \tau)\in \boldsymbol{C}^2: |\sigma|<\varepsilon,\ |\tau|<\varepsilon\}\,,$$

where ε is a small positive number. Define $M\rightarrow\tilde{D}$ to be the holomorphic fibre bundle of elliptic curves over D induced from $f^{-1}(D)\rightarrow D$ by the covering map $\lambda: \tilde{D}\rightarrow D$, $\lambda(\sigma, \tau)=(\sigma, \tau^m)=(u, v)$. Then M can be represented in the form

$$M=\tilde{D}\times\boldsymbol{C}/G\,,$$

where G is the group consisting of holomorphic automorphisms

$$((\sigma, \tau), \zeta)\longmapsto((\sigma, \tau),\ \zeta+n_1\omega+n_2)\,, \qquad n_1, n_2\in\boldsymbol{Z}$$

of $\tilde{D}\times\boldsymbol{C}$. Note that the holomorphic function ω ($\operatorname{Im}\omega\neq 0$) of (σ, τ) is constant since the general fibres F of M are isomorphic to each other. We denote by $[(\sigma, \tau), \zeta]$ the point of M corresponding to $((\sigma, \tau), \zeta)$. By a similar argument as in Kodaira [**6**] (pp. 767-768), we have

$$f^{-1}(D)=M/\mathfrak{G},$$

where $\mathfrak{G}$ is the cyclic group of order m generated by the holomorphic automorphism

$$h:[(\sigma, \tau), \zeta] \longmapsto [(\sigma, \rho\tau), \zeta+k/m], \qquad \rho=\exp(2\pi i/m),\ k\in \boldsymbol{Z},$$

of M. By the Künneth formula,

$$\begin{aligned} H^1(f^{-1}(D), \mathcal{O}_X) &= H^1(M, \mathcal{O}_M)^{\mathfrak{G}} \\ &= (H^0(\tilde{D}, \mathcal{O}_{\tilde{D}})\otimes H^1(F, \mathcal{O}_F))^{\mathfrak{G}} \\ &= H^0(\tilde{D}, \mathcal{O}_{\tilde{D}})^{\mathfrak{G}} \\ &= H^0(D, \mathcal{O}_D), \end{aligned}$$

since $\mathfrak{G}$ acts on $H^1(F, \mathcal{O}_F)$ trivially. This shows that $R^1f_*\mathcal{O}_{X|D}\cong\mathcal{O}_{V|D}$. Hence we infer that $R^1f_*\mathcal{O}_X$ *is a locally free sheaf of rank* 1 *on* V *except a finite number of stalks.*

PROPOSITION 1 (K. Akao). *If* (X, V, f) *satisfies* (b) *and* (c), *then there exists the following exact sequence of sheaves;*

$$(*) \qquad 0 \longrightarrow \mathcal{O}_V \longrightarrow R^1f_*\mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow 0,$$

where the support of $\mathcal{F}$ *is a finite set of points.*

PROOF. We shall prove this with the aid of the spectral sequence $E_2^{r,s}=H^r(V, R^sf_*\mathcal{O}_X)\Rightarrow E^{r+s}=H^{r+s}(X, \mathcal{O}_X)$. By (c) and the equality

$$\begin{aligned} q(X) = \dim H^1(X, \mathcal{O}_X) &= \dim E_2^{1,0}+\dim E_3^{0,1} \\ &= q(V)+\dim E_3^{0,1}, \end{aligned}$$

we have $\dim E_3^{0,1}\geqq 1$. This implies that $\dim E_2^{0,1}=\dim H^0(V, R^1f_*\mathcal{O}_X)\geqq 1$. Then, taking a non-zero section $\sigma\in H^0(V, R^1f_*\mathcal{O}_X)$, we form the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_V & \xrightarrow{j} & R^1f_*\mathcal{O}_X & \longrightarrow & \mathcal{F} & \longrightarrow & 0, \\ & & \Cup & & \Cup & & & & \\ & & s & \longmapsto & s\otimes\sigma & & & & \end{array}$$

where j is injective since $R^1f_*\mathcal{O}_X$ is locally free sheaf of rank 1 except on subvarieties of codimension $\geqq 2$. Let Δ be a general hyperplane section of V. Then, by (b), $S=f^{-1}(\Delta)$ is a non-singular elliptic surface over Δ every fibre of which is a (non-singular) elliptic curve. Then S is obtained from an elliptic surface S^* free from singular fibres by means of a finite number of logarithmic transformations ([6]). Note that S^* is an elliptic bundle with the projection $\pi: S^*\to\Delta$ corresponding to $f_{|S}: S\to\Delta$. Since $R^1\pi_*\mathcal{O}_{S^*}\cong\mathcal{O}_\Delta$ and $R^1\pi_*\mathcal{O}_{S^*}$ is invariant under logarithmic transformations, it follows that $\mathcal{O}_\Delta\cong R^1(f_{|S})_*\mathcal{O}_S\cong$

$R^1 f_* \mathcal{O}_{X|\Delta}$. Therefore the non-zero section $\sigma_{|\Delta} \in H^0(\Delta, \mathcal{O}_\Delta(R^1 f_* \mathcal{O}_{X|\Delta})) = H^0(\Delta, \mathcal{O}_\Delta)$ has no zero points. Hence we infer that σ has finitely many zero points on V. This implies that the support of $\mathcal{T}$ is a finite set. Q. E. D.

LEMMA 1. *If* (X, V, f) *satisfies* (b) *and* (c), *then the following equality and inequalities hold:*

(i) $p_g(X) = p_g(V)$,

(ii) $p(X) \leqq q(V) + p_g(V)$,

(iii) $1 + q(V) \leqq q(X) \leqq 1 + q(V) - T(X)$, $T(X) \leqq 0$.

PROOF. First we have $R^s f_* \mathcal{O}_X = 0$ for $s \geqq 2$, since every fibre is of dimension 1. This implies $E_t^{r,s} = 0$ for $s \geqq 2$ and $t \geqq 2$. Also we have $E_t^{r,0} = 0$ for $r \geqq 3$ and $t \geqq 2$, since $\dim V = 2$.

(i) By Proposition 1, we have $H^2(V, R^1 f_* \mathcal{O}_X) \cong H^2(V, \mathcal{O}_V)$. Hence

$$p_g(X) = \dim E_3^{2,1} = \dim E_2^{2,1} = \dim H^2(V, R^1 f_* \mathcal{O}_X)$$

$$= \dim H^2(V, \mathcal{O}_V) = p_g(V)\,.$$

(ii) By Proposition 1, we have $H^1(V, \mathcal{O}_V) \to H^1(V, R^1 f_* \mathcal{O}_X) \to 0$. This implies $\dim E_2^{1,1} \leqq q(V)$. Hence

$$p(X) = \dim E_3^{2,0} + \dim E_3^{1,1} \leqq \dim E_2^{2,0} + \dim E_2^{1,1}$$

$$\leqq p_g(V) + q(V)\,.$$

(iii) By (c) and the Riemann-Roch-Hirzebruch formula, we have

$$1 + q(V) \leqq q(X) = 1 + p(X) - p_g(X) - T(X) \leqq 1 + q(V) - T(X)\,.$$

Note that this implies $T(X) \leqq 0$, q. e. d.

LEMMA 2. *If* (X, V, f) *satisfies* (b)-(e), *then* $R^1 f_* \mathcal{O}_X \cong \mathcal{O}_V$ *and* $q(X) = 1$.

PROOF. By (b)-(e) and Lemma 1 (iii), we have $q(X) = 1$. By (e),

$$1 = q(X) = \dim E_2^{1,0} + \dim E_3^{0,1} = \dim E_2^{0,1}\,.$$

By (*) and (e), we have the exact sequence

$$0 \longrightarrow H^0(V, \mathcal{O}_V) \longrightarrow H^0(V, R^1 f_* \mathcal{O}_X) \longrightarrow H^0(V, \mathcal{T}) \longrightarrow 0\,.$$

It follows that $\dim H^0(V, \mathcal{T}) = 0$, therefore $\mathcal{T} = 0$, since the support of $\mathcal{T}$ is a set of finite points. Consequently we obtain $R^1 f_* \mathcal{O}_X \cong \mathcal{O}_V$, q. e. d.

LEMMA 3. *If* (X, V, f) *satisfies* (b) *and* (f), *then*

(i) $\chi(X) = 0$,

(ii) *for each* j, $b_2(F_j) = 1$ *and* $b_1(F_j) = 2$, *and* F_j *is irreducible.*

PROOF. For each j, we have $\chi(F_j) \geqq 0$, since $b_2(F_j) \geqq 1$, $b_1(F_j) \leqq 2$ and $b_0(F_j) = 1$. Hence $\chi(X) = \sum_{j=1}^{\rho} \chi(F_j) \geqq 0$. Therefore (f) implies $\chi(F_j) = 0$ for each j.

Thus we obtain $b_2(F_j)=1$ and $b_1(F_j)=2$, q. e. d.

LEMMA 4. *If* (X, V, f) *satisfies* (b)-(f), *then each* F_j *is a non-singular elliptic curve.*

PROOF. Letting $\mathfrak{m}$ be the maximal ideal sheaf of the point $a_j \in V$, we have the exact sequence of sheaves over X such that

$$0 \longrightarrow \mathfrak{m}\mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X \longrightarrow 0 .$$

Then, for any open set $U \ni a_j$ in V, there is the exact sequence

$$\cdots \to H^1(f^{-1}(U), \mathcal{O}_X) \to H^1(f^{-1}(U), \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X) \to H^2(f^{-1}(U), \mathfrak{m}\mathcal{O}_X) \to \cdots$$

and hence the exact sequence

$$\cdots \longrightarrow R^1 f_* \mathcal{O}_{X, a_j} \longrightarrow H^1(F_j, \mathcal{O}_{F_j}) \longrightarrow R^2 f_* \mathfrak{m}\mathcal{O}_{X, a_j} \longrightarrow \cdots ,$$

since $H^1(f^{-1}(U), \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X)=H^1(F_j, \mathcal{O}_{F_j})$, where, by definition, $\mathcal{O}_{F_j}=\mathcal{O}_X/\mathfrak{m}\mathcal{O}_X|_{F_j}$. Since $\dim F_j=1$, $R^2 f_* \mathfrak{m}\mathcal{O}_X=0$. Thus we obtain the exact sequence

$$R^1 f_* \mathcal{O}_{X, a_j} \longrightarrow H^1(F_j, \mathcal{O}_{F_j}) \longrightarrow 0 .$$

Since $R^1 f_* \mathcal{O}_{X, a_j} \cong \mathcal{O}_{V, a_j}$ by Lemma 2, we have $\dim H^1(F_j, \mathcal{O}_{F_j}) \leqq 1$. Indicating by $\mathcal{N}$ the nilradical of $\mathcal{O}_{F_j}$, we have the exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{O}_{F_j} \longrightarrow \mathcal{O}_{F_j, \mathrm{red}} \longrightarrow 0 .$$

It follows that

$$H^1(F_j, \mathcal{O}_{F_j}) \longrightarrow H^1(F_j, \mathcal{O}_{F_j, \mathrm{red}}) \longrightarrow H^2(F_j, \mathcal{N})=0$$

is exact. Hence $\dim H^1(F_j, \mathcal{O}_{F_j, \mathrm{red}}) \leqq 1$. If $\dim H^1(F_j, \mathcal{O}_{F_j, \mathrm{red}})=0$, then F_j is isomorphic to a non-singular rational curve. But this contradicts the equality $b_1(F_j)=2$ in Lemma 3. Thus we have $\dim H^1(F_j, \mathcal{O}_{F_j, \mathrm{red}})=1$ and F_j has at most one singular point. If F_j has a singular point, then F_j is a rational curve with either a cusp or an ordinary double point. If F_j has a cusp, then $b_1(F_j)=0$. If F_j has an ordinary double point, then $b_1(F_j)=1$. Thus we conclude that F_j is an elliptic curve, q. e. d.

§ 2. Preliminaries II.

In what follows except in § 4, we consider $(X, V, f)_a$. In this section we shall show that $(X, V, f)_a$ satisfies the five conditions (b)-(f) in § 1.

LEMMA 5. *For any subvariety* Y *of* X *with* $\dim Y=r \geqq 1$, *we have* $\dim f(Y) = \dim Y-1$.

PROOF. Since $b_2(X)=b_4(X)=0$, the cycle determined by Y is homologous to zero (modulo torsion), while, if $\dim f(Y)=\dim Y$, $f(Y)$ determines a real $2r$-

dimensional cycle which is not homologous to zero, since V is a projective algebraic surface. This is a contradiction. Hence we get $\dim f(Y)<\dim Y$. Since every fibre of f is of dimension 1, we have $\dim f(Y)=\dim Y-1$, q. e. d.

LEMMA 6. *Condition* (b) *holds.*

PROOF. We use the notation defined in § 1. Denote by S_i the set of all singular points of Σ_i. Since $\dim f(\Sigma_i\cap\Sigma_j)=0$ for $i\neq j$ and $\dim f(S_i)=0$ for any i by Lemma 5, the set of all singular points of Σ is contained in the union of finitely many fibres of f. Thus the fibre $F_c=f^{-1}(c)$ over a general point c of $\bigcup_{i=1}^{r} C_i$ is a non-singular curve by a theorem of Bertini. Let $\varDelta$ be a general hyperplane section of V. Then $S=f^{-1}(\varDelta)$ is a non-singular elliptic surface over $\varDelta$. We can assume that $c\in\varDelta$. Then F_c is a *non-singular* curve which appears as a fibre in a non-singular elliptic surface. By the classsification of singular fibres of elliptic surfaces [**5**], we conclude that F_c is an elliptic curve. Hence we infer that all but a finite number of the fibres are elliptic curves, q. e. d.

LEMMA 7. *Condition* (c) *holds. In particular,* $q(V)=0$.

PROOF. Since all the fibres are connected,

$$\pi_1(X) \longrightarrow \pi_1(V) \longrightarrow \{1\}$$

is exact. This implies that the sequence

$$H_1(X, \boldsymbol{Z}) \longrightarrow H_1(V, \boldsymbol{Z}) \longrightarrow 0$$

is exact. Hence $b_1(V)\leqq 1$ by (a). Since $b_1(V)$ is even, we have $b_1(V)=0$ and therefore $q(V)=0$.

Now it is sufficient to show that $q(X)\geqq 1$. We have the exact sequence of sheaves

$$0 \longrightarrow \boldsymbol{C} \xrightarrow{i} \mathcal{O}_X \xrightarrow{d} d\mathcal{O}_X \longrightarrow 0\,,$$

and the corresponding exact sequence

$$0 \longrightarrow H^0(X, d\mathcal{O}_X) \longrightarrow H^1(X, \boldsymbol{C}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \cdots.$$

Hence it is sufficient to show $H^0(X, d\mathcal{O}_X)=0$. Our method of proof is the same as that of Theorem 2 in [**6**]. We assume $\dim H^0(X, d\mathcal{O}_X)>0$ and derive a contradiction. Let φ be a non-zero element of $H^0(X, d\mathcal{O}_X)$. Since φ is d-closed, φ corresponds to an element of $H^1(X, \boldsymbol{C})$ by the de Rham theorem. Now we shall show that φ and its complex conjugate $\bar{\varphi}$ are cohomologically independent over $\boldsymbol{C}$. Assume that there exist complex numbers a, b such that

$$a\varphi+b\bar{\varphi}=d\mu$$

for some continuously differentiable function μ on X. Locally φ can be expressed as $\varphi=d\lambda$, where λ is a holomorphic function on a certain open set in X. Hence we have

$$\mu=a\lambda+b\bar{\lambda}+\text{constant}.$$

This shows that μ satisfies the mean value theorem. Hence μ is reduced to a constant and therefore

$$a\varphi+b\bar{\varphi}=0.$$

Consequently $a=b=0$. Hence φ and $\bar{\varphi}$ are cohomologically independent. This implies $\dim H^1(X, \boldsymbol{C})\geqq 2$. This contradicts (a). Hence $H^0(X, d\mathcal{O}_X)=0$, q. e. d.

LEMMA 8. *Condition* (d) *holds.*

PROOF. This is clear by (a).

LEMMA 9. *Condition* (f) *holds.*

PROOF. This follows from the following;

$$\chi(X)=2-2b_1(X)+2b_2(X)-b_3(X)=-b_3(X)\leqq 0 \quad (\text{by (a)}).$$

LEMMA 10. *Condition* (e) *holds.*

PROOF. It is shown in Lemma 7 that $q(V)=0$. By Lemmas 6, 7 and Lemma 1, we have $p_g(X)=p_g(V)$. By the Serre duality, we have $H^3(X, \mathcal{O}_X)\cong H^0(X, \Omega^3)$, where Ω^3 denotes the sheaf of germs of holomorphic 3-forms on X. Hence it is sufficient to show that $H^0(X, \Omega^3)=0$. Let φ be an element of $H^0(X, \Omega^3)$. The d-closed form φ corresponds to an element of $H^3(X, \boldsymbol{C})$ by the de Rham theorem. By Lemmas 6, 9 and Lemma 3 (i), we get $\chi(X)=0$. Hence $b_3(X)=-\chi(X)=0$ by (a). Hence there exists a differentiable 2-form ψ on X such that $d\psi=\varphi$. We have

$$0\leqq\int_X \varphi\wedge\bar{\varphi}=\int_X d\psi\wedge d\bar{\psi}=\int_X d(\psi\wedge d\bar{\psi})=0.$$

This implies $\varphi=0$. Therefore $H^0(X, \Omega^3)=0$, q. e. d.

An element of $H^1(X, \mathcal{O}_X^*)$ is called a *flat* line bundle if it is in the image of $H^1(X, \boldsymbol{C}^*)$ under the natural injection $j: \boldsymbol{C}^*\to\mathcal{O}_X^*$.

LEMMA 11. *Any line bundle on X is flat.*

PROOF. We already know that (b)-(e) hold. Hence $p(X)=0$ and $q(X)=1$ by Lemmas 1 and 2. In the proof of Lemma 7, we see that $i_*: H^1(X, \boldsymbol{C})\to H^1(X, \mathcal{O}_X)$ is an injection. Hence by (a) and $q(X)=1$, i_* is an isomorphism. Now the lemma follows from the following commutative diagram;

$$\begin{array}{ccccccccc}
H^1(X, \boldsymbol{Z}) & \longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{O}_X^*) & \longrightarrow & H^2(X, \boldsymbol{Z}) & \longrightarrow & 0 \\
\| & & \uparrow i_* & & \uparrow j_* & & \| & & \\
H^1(X, \boldsymbol{Z}) & \longrightarrow & H^1(X, \boldsymbol{C}) & \longrightarrow & H^1(X, \boldsymbol{C}^*) & \longrightarrow & H^2(X, \boldsymbol{Z}) & \longrightarrow & 0\,.
\end{array}$$

PROPOSITION 2. *Each fibre of f is a non-singular elliptic curve.*

PROOF. Since we have proved that (b)-(f) hold, the proposition follows from Lemma 4.

§ 3. Construction of $\Phi: X\to \boldsymbol{C}^N-\{0\}/\langle\alpha\rangle$.

Let H be a very ample line bundle on V. We represent H as a cocycle $\{h_{ij}\}\in Z^1(\mathcal{U}, \mathcal{O}_V^*)$, where $\mathcal{U}=\bigcup_i U_i$ is a finite open covering of V such that H is trivial on each U_i. Let $\varphi^{(1)}, \cdots, \varphi^{(N)}$ be a basis of $H^0(V, \mathcal{O}_V(H))$, where $N\geqq 3$. By Lemma 11, the line bundle f^*H over X has a flat representation corresponding to a group representation $\tilde{\rho}$ of $\pi_1(X)$ into $\boldsymbol{C}^*$. Clearly $\tilde{\rho}$ factors into the canonical surjection $\pi_1(X)\to H_1(X, \boldsymbol{Z})$ and a representation ρ of $H_1(X, \boldsymbol{Z})$ into $\boldsymbol{C}^*$. Let γ be a Betti base of $H_1(X, \boldsymbol{Z})$ and put $\alpha=\rho(\gamma)\in \boldsymbol{C}^*$. Replacing H by its suitable tensor product $H^{\otimes k}$ if necessary, we may assume that $\rho(\tau)=1$ for each torsion element τ of $H_1(X, \boldsymbol{Z})$. Choosing a suitable finite open covering $\mathcal{V}=\{U_{i\lambda}\}$ of X such that $f(U_{i\lambda})=U_i$ for all i and λ, we write f^*H in the form

$$f^*h_{ij}=\xi_{i\lambda}^{-1}\cdot\alpha^{m_{i\lambda,j\mu}}\cdot\xi_{j\mu} \qquad \text{on} \quad U_{i\lambda}\cap U_{j\mu},$$

where $\{\xi_{i\lambda}\}\in C^0(\mathcal{V}, \mathcal{O}_X^*)$ and $\{m_{i\lambda,j\mu}\}\in Z^1(\mathcal{V}, \boldsymbol{Z})$.

LEMMA 12. $|\alpha|\neq 1$.

PROOF. Let $\psi=\{\psi_{i\lambda}\}$ be an element of $H^0(X, \mathcal{O}_X(f^*H))$. If $|\alpha|=1$, we have

$$|\xi_{i\lambda}\psi_{i\lambda}|=|\xi_{j\mu}\psi_{j\mu}|=\cdots \qquad \text{on } X.$$

Hence $\{\xi_{i\lambda}\psi_{i\lambda}\}$ is reduced to a constant on X. This implies $\dim H^0(X, \mathcal{O}_X(f^*H))\leqq 1$. This contradicts that $\dim H^0(X, \mathcal{O}_X(f^*H))=\dim H^0(V, \mathcal{O}_V(H))\geqq 3$, q. e. d.

Denote by the same letter α the linear transformation of $\boldsymbol{C}^N$ defined by

$$(z_1, \cdots, z_N)\longmapsto(\alpha z_1, \cdots, \alpha z_N).$$

Let $\langle\alpha\rangle$ be the infinite cyclic group generated by α. Then the quotient space $\boldsymbol{C}^N-\{0\}/\langle\alpha\rangle$ is a compact complex manifold.

Now we construct a holomorphic mapping $\Phi: X\to\boldsymbol{C}^N-\{0\}/\langle\alpha\rangle$. We represent the sections $\varphi^{(l)}$ as follows:

$$\varphi_i^{(l)}(v)=h_{ij}(v)\varphi_j^{(l)}(v) \qquad \text{for} \quad v\in U_i\cap U_j \ (l=1, 2, \cdots, N).$$

Define Φ to be the holomorphic mapping

$$\Phi: x\longmapsto(\xi_{i\lambda}(x)\varphi_i^{(1)}(f(x)), \cdots, \xi_{i\lambda}(x)\varphi_i^{(N)}(f(x)))$$

for $x\in U_{i\lambda}$. Note that $\xi_{i\lambda}(x)$ never vanishes and that $\varphi_i^{(1)}(f(x)), \cdots, \varphi_i^{(N)}(f(x))$

never vanish simultaneously. If $x\in U_{i\lambda}\cap U_{j\mu}$, then, by the equation

$$\varphi_i^{(l)}(f(x))=\xi_{i\lambda}^{-1}(x)\cdot\alpha^{m_{i\lambda,j\mu}}\cdot\xi_{j\mu}(x)\cdot\varphi_j^{(l)}(f(x)),$$

$(\xi_{i\lambda}(x)\varphi_i^{(1)}(f(x)),\cdots,\xi_{i\lambda}(x)\varphi_i^{(N)}(f(x)))$ and $(\xi_{j\mu}(x)\varphi_j^{(1)}(f(x)),\cdots,\xi_{j\mu}(x)\varphi_j^{(N)}(f(x)))$ define the same point on $\boldsymbol{C}^N-\{0\}/\langle\alpha\rangle$. Thus Φ is a well-defined holomorphic mapping. Let $\tilde{p}:\boldsymbol{C}^N-\{0\}\to\boldsymbol{C}^N-\{0\}/\langle\alpha\rangle$ be the canonical projection and $p:\boldsymbol{C}^N-\{0\}\to\boldsymbol{P}^{N-1}$ the Hopf fibering. Then p induces $p_0:\boldsymbol{C}^N-\{0\}/\langle\alpha\rangle\to\boldsymbol{P}^{N-1}$ such that $p_0\circ\tilde{p}=p$. By the definition of Φ we have the following commutative diagram;

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & \boldsymbol{C}^N-\{0\}/\langle\alpha\rangle \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_0} \\ V & \xrightarrow{\varphi} & \boldsymbol{P}^{N-1} \end{array}$$

where φ is the embedding of V into $\boldsymbol{P}^{N-1}$ defined by the complete linear system of H. Put $X'=p_0^{-1}(\varphi(V))$. Then X' is an elliptic fibre bundle over $\varphi(V)$. It is clear that the algebraic dimension $a(X')$ of X' is equal to 2. We infer that $\Phi(X)=X'$. In fact, if $\dim\Phi(X)=2$, then $\Phi(X)$ would be a branched covering of $\varphi(V)$ and this would imply $a(X')=3$. This is a contradiction. Hence $\dim\Phi(X)=3$ and consequently $\Phi(X)=X'$. Since any fibres of f and p_0 are both non-singular elliptic curves, Φ is an unramified covering on each fibre. Hence we obtain the following

PROPOSITION 3. *The holomorphic mapping $\Phi: X\to\boldsymbol{C}^N-\{0\}/\langle\alpha\rangle$ is a branched coverig on its image $\Phi(X)$ and its singular locus coincides with the set of points where the Jacobian matrix of f does not have the maximal rank. Moreover Φ satisfies the following commutative diagram;*

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & \boldsymbol{C}^N-\{0\}/\langle\alpha\rangle \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p_0} \\ V & \xrightarrow{\varphi} & \boldsymbol{P}^{N-1}. \end{array}$$

§4. A Theorem.

Let X be any (reduced Hausdorff) complex space. Assume that X admits a holomorphic automorphism g with a unique fixed point $0\in X$. The automorphism g is called a *contraction* to 0 if g has the following two properties;

(i) $\lim_{\nu\to+\infty}g^{\nu}(x)=0$ *for any point* $x\in X$,

(ii) *for any small neighborhood U of 0 there exists a positive integer ν_0 such that $g^{\nu}(U)\subset U$ for all $\nu\geqq\nu_0$.*

In this section we shall prove the following

THEOREM 1.[3] *If a complex space X admits a contracting holomorphic automorphism g such that $g(0)=0$, then X is holomorphically isomorphic to an affine algebraic set. If, moreover, X is non-singular at* 0 *then $X\cong \boldsymbol{C}^m$ (m=the dimension of X at* 0).

To prove the theorem, we need several lemmas. Denote by $\mathfrak{m}_0$ the maximal ideal of $\mathcal{O}_{X,0}$, which indicates the stalk of $\mathcal{O}_X$ at 0. Put $n=\dim \mathfrak{m}_0/\mathfrak{m}_0^2$.

LEMMA 13. *X can be embedded in $\boldsymbol{C}^n$ as an analytic subset which is invariant under a contracting holomorphic automorphism $\tilde{g}$ of $\boldsymbol{C}^n$ having the form* (1) *below.*

PROOF. Let α_i ($i=1, 2, \cdots, n$) be the eigenvalues of the linear transformation $g^*: \mathfrak{m}_0/\mathfrak{m}_0^2\to\mathfrak{m}_0/\mathfrak{m}_0^2$ induced by g. We shall show that $0<|\alpha_i|<1$ for all i. Choose a small neighborhood U of 0 in X such that $g^\nu(U)\subset U$ for $\nu\geqq\nu_0$, where ν_0 is a fixed positive integer. We can assume without loss of generality that there exists a holomorphic embedding j of U into the unit ball B in $\boldsymbol{C}^n$ such that the image $N=j(U)$ is an analytic subset of B and $j(0)=0$. Put $f_\nu=j\circ g^\nu\circ j^{-1}$ for $\nu\geqq\nu_0$. Then each f_ν is a holomorphic mapping of N into itself. Since the sequence $\{f_\nu\}$ is uniformly bounded, there exists a uniformly bounded sequence $\{\tilde{f}_\nu\}$ of holomorphic functions on a certain polydisk $\Delta\ni 0$ in B such that $\tilde{f}_{\nu|N\cap\Delta}=f_\nu$ ([2] pp. 290-292). We can choose a subsequence $\{\tilde{f}_{\nu_j}\}$ of $\{\tilde{f}_\nu\}$ which converges uniformly to a holomorphic function $\tilde{f}$ on a certain polydisk Δ_0 of 0. The sequence of Jacobian matrices $\{d\tilde{f}_{\nu_j}(0)\}$ also converges to $d\tilde{f}(0)$. Since the eigenvalues of $d\tilde{f}_{\nu_j}(0)$ are equal to $\alpha_1^{\nu_j}, \cdots, \alpha_n^{\nu_j}$, it follows that $0<|\alpha_i|\leqq 1$ for all i. Assume that $|\alpha_i|=1$ for some i. Then $\operatorname{rank} d\tilde{f}(0)>0$. Since g is a contraction to $0\in X$ and $j(0)=0$, $\tilde{f}_{|N\cap\Delta_0}=0$. This implies that the point $0\in X$ is embedded in a non-singular manifold of dimension $n-\operatorname{rank} df(0)$ which is less than n. This is a contradiction and we obtain $0<|\alpha_i|<1$ for all i. Put $g_0=j\circ g\circ j^{-1}$. Then g_0 is defined in a neighborhood of $0\in\boldsymbol{C}^n$. Then, by L. Reich [11, 12], there exists a local coordinate transformation T at 0 such that $T(0)=0$ and $\tilde{g}=T^{-1}\circ g_0\circ T$ has the following form:

$$
\begin{aligned}
&z_1'=\alpha_1 z_1\\
&z_2'=z_1+\alpha_2 z_2\\
&\quad\cdots\cdots\\
&z_{r_1}'=z_{r_1-1}+\alpha_{r_1}z_{r_1}\\
&z_{r_1+1}'=\alpha_{r_1+1}z_{r_1+1}+P_{r_1+1}(z_1, \cdots, z_{r_1})
\end{aligned}
\tag{1}
$$

3) This result was announced in [4], where the condition (ii) was forgotten. We can replace (ii) by the weaker condition (ii)′ to get the same result: (ii)′ *there exists a neighborhood U of* 0 *such that U is isomorphic to an analytic subset of an open set of $\boldsymbol{C}^k$ for some k and such that $g^\nu(U)\subset U$ for some $\nu>0$.* But in our case (ii) is sufficient.

$$\cdots\cdots$$
$$z'_{r_1+r_2}=z_{r_1+r_2-1}+\alpha_{r_1+r_2}z_{r_1+r_2}+P_{r_1+r_2}(z_1, \cdots, z_{r_1})$$
$$z'_{r_1+r_2+1}=\alpha_{r_1+r_2+1}z_{r_1+r_2+1}+P_{r_1+r_2+1}(z_1, \cdots, z_{r_1+r_2})$$
$$\cdots\cdots$$
$$z'_n=z_{n-1}+\alpha_n z_n+P_n(z_1, \cdots, z_{r_1+\cdots+r_{\kappa-1}}),$$

where $1>|\alpha_1|\geqq|\alpha_2|\geqq\cdots\geqq|\alpha_n|>0$, κ is the number of the Jordan block of the linear part of g_0, and P_r $(r(\sigma)<r\leqq r(\sigma+1)$, $r(\sigma)=r_1+\cdots+r_\sigma)$ is a finite sum of monomials $z_1^{m_1}\cdots z_{r(\sigma)}^{m_{r(\sigma)}}$ which satisfy $\alpha_r=\alpha_1^{m_1}\cdots\alpha_{r(\sigma)}^{m_{r(\sigma)}}$, $m_1+\cdots+m_{r(\sigma)}\geqq 2$ and all $m_j>0$.

Note that $\tilde{g}$ is a contracting holomorphic automorphism of $\boldsymbol{C}^n$ such that $\tilde{g}(0)=0$. Let U_1 be a neighborhood of $0\in X$ on which $j_0=T^{-1}\circ j$ is defined. Let $U_0(\subset U_1)$ be a neighborhood of $0\in X$ such that $g^\nu(U_0)\subset U_1$ for all $\nu\geqq 0$. Then $\tilde{g}^\nu\circ j_0=j_0\circ g^\nu$ on U_0 for all $\nu\geqq 0$. Now we define a holomorphic mapping J of X into $\boldsymbol{C}^n$ by

$$J(x)=\tilde{g}^{-\nu}\circ j_0\circ g^\nu(x) \qquad (x\in X),$$

where $g^\nu(x)\in U_0$. If $g^\mu(x)\in U_0$ $(\nu\geqq\mu)$, then $\tilde{g}^{-\nu}\circ j_0\circ g^\nu(x)=\tilde{g}^{-\nu}\circ j_0\circ g^{\nu-\mu}\circ g^\mu(x)=\tilde{g}^{-\nu}\circ\tilde{g}^{\nu-\mu}\circ j_0\circ g^\mu(x)=\tilde{g}^{-\mu}\circ j_0\circ g^\mu(x)$. Hence J is well-defined. It is easy to check that J is a holomorphic embedding of X and that $J(X)$ is a $\tilde{g}$-invariant *analytic* subset of $\boldsymbol{C}^n$, q. e. d.

It X is non-singular at 0, then $n=m$. Hence $J(X)=\boldsymbol{C}^m$. This proves the latter statement of the theorem.

From now on we identify X with $J(X)$. Since the number of the irreducible branches of X at 0 is finite, X has a finite number of irreducible components X_j $(j=1, 2, \cdots)$. Hence there exists a positive integer l such that $\tilde{g}^l$ acts on each X_j as a contracting automorphism which has the similar form to (1). Therefore we may assume that X is irreducible. For an analytic subset Z in $\boldsymbol{C}^n$, we define $\dim Z$ to be the maximum of the dimensions of irreducible components of Z.

LEMMA 14. *Let Z be a $\tilde{g}$-invariant analytic subset in $\boldsymbol{C}^n$ such that $Z\supset X$ and* $\dim Z>\dim X$. *Then there exists a non-constant holomorphic function f on Z such that $\tilde{g}^*f=\alpha f$ $(0<|\alpha|<1)$ and $f_{|X}=0$.*

PROOF. It is clear that both Z and X contains the origin $0\in\boldsymbol{C}^n$. In view of (1) we can choose a small relatively compact neighborhood D of 0 in Z such that $\tilde{g}(\bar{D})\subset D$, where $\bar{D}$ denotes the closure of D in Z. Let $\mathscr{B}$ be a vector space of holomorphic functions defined by

$$\mathscr{B}=\left\{f:\begin{array}{l} f \text{ is a bounded holomorphic function} \\ \text{on } D \text{ such that } f_{|X\cap D}=0\end{array}\right\}.$$

We define the norm $\|\ \|_D$ for $f\in\mathscr{B}$ by

$$\|f\|_D = \sup_{z \in D} |f(z)| .$$

Then $(\mathscr{B}, \| \ \|_D)$ is clearly a Banach space. The linear mapping $\tilde{g}^* : \mathscr{B} \to \mathscr{B}$ defined by $(\tilde{g}^* f)(z) = f(\tilde{g}(z))$ is a compact operator by Vitali's theorem. It is easy to see that $\|\tilde{g}^*\|_D \leqq 1$ and $\|\tilde{g}^* f\|_D = \|f\|_D$ implies $f=0$. Now we shall show that there exists a non-zero element $f_0 \in \mathscr{B}$ such that

$$\tilde{g}^* f_0 = \alpha f_0 \qquad (0 < |\alpha| < 1) .$$

Put

$$R(\lambda) = (I - \lambda \tilde{g}^*)^{-1} , \tag{2}$$

where I denotes the identity operator. Since $\tilde{g}^*$ is a compact operator, any spectrum except 0 is an eigenvalue ([**16**]). Hence if there is no such f_0, then $R(\lambda)$ is an entire function of λ on $\boldsymbol{C}$. This implies that the radius of the circle of convergence of the Taylor expansion of (2) is infinite, i. e., $\lim_{\nu \to +\infty} \sqrt[\nu]{\|\tilde{g}^{*\nu}\|_D} = 0$. This is equivalent to saying that for any $\varepsilon > 0$, there exists an integer ν_1 such that

$$\| \tilde{g}^{*\nu} \|_D < \varepsilon^\nu \tag{3}$$

for $\nu > \nu_1$. Let $\mathfrak{m}_{Z,0}$ denote the maximal ideal of $\mathcal{O}_{Z,0}$, ρ a positive integer such that there exists an element $h \in \mathscr{B}$ which is not contained in $\mathfrak{m}_{Z,0}^{\rho+1}$. Fix a positive number δ such that $\delta < |\alpha_i|^{\rho+1}$ for all i $(i=1, 2, \cdots, n)$. Then

$$\|\delta^{-\nu} \tilde{g}^{*\nu} h\|_D > \|h\|_D$$

for sufficiently large ν. But this contradicts (3). Hence there exists a non-zero element $f_0 \in \mathscr{B}$ such that $\tilde{g}^* f_0 = \alpha f_0$ $(0 < |\alpha| < 1)$. For every positive integer ν, we have

$$f_0(z) = \alpha^{-\nu} f_0(\tilde{g}^\nu(z)) \qquad (z \in D) . \tag{4}$$

Since $\alpha^{-\nu} \tilde{g}^{*\nu} f_0$ is defined on $\tilde{g}^{-\nu}(D)$ and $\bigcup_\nu \tilde{g}^{-\nu}(D) = Z$, it follows from (4) that f_0 can be continued analytically to a holomorphic function f on Z such that $\tilde{g}^* f = \alpha f$. It is clear that $f_{|X} = 0$, q. e. d.

Denote by $\|z\|$ the norm of the point $z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n$ defined by $\|z\| = |z_1| + \cdots + |z_n|$.

LEMMA 15. *Let Z be a $\tilde{g}$-invariant analytic subset of $\boldsymbol{C}^n$ and f a holomorphic function on Z which satisfies the equality*

$$\tilde{g}^* f = \alpha f \qquad (0 < |\alpha| < 1) . \tag{5}$$

Then f satisfies the following inequality:

$$|f(z)| \leqq M(1 + \|z\|)^N , \tag{6}$$

where M and N are positive constants which are independent of $z\in Z$.

PROOF. Let K be a closed small neighborhood of $0\in \boldsymbol{C}^n$ defined by $\|z\|\leqq\varepsilon$. First for a point $z\in Z-K$ we estimate by $\|z\|$ the minimum non-negative integer ν such that $\tilde{g}^\nu(z)\in K$. By (1), the r-th coordinate $(r(\sigma)<r\leqq r(\sigma+1))$ of the point $\tilde{g}^\nu(z)$ is given by

$$(\tilde{g}^\nu(z))_r=\alpha_r^\nu(z_r+Q_r(\nu, z_1, \cdots, z_{r(\sigma)}))\,,$$

where Q_r is a polynomial of ν, $z_1, \cdots, z_{r(\sigma)}$. Hence we get

$$\|\tilde{g}^\nu(z)\|\leqq\sum_r|\alpha_r|^\nu(|z_r|+|Q_r(\nu, z_1, \cdots, z_{r(\sigma)})|)\,.$$

Then it is easy to see that, for some positive constants A, B and β ($|\alpha_i|<\beta<1$ for all i), the following inequality holds:

$$\|\tilde{g}^\nu(z)\|\leqq A\beta^\nu(1+\|z\|)^B\,.$$

Let ν be the least integer such that $\nu>-(\log\beta)^{-1}\log(A(1+\|z\|)^B/\varepsilon)$, which is positive since we have chosen ε to be small enough. Then $\|\tilde{g}^\nu(z)\|\leqq\varepsilon$, therefore $\tilde{g}^\nu(z)\in K$. Then, by (5),

$$\begin{aligned}|f(z)|&=|\alpha^{-\nu}f(\tilde{g}^\nu(z))|\\&\leqq|\alpha|^{-\nu}\|f\|_K \qquad (\|f\|_K=\sup_{z\in K}|f(z)|)\\&\leqq|\alpha|^{(\log\beta)^{-1}\cdot\log(A(1+\|z\|)^B/\varepsilon)-1}\cdot\|f\|_K\\&=(A(1+\|z\|)^B/\varepsilon)^{(\log|\alpha|)/\log\beta}\cdot|\alpha^{-1}|\cdot\|f\|_K\,.\end{aligned}$$

Putting $N=B(\log|\alpha|)/\log\beta$ and $M=(A/\varepsilon)^{N/B}\cdot|\alpha^{-1}|\cdot\|f\|_K$, we get

$$|f(z)|\leqq M(1+\|z\|)^N\,, \qquad \text{q. e. d.}$$

W. Rudin [**13**] proved the following

THEOREM 2. *An analytic subset V of pure dimension k in $\boldsymbol{C}^n$ is algebraic if and only if V lies in some algebraic region of type (k, n).*

By [**13**], a set Ω in $\boldsymbol{C}^n$ will be called an *algebraic region of type* (k, n) if there are vector subspaces E, F in $\boldsymbol{C}^n$ and a positive real numbers A, B such that the following conditions hold: $\dim E=k$, $\dim F=n-k$, $\boldsymbol{C}^n=E\oplus F$ (direct sum) and Ω consists precisely of the points $z\in\boldsymbol{C}^n$ satisfying the inequality

$$\|z''\|\leqq A(1+\|z'\|)^B\,,$$

where $z=z'+z''$, $z'\in E$ and $z''\in F$.

LEMMA 16. *Let Z be a $\tilde{g}$-invariant pure k-dimensional affine algebraic subset of $\boldsymbol{C}^n$, f a holomorphic function on Z such that $\tilde{g}^*f=\alpha f$ $(0<|\alpha|<1)$. Then* *the zero locus* $Y=\{z\in Z: f(z)=0\}$ *is a* $\tilde{g}$*-invariant affine algebraic subset of* $\boldsymbol{C}^n$.

PROOF. It is sufficient to show that the graph

$$\Gamma_f=\{(z, w)\in Z\times \boldsymbol{C}: w=f(z)\}$$

is an affine algebraic subset of $\boldsymbol{C}^n\times\boldsymbol{C}=\boldsymbol{C}^{n+1}$. By Theorem 2, there exists an algebraic region of type (k, n) such that

$$Z\subset\{(z', z'')\in\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}: \|z''\|\leqq A(1+\|z'\|)^B\}\,, \tag{7}$$

where we can choose the subspace $\boldsymbol{C}^k$ so that there exists an algebraic branched covering $Z\to\boldsymbol{C}^k$. By (6), we have

$$\Gamma_f\subset\{(z, w)\in Z\times\boldsymbol{C}: |w|\leqq M(1+\|z\|)^N\}\subset\boldsymbol{C}^{n+1}\,.$$

Hence, by (7), for points $(z, w)\in\Gamma_f$,

$$\begin{aligned}|w|&\leqq M(1+\|z'\|+\|z''\|)^N\leqq M(1+\|z'\|+A(1+\|z'\|)^B)^N\\&\leqq M_1(1+\|z'\|)^{N_1}\,,\end{aligned}\tag{8}$$

where M_1 and N_1 are some positive constants. Thus combining (8) with (7), we get

$$\begin{aligned}\|(z'', w)\|&\leqq A(1+\|z'\|)^B+M_1(1+\|z'\|)^{N_1}\\&\leqq M_2(1+\|z'\|)^{N_2}\qquad(\text{for some } M_2 \text{ and } N_2)\,.\end{aligned}$$

Hence the graph Γ_f is contained in an algebraic region of type $(k, n+1)$ ($k=\dim Z$). Hence, by Theorem 2, Γ_f is an affine algebraic subset, since Γ_f is pure dimensional, q. e. d.

LEMMA 17. *Let Z be a $\tilde{g}$-invariant pure dimensional affine algebraic subset of $\boldsymbol{C}^n$ such that $Z\supset X$ and* $\dim Z>\dim X$. *Then there exists a $\tilde{g}$-invariant pure dimensional affine algebraic subset Y of $\boldsymbol{C}^n$ such that $Z\supset Y\supset X$ and* $\dim Z=\dim Y+1$. *If* $\dim Z=\dim X+1$, *then X is an affine algebraic subset of $\boldsymbol{C}^n$.*

PROOF. Let Z_0 be an irreducible component of Z such that $Z_0\supset X$. Put $V=\bigcup_{\nu\in\boldsymbol{Z}}\tilde{g}^\nu(Z_0)$. Then V is a $\tilde{g}$-invariant pure dimensional affine algebraic subset of $\boldsymbol{C}^n$ which consists of the irreducible components of Z. Applying Lemmas 14 and 16 to V, we get a $\tilde{g}$-invariant affine algebraic subset Y defined by a non-constant holomorphic function f on V. Now Y contains no irreducible components of V. In fact, if Y contains an irreducible component of V, then f vanishes identically on V. Hence Y is pure dimensional and $\dim Z=\dim Y+1$. The latter statement is clear, since X is an irreducible component of the affine algebraic subset Y, q. e. d.

PROOF OF THEOREM 1. Note that $\boldsymbol{C}^n$ itself is a $\tilde{g}$-invariant pure dimensional affine algebraic "subset" of $\boldsymbol{C}^n$. Hence, by Lemma 17, the theorem is easily proved by the induction on the codimension of X in C^n. Q. E. D.

§5. Complex structure of $(X, V, f)_a$.

Now consider $(X, V, f)_a$. By Proposition 3, the holomorphic mapping $\Phi: X \to C^N - \{0\}/\langle\alpha\rangle$ induces a non-trivial homomorphism

$$\Phi_*: \pi_1(X) \longrightarrow \pi_1(C^N - \{0\}/\langle\alpha\rangle) \cong Z.$$

Then $\Phi_*(\pi_1(X)) = \langle\alpha^m\rangle$ for a certain non-zero integer m. We can assume that $|\alpha^m| < 1$. Let W be the unramified covering manifold of X such that $\pi_1(W)$ is equal to the kernel of Φ_*. Then W is the infinite cyclic covering manifold of X. Lifting up Φ to W, we have a finite branched covering

$$\Psi: W \longrightarrow \Psi(W) \subset C^N - \{0\}.$$

Let g be a generator of $\pi_1(X)/\pi_1(W) \cong Z$ such that $\Phi_*(g) = \alpha^m$. Then g can be regarded as a holomorphic automorphism of W such that $W/\langle g\rangle = X$. Put $Y = p^{-1}(\varphi(V))$ and $\hat{Y} = Y \cup \{0\}$, where $\hat{Y}$ is an affine variety of dimension 3 which is non-singular at everywhere except 0.

THEOREM 3. *$\hat{W} = W \cup \{$one point$\}$ has a complex structure of an affine variety which admits an algebraic C^*-action.*

PROOF. Let $\lambda: \hat{Y}^* \to \hat{Y}$ be the normalization of $\hat{Y}$. Since $\hat{Y}$ has one irreducible branch at 0, λ is an isomorphism of $\hat{Y}^* - \{0^*\}$ onto $\hat{Y} - \{0\} = Y$, where $0^* = \lambda^{-1}(0)$ is a point. By λ we identify $\hat{Y}^* - \{0^*\}$ with Y. It is clear by the definition of Ψ that the image of the branch loci in Y can be extended to subvarieties of $\hat{Y}^*$. By a theorem of Grauert-Remmert [2] Satz 8 (see also Raynaud [10] Théorèm 5.4), there uniquely exist a normal complex space $\hat{W}$ and a finite branched covering $\hat{\Psi}: \hat{W} \to \hat{Y}^*$ such that $\hat{\Psi}_{|W} = \Psi$, where $\hat{W}$ is obtained by attaching analytic sets to W. Simple topological argument shows that $\hat{\Psi}^{-1}(0^*)$ consists of one point, which we denote by 0. By the same theorem of Grauert-Remmert, the automorphism g of W extends uniquely to a holomorphic automorphism of $\hat{W}$ which fixes 0. We denote this automorphism by the same letter g. It is easy to see that g is a contraction to 0. Hence, by Theorem 1, $\hat{W} = W \cup \{0\}$ is isomorphic to an affine variety. Let θ be the vector field on $\Phi(X)$ which induces the natural complex torus action on $\Phi(X)$. By the construction of Φ and the equation (0) in §1, we have the vector field $\tilde{\theta}$ on $X - S$ such that $\Phi_*\tilde{\theta} = \theta$, where S is the finite union of the fibres of f. Integrating this vector field, we obtain a complex torus action on $X - S$. Since S is of codimension 2, this action uniquely extends to the whole X. Hence W has a C^*-action σ which induces the natural algebraic C^*-action on Y. We embed $\hat{W}$ to C^n for some n so that 0 coincides with the origin of C^n. Let Γ

be the graph of $\sigma: \boldsymbol{C}^*\times W\to W$. Take a point $(t, w)\in\boldsymbol{C}^*\times W$. Then $z=\sigma(t, w)$ is a vector-valued holomorphic function defined on $\boldsymbol{C}^*\times W$ which is bounded on $D^*=\{(t, w)\in\boldsymbol{C}\times U: 0<|t|<\varepsilon,\ w\neq 0\}$, where U is an arbitrary relatively compact neighborhood of 0 in $\hat{W}$ and ε is a small positive number. Hence σ extends to a holomorphic function on $D=\{(t, w)\in\boldsymbol{C}\times U: |t|<\varepsilon\}$ such that $\sigma(t, 0)=0$ and $\sigma(0, w)=0$. Hence we infer that the closure $\bar{\Gamma}$ of Γ in $\boldsymbol{C}\times\hat{W}\times\hat{W}$ is an analytic subset such that $\bar{\Gamma}=\Gamma\cup(\boldsymbol{C}\times\{0\}\times\{0\})\cup(\{0\}\times\hat{W}\times\{0\})$. We define a holomorphic automorphism h of $\boldsymbol{C}\times\hat{W}\times\hat{W}$ as follows;

$$\begin{array}{ccc} h:\boldsymbol{C}\times\hat{W}\times\hat{W} & \longrightarrow & \boldsymbol{C}\times\hat{W}\times\hat{W} \\ \Cup & & \Cup \\ (t, w, z) & \longmapsto & ((1/2)t,\ \sigma(1/2, w),\ \sigma(1/4, z)). \end{array}$$

Then h is a contraction of $\boldsymbol{C}\times\hat{W}\times\hat{W}$ to $(0, 0, 0)$. Moreover h leaves $\bar{\Gamma}$ invariant. Hence $\bar{\Gamma}$ is an algebraic subvariety of $\boldsymbol{C}\times\hat{W}\times\hat{W}$ as the proof of Theorem 1 shows. Therefore the $\boldsymbol{C}^*$-action σ on W extends to an algebraic $\boldsymbol{C}^*$-action on $\hat{W}$. Q. E. D.

Let $\boldsymbol{C}^M-\{0\}/\langle\beta\rangle$ be a compact complex manifold defined by factoring $\boldsymbol{C}^M-\{0\}$ by the linear transformation β of $\boldsymbol{C}^M$;

$$\beta:(z_1, \cdots, z_M)\longrightarrow(\beta^{q_1}z_1, \cdots, \beta^{q_M}z_M),$$

where $0<|\beta|<1$ and all q_i's are positive integers.

PROPOSITION 4. *There exists a finite cyclic unramified covering manifold of X which is holomorphically isomorphic to a submanifold of $\boldsymbol{C}^M-\{0\}/\langle\beta\rangle$ for some M and β.*

PROOF. Since the $\boldsymbol{C}^*$-action on $\hat{W}$ is algebraic, by Proposition (1, 1, 3) in Orlik-Wagreich [**9**], there is an embedding $j: \hat{W}\to\boldsymbol{C}^M$ for some M and an algebraic $\boldsymbol{C}^*$-action $\tilde{\sigma}$ on $\boldsymbol{C}^M$ such that $j(\hat{W})$ is $\tilde{\sigma}$-invariant and $j^*\tilde{\sigma}=\sigma$, and moreover, by a suitable choice of coordinates on $\boldsymbol{C}^M$, we can write $\tilde{\sigma}$ as $\tilde{\sigma}(t, (z_1, \cdots, z_M))=(t^{q_1}z_1, \cdots, t^{q_M}z_M)$, where the q_i's are positive integers. For a suitable positive integer ν_0, g^{ν_0} fixes every component of the fibres of $f\circ\omega: W\to V$, where ω is the canonical covering map $W\to X=W/\langle g\rangle$. Then it is easy to see that the action of the group $\langle g^{\nu_0}\rangle$ is compatible with that of $\boldsymbol{C}^*$, i. e., there exists some β such that $\beta=g^{\nu_0}$. This implies that $W/\langle g^{\nu_0}\rangle$ is a submanifold of $\boldsymbol{C}^M-\{0\}/\langle\beta\rangle$. Hence $W/\langle g^{\nu_0}\rangle$ is the desired manifold. Q. E. D.

PROPOSITION 5. *V is holomorphically isomorphic to either a projective plane or a surface of general type with $b_2(V)=1$.*

POOOF. Let L be a line bundle on V. We denote by ρ_L the representation of $H_1(X, \boldsymbol{Z})$ into $\boldsymbol{C}^*$ corresponding to the flat line bundle f^*L on X. Put $\alpha_L=\rho_L(\gamma)$, where γ is a fixed Betti base of $H_1(X, \boldsymbol{Z})$.

Let H be an ample line bundle on V. Then, for any line bundle L on V, both $H_1=H^{\otimes m}$ and $H_2=H^{\otimes m}\otimes L$ are very ample if we choose m to be sufficiently large. By Proposition 3, we have, for some positive integer n, the following holomorphic mappings Φ_1 and Φ_2 corresponding to $H_1^{\otimes n}$ and $H_2^{\otimes n}$, respectively;

$$\begin{tikzcd} X \arrow[r, "\Phi_1"] \arrow[dr, "f_1"'] & \Phi_1(X) \arrow[d, "p_1"] \subset & \boldsymbol{C}^{N_1}-\{0\}/\langle\alpha_{H_1^{\otimes n}}\rangle \arrow[d, "p_1"] \\ & V \subset & \boldsymbol{P}^{N_1-1}, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\Phi_2"] \arrow[dr, "f_2"'] & \Phi_2(X) \arrow[d, "p_2"] \subset & \boldsymbol{C}^{N_2}-\{0\}/\langle\alpha_{H_2^{\otimes n}}\rangle \arrow[d, "p_2"] \\ & V \subset & \boldsymbol{P}^{N_2-1}. \end{tikzcd}$$

Since the fibre F_v on $v\in V$ is an unramified covering of both $\boldsymbol{C}^*/\langle\alpha_{H_1^{\otimes n}}\rangle$ and $\boldsymbol{C}^*/\langle\alpha_{H_2^{\otimes n}}\rangle$, there exist integers m_1, m_2 such that $\alpha_{H_1}^{nm_1}=\alpha_{H_2}^{nm_2}$. This implies that $f^*(H_1^{\otimes nm_1})=f^*(H_2^{\otimes nm_2})$ and, consequently, $L^{\otimes nm_2}=H^{\otimes nm(m_1-m_2)}$, since $f^*: H^1(V, \mathcal{O}_V^*)\to H^1(X, \mathcal{O}_X^*)$ is an injection. This shows that $b_2(V)=1$, since we know that $H^1(V, \mathcal{O}_V^*)\cong H^2(V, \boldsymbol{Z})$ by $q(V)=p_g(V)=0$. Therefore we conclude that V is a projective plane or a surface of general type by the classification theory of surfaces [7].

§ 6. Small deformations.

Denote by D_ε a polydisk $\{t=(t_1, \cdots, t_n): |t_i|<\varepsilon, i=1, 2, \cdots, n\}$ in $\boldsymbol{C}^n$ $(n\geqq 1)$. Let $\pi': \mathcal{X}\to D_\varepsilon$ $(\pi'^{-1}(t)=X_t)$ be a complex analytic family of small deformations of $X_0=X$. Then we have another complex analytic family $\pi: \mathcal{W}\to D_\varepsilon$ $(\pi^{-1}(t)=W_t)$ of small deformations of $W_0=W$ with the following commutative diagram;

$$\begin{tikzcd} \mathcal{W} \arrow[rr, "\tilde{\omega}"] \arrow[dr, "\pi"'] & & \mathcal{X} \arrow[dl, "\pi'"] \\ & D_\varepsilon & \end{tikzcd}$$

where $\tilde{\omega}: \mathcal{W}\to\mathcal{X}$ is the unramified infinite cyclic covering such that $\tilde{\omega}_{|W_0}=\omega$ is the canonical projection $W\to X$.

THEOREM 4. *There exists a complex fibre space* $\hat{\pi}: \hat{\mathcal{W}}\to D_\varepsilon$ *with the projection* $\hat{\pi}$ *which has the following properties:*

(1) $\hat{\mathcal{W}}=\mathcal{W}\cup\mathcal{S}$, *where* $\mathcal{S}$ *is an analytic subvariety of* $\hat{\mathcal{W}}$;

(2) $\hat{\pi}_{|\mathcal{W}}=\pi$;

(3) $\hat{\pi}^{-1}(t)\cap\mathcal{S}=\{$*one point*$\}$ *for any* $t\in D_\varepsilon$;

(4) $\hat{\pi}^{-1}(t)$ *is holomorphically isomorphic to an affine variety for any* $t\in D_\varepsilon$.

PROOF. Let $F': \mathcal{X}\to X\times D_\varepsilon$ be a differentiable trivialization of $\mathcal{X}$ and F:

$\mathcal{W} \to W \times D_\varepsilon$ the corresponding trivialization. Then there is the commutative diagram;

$$\begin{tikzcd}
\mathcal{W} \arrow[rr, "F"] \arrow[dr, "\pi"] \arrow[dd, "\tilde{\omega}"'] & & W \times D_\varepsilon \arrow[dl, "q"'] \arrow[dd, "\omega \times \mathrm{id}_{D_\varepsilon}"] \\
& D_\varepsilon & \\
\mathcal{X} \arrow[rr, "F'"'] \arrow[ur, "\pi'"] & & X \times D_\varepsilon \arrow[ul, "q'"]
\end{tikzcd},$$

where q and q' are the projections to the 2nd components. Let $\mathcal{G}$ be the automorphism of $\mathcal{W}$ such that $\mathcal{W}/\langle\mathcal{G}\rangle = \mathcal{X}$ and $\mathcal{G}_{|W} = g$. Put $g_t = \mathcal{G}_{|W_t}$ for $t \in D_\varepsilon$.

Now we embed $\hat{W}$ into $\boldsymbol{C}^M$ by the holomorphic mapping J which was constructed in §4. Let $(z_1, \cdots, z_M)$ be a standard system of coordinates on $\boldsymbol{C}^M$. By the definition of J, there exists a contracting holomorphic automorphism $\tilde{g}$ of $\boldsymbol{C}^M$ such that $\tilde{g}(0)=0$ and $\tilde{g}_{|J(W)} = J \circ g \circ J^{-1}$. Let $q_W : W \times D_\varepsilon \to W$ be the projection. We define a C^∞-function on $\mathcal{W}$ by

$$\theta(w) = -1 + \sum_{i=1}^{M} |z_i(q_W \circ F(w))|^2 + c \sum_{j=1}^{n} |t_j(\pi(w))|^2 \qquad (w \in \mathcal{W}),$$

where c is a positive number which will be defined later. Note that $-1 < \theta < +\infty$ on $\mathcal{W}$. Put

$$\mathcal{B} = \{w \in \mathcal{W} : \theta(w) = 0\},$$

$$\mathcal{W}^+ = \{w \in \mathcal{W} : \theta(w) > 0\},$$

$$\mathcal{W}^- = \{w \in \mathcal{W} : \theta(w) < 0\},$$

$$B = \mathcal{B} \cap W,$$

$$W^+ = \mathcal{W}^+ \cap W,$$

and

$$W^- = \mathcal{W}^- \cap W.$$

Choose a positive integer ν_0 such that $g^{\nu_0}(\overline{W^-}) \subset W^-$, where we indicate by $\bar{A}$ the topological closure of the set A in $\mathcal{W}$. Put $\mathcal{H} = \mathcal{G}^{\nu_0}$ and $h = g^{\nu_0}$. Let a be a negative number such that $-1 < a < r = \min\{\theta(w) : w \in h^2(B)\}$ and b a positive number. Put

$$\mathcal{K} = \{w \in \mathcal{W} : a < \theta(w) < b\},$$

and

$$K = \mathcal{K} \cap W.$$

Clearly K is defined independently to c. Fix c so that θ defines a positive definite Levi form on an open neighborhood $\mathcal{U}$ of $\bar{K}$ in $\mathcal{W}$. Since we consider only small deformations of X, replacing $\mathcal{X}$, if necessary, by $\pi'^{-1}(D_{\varepsilon'})$ for some ε' $(0<\varepsilon'<\varepsilon)$, we can assume the following;

$$\mathcal{K}\subset\mathcal{U}\,,$$

$$\mathcal{H}^{\nu}(\overline{\mathcal{W}}^{-})\subset\mathcal{H}^{\nu-1}(\mathcal{W}^{-})\qquad(\nu=1,2)\,,$$

$$\{w\in\mathcal{W}:\theta(w)\geqq b\}\subset\mathcal{W}^{+}\,,$$

and

$$\{w\in\mathcal{W}:\theta(w)\leqq a\}\subset\mathcal{H}^{2}(\mathcal{W}^{-})\,.$$

Put $D=D_{\varepsilon}$. Then $\pi:\mathcal{K}\to D$ is a $(1,1)$ convex-concave holomorphic mapping with exhaustion function θ (for the definition of the terminologies, see Ling [8]). Since $\mathcal{K}$ is non-singular and of dimension $n+3$, by Theorem $(\mathrm{I})_n$ [8], we have a unique $(n+1)$-normal *Stein space* $\hat{\mathcal{K}}$ with the following properties;

(i) *there exist a projection* $\hat{\pi}:\hat{\mathcal{K}}\to D$ *and a holomorphic homeomorphism* i *of* $\mathcal{K}$ *onto an open subset of* $\hat{\mathcal{K}}$ *which makes the diagram*

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, "i"] \arrow[dr, "\pi"'] & & \hat{\mathcal{K}} \arrow[dl, "\hat{\pi}"] \\
& D &
\end{tikzcd}$$

commutative,

(ii) *the projection* $\hat{\pi}$ *restricted to*

$$\hat{\mathcal{K}}-\{w\in i(\mathcal{K}):\theta(i^{-1}(w))>d\}$$

is proper for any d $(a<d<b)$.

Put

$$\hat{\mathcal{L}}_1=\hat{\mathcal{K}}-i(\overline{\mathcal{W}}^{+}\cap\mathcal{K})\,,$$

$$\hat{\mathcal{L}}_2=\hat{\mathcal{K}}-i(\mathcal{H}(\overline{\mathcal{W}}^{+})\cap\mathcal{K})\,,$$

$$\partial\hat{\mathcal{L}}_1=\text{the boundary of }\hat{\mathcal{L}}_1\text{ in }\hat{\mathcal{K}}=\mathcal{B}\,,$$

and

$$\partial\hat{\mathcal{L}}_2=\text{the boundary of }\hat{\mathcal{L}}_2\text{ in }\hat{\mathcal{K}}=\mathcal{H}(\mathcal{B})\,.$$

We claim that $\hat{\mathcal{L}}_1$ and $\hat{\mathcal{L}}_2$ are Stein spaces. In fact, we can find a C^{∞}[4] real valued function $\hat{\theta}_1$ on $\hat{\mathcal{L}}_1$ which agrees with $-1/(\theta\circ i^{-1})$ on the neighborhood of $\partial\hat{\mathcal{L}}_1$. Then $\hat{\pi}_{|\hat{\mathcal{L}}_1}:\hat{\mathcal{L}}_1\to D$ is a 1-convex holomorphic mapping with exhaustion function $\hat{\theta}_1$. Hence, by Proposition 3.6 and the proof of Main Theorem (p. 213) in [**15**], $\hat{\mathcal{L}}_1$ is holomorphically convex. Since $\hat{\mathcal{L}}_1$ contains no positive

4) See [8] for the definition.

dimensional compact subvarieties, $\hat{\mathcal{L}}_1$ is a Stein space. Similarly, we have a 1-convex holomorphic mapping $\hat{\pi}_{|\hat{\mathcal{L}}_2}:\hat{\mathcal{L}}_2\to D$ with exhaustion function $\hat{\theta}_2$ which agrees with $-1/(\theta\circ\mathcal{H}^{-1}\circ i^{-1})$ on the neighborhood of $\partial\hat{\mathcal{L}}_2$. Hence we infer as above that $\hat{\mathcal{L}}_2$ is a Stein space. Let $\mathcal{E}=\mathcal{H}(\mathcal{W}^-)-\mathcal{H}^2(\mathcal{W}^-)$ and $\mathcal{E}_0=\mathcal{H}(\mathcal{W}^-)-\mathcal{H}^2(\overline{\mathcal{W}}^-)$. Then $\mathcal{E}$ is a fundamental domain of $\mathcal{H}$ in $\mathcal{W}$ and $\mathcal{E}_0$ is the interior of $\mathcal{E}$. Put $\mathcal{H}_0=i\circ\mathcal{H}\circ i^{-1}$. Then $\mathcal{H}_0$ is defined on a neighborhood of $i(\mathcal{H}^{-1}(\bar{\mathcal{E}}))$. Put $\mathcal{N}_1=i(\mathcal{H}^{-1}(\mathcal{E}_0))$ and $\mathcal{N}_2=i(\mathcal{E}_0)$. Denote by $\tilde{\mathcal{L}}$ the fibre product $\hat{\mathcal{L}}_1\times_D\hat{\mathcal{L}}_2$ $(\subset\hat{\mathcal{L}}_1\times\hat{\mathcal{L}}_2)$ over D and $\tilde{\pi}:\tilde{\mathcal{L}}\to D$ the canonical projection induced by $\hat{\pi}$. Put $\mathcal{N}=(\mathcal{N}_1\times_D\hat{\mathcal{L}}_2)\cup(\hat{\mathcal{L}}_1\times_D\mathcal{N}_2)$. Let $\iota:\mathcal{N}\to\tilde{\mathcal{L}}$ be the natural inclusion and $\rho_i:\hat{\mathcal{L}}_1\times\hat{\mathcal{L}}_2\to\hat{\mathcal{L}}_i$ $(i=1,2)$ the canonical projection to the i-th component. Put $\tilde{\theta}=\iota^*((\rho_1^*\hat{\theta}_1+\rho_2^*\hat{\theta}_2)_{|\tilde{\mathcal{L}}})$. Then $\tilde{\pi}:\tilde{\mathcal{L}}\to D$ is a Stein completion of the fibre space $\tilde{\pi}_{|\mathcal{N}}:\mathcal{N}\to D$ with exhaustion function $\tilde{\theta}$. Take the $(n+1)$-normalization $\nu:\tilde{\mathcal{L}}^*\to\tilde{\mathcal{L}}$ of $\tilde{\mathcal{L}}$. Since $\tilde{\mathcal{L}}$ is non-singular in $\mathcal{N}$, $\mathcal{N}$ can be seen as an open subset of $\tilde{\mathcal{L}}^*$. Let Γ be the graph of $\mathcal{H}_{0|\mathcal{N}_1}$ in $\mathcal{N}$. By Theorem (4.4.5) [8], the ideal sheaf $\mathcal{J}$ of Γ on $\mathcal{N}$ extends uniquely to a coherent analytic sheaf $\tilde{\mathcal{J}}$ on $\tilde{\mathcal{L}}^*$ of which the $(n+1)$-th absolute gap sheaf agrees with $\tilde{\mathcal{J}}$. By Proposition (4.4.3) [8], we have

$$(**)\qquad \Gamma(\tilde{\mathcal{L}}^*,\tilde{\mathcal{J}})\xrightarrow[\sim]{r_1}\Gamma(\mathcal{N},\mathcal{J})\xrightarrow{j}\Gamma(\mathcal{N},\mathcal{O}_{\mathcal{N}})\xleftarrow[\sim]{r_2}\Gamma(\tilde{\mathcal{L}}^*,\mathcal{O}_{\tilde{\mathcal{L}}^*})$$

where r_i $(i=1,2)$ are the restrictions and j is the natural inclusion. Put $I=r_2^{-1}\circ j(\Gamma(\mathcal{N},\mathcal{J}))$ and

$$\tilde{\Gamma}^*=\{x\in\tilde{\mathcal{L}}:f(x)=0\ \text{for all}\ f\in I\}\,.$$

Since $\tilde{\mathcal{L}}^*$ is a Stein space, in view of (**), every stalk $\mathcal{J}_x$ $(x\in\mathcal{N})$ of $\mathcal{J}$ is generated over $\mathcal{O}_{\tilde{\mathcal{L}}^*,x}$ by finitely many holomorphic functions on $\tilde{\mathcal{L}}^*$ which vanish at least on Γ. Hence the analytic subset $\tilde{\Gamma}^*$ of $\tilde{\mathcal{L}}^*$ coincides with Γ in $\mathcal{N}$. Take the irreducible component $\tilde{\Gamma}_0^*$ of $\tilde{\Gamma}^*$ such that $\tilde{\Gamma}_0^*\supset\Gamma$. Put $\nu(\tilde{\Gamma}_0^*)=\tilde{\Gamma}$. Then $\tilde{\Gamma}$ is a subvariety of $\tilde{\mathcal{L}}$ which extends Γ.[5)] Let $\rho_i'=\rho_{i|\tilde{\Gamma}}$. Then $\rho_i':\tilde{\Gamma}\to\hat{\mathcal{L}}_i$ is the branched covering and ρ_i' maps $\tilde{\Gamma}\cap\mathcal{N}$ homeomorphically onto $\mathcal{N}_i$. Let A_i be the singular locus of ρ_i' in $\tilde{\Gamma}$. Since $\tilde{\Gamma}$ contains no positive dimensional compact subvarieties, A_i intersects each fibre of $\tilde{\pi}:\tilde{\mathcal{L}}\to D$ at most finitely many points. Hence we infer that $\mathcal{H}_{0|\mathcal{N}_1}:\mathcal{N}_1\to\mathcal{N}_2$ extends to a biholomorphic mapping $\hat{\mathcal{H}}:\hat{\mathcal{L}}_1-B_1\to\hat{\mathcal{L}}_2-B_2$, where $B_i=\rho_i'(A_1)\cup\rho_i'(A_2)$ $(i=1,2)$. Since $\dim B_i\leqq n$ and $\hat{\mathcal{L}}_i$ is $(n+1)$-normal, $\hat{\mathcal{H}}$ extends uniquely to a holomorphic mapping $\hat{\mathcal{L}}_1\to\hat{\mathcal{L}}_2$, which we denote by the same letter $\hat{\mathcal{H}}$. By the uniqueness of the $(n+1)$-normalization, $\hat{\mathcal{H}}$ is an isomorphism of $\hat{\mathcal{L}}_1$ onto $\hat{\mathcal{L}}_2$. It is clear that

5) By a similar method as in the proof of Theorem I_n in Siu [**14**], we can prove directly that $\tilde{\mathcal{J}}$ is an ideal sheaf whose zero locus extends Γ.

$\hat{\pi}_{|\hat{\mathcal{L}}_2} \circ \hat{\mathcal{H}} = \hat{\pi}_{|\hat{\mathcal{L}}_1}$.

Identify the point $w \in \mathcal{K}$ with $i(w) \in \hat{\mathcal{K}}$. Then we obtain a complex space $\hat{\mathcal{W}} = \mathcal{W}^+ \cup \hat{\mathcal{K}}$. Denote by the same symbol $\hat{\pi}$ the projection $\hat{\mathcal{W}} \to D$ which can be defined naturally by using $\pi: \mathcal{W} \to D$ and $\hat{\pi}: \mathcal{K} \to D$. We shall show that $\hat{\mathcal{W}}$ is the desired fibre space.

Note that

$$(***) \qquad i = \hat{\mathcal{H}} \circ i \circ \mathcal{H}^{-1}$$

on an open neighborhood of $\mathcal{H}^2(\overline{\mathcal{W}}^+) \cap \mathcal{H}(\overline{\mathcal{W}}^-)$ $(=\bar{\mathcal{E}})$. For any point $w \in \mathcal{H}(\mathcal{W}^-)$ there exists a unique integer $\mu \geqq 0$ such that $\mathcal{H}^{-\mu}(w) \in \mathcal{E}$. Now define a holomorphic mapping $\hat{i}: \mathcal{W} = \mathcal{H}^2(\mathcal{W}^+) \cup \mathcal{H}(\mathcal{W}^-) \to \hat{\mathcal{W}}$ as follows:

$$\hat{i}(w) = \begin{cases} w & \text{if } w \in \mathcal{H}^2(\mathcal{W}^+)\,, \\ \hat{\mathcal{H}}^{\mu} \circ i \circ \mathcal{H}^{-\mu}(w) \quad (\mathcal{H}^{-\mu}(w) \in \mathcal{E},\ \mu \geqq 0) & \text{if } w \in \mathcal{H}(\mathcal{W}^-)\,. \end{cases}$$

By (***), $\hat{i}$ is a well-defined holomorphic mapping. Moreover $\hat{i}$ is a local homeomorphism. We claim that $\hat{i}: \mathcal{W} \to \hat{\mathcal{W}}$ is injective. Put $\mathcal{E}_\nu = \mathcal{H}^\nu(\mathcal{E})$ and $\mathcal{D}_\nu = \hat{\mathcal{H}}^\nu(i(\mathcal{E}))$ $(\nu \geqq 0)$. Since $\mathcal{E}_\nu \cap \mathcal{E}_\mu = \emptyset$, $\mathcal{D}_\nu \cap \mathcal{D}_\mu = \emptyset$ $(\nu \neq \mu)$ and $\hat{i}$ maps injectively $\mathcal{E}_\nu$ onto $\mathcal{D}_\nu$, $\hat{i}$ is injective on $\mathcal{H}(\mathcal{W}^-)$. By the definition of $\hat{\mathcal{W}}$, $\hat{i}$ is injective on $\mathcal{H}(\overline{\mathcal{W}}^+)$. Since $\hat{i}(\mathcal{H}(\overline{\mathcal{W}}^+)) \cap \hat{i}(\mathcal{H}(\mathcal{W}^-)) = \emptyset$, it follows that $\hat{i}: \mathcal{W} \to \hat{\mathcal{W}}$ is injective. Identify $\mathcal{W}$ with the open subset $\hat{i}(\mathcal{W}) \subset \hat{\mathcal{W}}$. Thus we have the complex fibre space $\hat{\mathcal{W}}$ over D with projection $\hat{\pi}$ such that $\hat{\pi}_{|\mathcal{W}} = \pi$. Hence (2) is proved.

PROOF OF (3). Put $\mathcal{S} = \hat{\mathcal{W}} - \mathcal{W}$. Take any $t \in D$. Put $\hat{\mathcal{L}}_1 \cap \hat{\pi}^{-1}(t) = L$, $\mathcal{S} \cap \hat{\pi}^{-1}(t) = S$ and $\hat{\mathcal{H}}_{|L} = u$. Since L is non-singular outside of S and S is compact, L has at most finitely many singular points. Hence we can consider L as an analytic subvariety of an affine space. Therefore $u^\nu: L \to L$ $(\nu > 0)$ can be regarded as (vector-valued) holomorphic functions on L. Since $u^\nu(L)$ is relatively compact in L, the sequence $\{u^\nu\}_{\nu>0}$ is uniformly bounded on L. Put $u(L) = L'$. Then there exists a subsequence of $\{u^\nu\}_{\nu>0}$ which converges uniformly on L'. Let u_0 be the limit function which is holomorphic on L'. Note that $u_0(L' - S) = \partial S$. This implies that each component of the vector-valued holomorphic function u_0 attains its maximum on an open set $L' - S$. Hence u_0 is constant. Since $L' - S$ is connected, S consists of one point. This implies that $\mathcal{S} \cap \hat{\pi}^{-1}(t) = \{\text{one point}\}$ and thus (3) is proved.

PROOF OF (1). Define the subvariety $\mathcal{T}$ of $\hat{\mathcal{L}}_1$ by

$$\mathcal{T} = \{w \in \hat{\mathcal{L}}_1 : \hat{\mathcal{H}}^* f(w) = f(w) \quad \text{for all } f \in \Gamma(\hat{\mathcal{L}}_1, \mathcal{O}_{\hat{\mathcal{L}}_1})\}\,.$$

Clearly $\mathcal{S} \subset \mathcal{T}$. Conversely, take any point $w \notin \mathcal{S}$. Then $\hat{\mathcal{H}}(w) \neq w$. Hence there exists $f \in \Gamma(\hat{\mathcal{L}}_1, \mathcal{O}_{\hat{\mathcal{L}}_1})$ such that $f(w) \neq f(\hat{\mathcal{H}}(w)) = \hat{\mathcal{H}}^* f(w)$, since $\hat{\mathcal{L}}_1$ is a Stein space. This shows $\mathcal{T} \subset \mathcal{S}$. Consequently $\mathcal{T} = \mathcal{S}$.

PROOF OF (4). The holomorphic automorphism $\mathcal{H}$ of $\mathcal{W}$ extends to a holomorphic automorphism of $\hat{\mathcal{W}}$. Denote this extended automorphism by the same symbol $\mathcal{H}$. Then $\mathcal{H}$ acts on each fibre $\hat{\pi}^{-1}(t)$ as a contracting holomorphic automorphism which fixes 0_t. Hence $\hat{\pi}^{-1}(t)$ is holomorphically isomorphic to an affine variety by Theorem 1. Q. E. D.

As a corollary we obtain

THEOREM 5. *X_t is holomorphically isomorphic to a submaniold of $\boldsymbol{C}^{n_t}-\{0\}/\langle\tilde{g}_t\rangle$, where $\tilde{g}_t$ is a contracting holomorphic automorphism of $\boldsymbol{C}^{n_t}$ with $\tilde{g}_t(0)=0$ and n_t is the dimension of the Zariski tangent space at the point 0_t attached to W_t.*

§ 7. Applications.

(A) Let X be a 3-dimensional compact complex manifold which is topologically homeomorphic to $S^1\times S^5$, where S^m denotes the real m-dimensional topological sphere. Then $b_1(X)=1$ and $b_2(X)=0$. Assume that X admits the fibre space structure $(X, V, f)_a$.

Let $\tilde{X}$ be the universal covering manifold of X. Since $\pi_1(X)\cong\boldsymbol{Z}$, we obtain

COROLLARY TO THEOREM 3. *$\hat{W}=\tilde{X}\cup\{$one point$\}$ admits a complex structure such that $\hat{W}$ is holomorphically isomorphic to an affine variety which has an algebraic $\boldsymbol{C}^*$-action.*

Let X_t be any small deformation of X and $\tilde{X}_t$ the universal covering manifold of X_t.

COROLLARY TO THEOREM 4. *Attaching one point 0_t to $\tilde{X}_t$, we obtain a complex analytic family of complex spaces $\bigcup_t(\tilde{X}_t\cup\{0_t\})$ such that, for each t, $\tilde{X}_t\cup\{0_t\}$ is holomorphically isomorphic to an affine variety.*

COROLLARY TO THEOREM 5. *X_t is holomorphically isomorphic to a submanifold of $\boldsymbol{C}^{n_t}-\{0\}/\langle\tilde{g}_t\rangle$, where $\tilde{g}_t$ is a contracting holomorphic automorphism of $\boldsymbol{C}^{n_t}$ with $\tilde{g}_t(0)=0$.*

An example of the complex structure $(X, V, f)_a$ on $S^1\times S^5$ was constructed by Brieskorn-Van de Ven [**1**].

(B) Consider a compact complex surface S satisfying the three conditions;

(i) *$b_1(S)=odd$,*

(ii) *S is an elliptic surface,*

(iii) *S is relatively minimal.*

Note that, by a theorem of Kodaira [**7**], if $b_1(S)=$odd and not equal to 1, then the condition (ii) if redundant. Let f be a holomorphic mapping from S onto a curve C of which the general fibres are connected non-singular elliptic curves. By (i) and (iii), every fibre of f is a non-singular elliptic curve, i. e., all singular fibres are multiple of elliptic curves ([**6**]).

Now we can prove statements corresponding to Theorems 3, 5 by a similar method as in §5. Namely we have

THEOREM 6. *There exists an infinite cyclic unramified covering manifold* W *of* S *such that* $\hat{W}=W\cup\{\text{one point}\}$ *is holomorphically isomorphic to an affine variety of dimension* 2 *which admits an algebraic* $\boldsymbol{C}^*$*-action.*

Since $\hat{W}$ admits a contracting holomorphic automorphism, we get

THEOREM 7. *The surface* S *is holomorphically isomorphic to a submanifold of* $\boldsymbol{C}^n-\{0\}/\langle\tilde{g}\rangle$ *for some* n, *where* $\tilde{g}$ *is a contracting holomorphic automorphism of* $\boldsymbol{C}^n$ *such that* $\tilde{g}(0)=0$.

To prove Theorem 6, we need several lemmas.

LEMMA 18. $H^2(C, \boldsymbol{C}) \xrightarrow{f^*} H^2(S, \boldsymbol{C})$ *is a zero mapping.*

PROOF. We prove this with the aid of the spectral sequence $E_2^{r,s}=H^r(C, R^s f_*\boldsymbol{C}) \Rightarrow E^{r+s}=H^{r+s}(S, \boldsymbol{C})$. We have

$$H^2(S, \boldsymbol{C})=E_3^{2,0}+E_3^{1,1}+E_4^{0,2},\ E_3^{1,1}=E_2^{1,1},$$

$$E_3^{2,0}=\frac{E_2^{2,0}}{\operatorname{Im}(E_2^{0,1}\to E_2^{2,0})},\ \dim E_2^{0,1}=2,\ \dim E_2^{2,0}=1,$$

$$\dim E_2^{1,1}=2b_1(C).$$

We remark that $R^1f_*\boldsymbol{C}=\boldsymbol{C}^2$. On the other hand,

$$H^1(S, \boldsymbol{C})=E_2^{1,0}+E_3^{0,1}=H^1(C, \boldsymbol{C})+E_3^{0,1}.$$

Since $b_1(S)=\text{odd}$, $b_1(C)=\text{even}$ and $\dim E_3^{0,1}\leqq 2$, we have $\dim E_3^{0,1}=1$. Hence it follows that $1=\dim E_3^{0,1}=\dim\ker(E_2^{0,1}\to E_2^{2,0})$, i.e., $\dim\operatorname{Im}(E_2^{0,1}\to E_2^{2,0})=1$. Thus we obtain $E_3^{2,0}=0$. This proves the lemma.

We have $H^2(S, \boldsymbol{C})=E_2^{1,1}+\ker(E_2^{0,2}\to E_2^{2,1})$ and hence $b_2(S)=2b_1(C)+\dim\ker(E_2^{0,2}\to E_2^{2,1})$. The dimension of $E_2^{0,2}$ is at most 1. By $\chi(S)=0$, $b_2(S)=$ even. Hence we have $\dim\ker(E_2^{0,2}\to E_2^{2,1})=0$ and $b_2(S)=2b_1(C)$. Thus we get the following lemma.

LEMMA 19. $b_2(S)=2b_1(C)$, $b_1(S)=b_1(C)+1$.

LEMMA 20. $\dim H^1(S, \mathcal{O}_S)=\dim H^1(C, \mathcal{O}_C)+1$.

PROOF. We use the spectral sequence $E_2^{r,s}=H^r(C, R^sf_*\mathcal{O}_S) \Rightarrow H^{r+s}(S, \mathcal{O}_S)$. The lemma follows from the following equalities;

$$H^1(S, \mathcal{O}_S)=E_2^{1,0}+E_3^{0,1}=H^1(C, \mathcal{O}_C)+E_3^{0,1},$$

$$E_3^{0,1}=\ker(E_2^{0,1}\longrightarrow E_2^{2,0})=E_2^{0,1}=\boldsymbol{C}.$$

LEMMA 21. $0\longrightarrow H^0(S, d\mathcal{O}_S)\longrightarrow H^1(S, \boldsymbol{C})\xrightarrow{i} H^1(S, \mathcal{O}_S)\longrightarrow 0$ (*exact*).

PROOF. It is sufficient to show that i is surjective. By Kodaira [**6**] (pp.

754-755), we have $\dim H^0(S, d\mathcal{O}_S)=\dim H^1(S, \mathcal{O}_S)-1$. Hence, by Lemma 20, $\dim H^0(S, d\mathcal{O}_S)=\dim H^1(C, \mathcal{O}_C)=(1/2)b_1(C)$. On the other hand, $\dim H^1(S, \boldsymbol{C})=b_1(C)+1$ by Lemma 19. Hence we have $\dim \operatorname{Im} i=b_1(C)+1-(1/2)b_1(C)=(1/2)b_1(C)+1$. This is equal to the dimension of $H^1(S, \mathcal{O}_S)$ by Lemma 20, q. e. d.

LEMMA 22. *Let η be any line bundle on C. Then $f^*\eta$ is a flat line bundle on S.*

PROOF. By Lemma 21, we have the following commutative diagram;

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
\cdots \longrightarrow & & H^1(S, \mathcal{O}_S) & \xrightarrow{e} & H^1(S, \mathcal{O}_S^*) & \xrightarrow{c} & H^2(S, \boldsymbol{Z}) & \longrightarrow \cdots & \text{(exact)} \\
 & & \uparrow i & & \uparrow j & & \| & & \\
\cdots \longrightarrow & & H^1(S, \boldsymbol{C}) & \xrightarrow{e} & H^1(S, \boldsymbol{C}^*) & \xrightarrow{c} & H^2(S, \boldsymbol{Z}) & \longrightarrow \cdots & \text{(exact)} \\
 & & \uparrow & & & & & & \\
 & & H^0(S, d\mathcal{O}_S) & & & & & & \\
 & & \uparrow & & & & & & \\
 & & 0 & & & & & & \\
 & & \text{(exact)} & & & & & &
\end{array}$$

By Lemma 18, $c(f^*\eta)\equiv 0$ (modulo torsion). Hence we can choose a positive integer m such that $c(f^*\eta^{\otimes m})=0$. Then there exists $\omega\in H^1(S, \boldsymbol{C})$ such that $j\circ e(\omega)=f^*\eta^{\otimes m}$. Thus the line bundle $f^*\eta^{\otimes m}$ has a flat representation $e(\omega)$. Hence $f^*\eta$ is also flat, q. e. d.

Now let η be an ample line bundle on C. Then $f^*\eta^{\otimes m}$ is raised from a group representation $\rho: H_1(S, \boldsymbol{Z})\rightarrow \boldsymbol{C}^*$. Choosing suitable m, we can assume that $\rho(\operatorname{Tor} H_1(S, \boldsymbol{Z}))=1$. Let $\{\gamma_0, \gamma_1, \cdots, \gamma_s\}$ $(s=b_1(S)-1)$ be a Betti basis of $H_1(S, \boldsymbol{Z})$ such that γ_0 vanishes under the mapping $f_*: H_1(S, \boldsymbol{Z})\rightarrow H_1(C, \boldsymbol{Z})$. Note that $f_*\gamma_i\neq 0$ for $i\neq 0$. Let ρ_1 be the representation of $H_1(S, \boldsymbol{Z})$ defined by

$$\begin{cases} \rho_1(\gamma_0)=1 \\ \rho_1(\gamma_i)=\rho(\gamma_i) \qquad (i>0). \end{cases}$$

Let F_m be a flat line bundle on S raised from ρ_1. Then there exists a flat line bundle ξ_m on C such that $f^*\xi_m=F_m$. Put $H_m=(f^*\eta)^{\otimes m}\otimes F_m^{-1}$. Then the line bundle H_m is raised from the representation ρ_0 defined by

$$\begin{cases} \rho_0(\gamma_0)=\rho(\gamma_0)=\alpha, \\ \rho_0(\gamma_i)=1 \qquad (i>0). \end{cases}$$

Choosing m to be sufficiently large, we can assume that $\eta^{\otimes m}\otimes\xi_m^{-1}$ is very ample. Let $\{\varphi_1, \cdots, \varphi_N\}$ be a basis of $H^0(C, \mathcal{O}_C(\eta^{\otimes m}\otimes\xi_m^{-1}))$. Now, as in § 3, we can construct a holomorphic mapping which makes the following diagram commutative;

$$\begin{tikzcd} S \arrow[r, "\Phi"] \arrow[d, "f"] & \boldsymbol{C}^N-\{0\}/\langle\alpha\rangle \arrow[d, "p_0"'] \\ C \arrow[r, "\varphi"] & \boldsymbol{P}^{N-1}. \end{tikzcd}$$

Then the theorem follows by the same method of proof as in § 5.

References

[1] E. Brieskorn and A. Van de Ven, Some complex structures on products of homotopy spheres, Topology, **7** (1967), 389-393.
[2] H. Grauert and R. Remmert, Komplexe Räume, Math. Ann., **136** (1958), 245-318.
[3] Ma. Kato, Complex structures on $S^1 \times S^5$, Proc. Japan Acad., **49** (1973), 575-577.
[4] Ma. Kato, A generalization of Bieberbach's example, Proc. Japan Acad., **50** (1974), 329-333.
[5] K. Kodaira, On compact analytic surfaces II, Ann. of Math., **77** (1963), 563-626.
[6] K. Kodaira, On the structure of compact complex analytic surfaces I, Amer. J. Math., **86** (1964), 751-798.
[7] K. Kodaira, On the structure of compact complex analytic surfaces IV, Amer. J. Math., **90** (1968), 1048-1066.
[8] H.-S. Ling, Extending families of pseudoconcave complex spaces, Math. Ann., **204** (1973), 13-48.
[9] P. Orlik and P. Wagreich, Isolated singularities of algebraic surfaces with $\boldsymbol{C}^*$-action, Ann. of Math., **93** (1971), 205-228.
[10] M. Raynaud, Géométrie Algébrique et Géométrie Analytique, Séminaire de Géométrie Algébrique 1, Exposé XII, (1960-61), IHES, Lecture Notes in Math., **224**, Springer, 1971.
[11] L. Reich, Das Typenproblem bei formal-biholomorphen Abbildungen mit anziehendem Fixpunkt, Math. Ann., **179** (1969), 227-250.
[12] L. Reich, Normalformen biholomorpher Abbildungen mit anziehendem Fixpunkt, Math. Ann., **180** (1969), 233-255.
[13] W. Rudin, A geometric criterion for algebraic varieties, J. Math. Mech., **17**(7) (1968), 671-683.
[14] Y.-T. Siu, Extending coherent analytic sheaves, Ann. of Math., **90** (1969), 108-143.
[15] Y.-T. Siu, The 1-convex generalization of Grauert's direct image theorem, Math. Ann., **190** (1971), 203-214.
[16] K. Yosida, Functional Analysis, Springer, 1964.

Masahide KATO
Department of Mathematics
Faculty of Science
Rikkyo University
Nishi-Ikebukuro, Tokyo
Japan