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An odd characterization of some simple groups

By Tomoyuki YOSHIDA

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§1. Introduction.

In his lecture note [**10**], G. Higman proposed characterizations of finite simple groups in terms of properties concerning with odd primes, which are called odd characterizations. He discussed the recent results on odd characterizations and also illustrated some techniques which are used in such characterizations. For instance, the classification problem of $C\theta\theta$-groups is the most famous example (Fletcher [**3**], [**4**]). However, as a more typical situation for such problems, we consider the structure of the centralizers of elements of order 3. Actually, we are interested in such centralizers in known sporadic simple groups which are given in Conway [**2**] and Tits [**12**] and we notice that those have some particular properties.

The purpose of this paper is to prove a result in this direction:

THEOREM. *Let G be a finite simple group and assume that for any element s of G of order* 3, $C_G(s)\cong Z_3\times Z_3$ *or* $Z_3\times A_4$. *Then* $G\cong A_6$, A_7, $L_3(4)$, $L_3(7)$, $U_3(5)$, *or* M_{22}.

All groups considered in this paper are finite. Most of our notation is standard and taken from Gorenstein [**6**]. For each p-subgroup P of a group G, $\mathcal{V}(P)$ denotes the set of p'-subgroups of G normalized by P, and $\mathcal{V}^*(P)$ denotes the set of the maximal elements in $\mathcal{V}(P)$. For any 2-group T, $m(T)$ is the maximum rank of an abelian subgroup of T.

§2. Preliminary lemmas.

LEMMA 2.1. *The following hold:*

(1) $L_3(4)\not\leqq GL(8, 2)$;

(2) M_{10}, $PGL(2, 9)\not\leqq GL(7, 2)$;

(3) *Let X be an elementary abelian group of order 2^8 and G be a subgroup of* Aut(X) *isomorphic to* A_6, M_{10} *or* $PGL(2, 9)$. *Assume that* $|C_X(s)|\leqq 4$ *for each element s of order* 3. *Then* $|X|=|C_X(t)|^2$ *for any involution t of G.*

PROOF. (1) Suppose $L=L_3(4)$ acts faithfully on an elementary abelian group V of order 2^8. Since L has an elementary abelian Sylow 3-subgroup of order 9 and all elements of L of order 3 are conjugate, we have that $|C_X(s)|=4$ for any element s of L of order 3. Clearly, a Sylow 7-subgroup of L centralizes an element $v\in V^{\#}$. Since $|L:C_L(v)|=|v^L|<|V|=2^8$, we have $|C_L(v)|>70$. Thus $C_L(v)$ has no normal 7-complement, as otherwise $|O_{7'}(C_L(v))|\leqq 8$. Thus $C_L(v)$ contains a Frobenius group of order 21. But then for some element s of L of order 3, $|C_V(s)|=16$, a contradiction.

(2) This follows from the fact that all elements of order 3 are conjugate in either group.

(3) It follows easily that a Sylow 3-subgroup of G acts fixed-point-free on X. Assume first that the involution t normalizes a Sylow 3-subgroup S of G. Since $X=\langle C_X(s) \mid s\in S^{\#}\rangle$ and $|C_X(s)|=|C_X(s)\cap C_X(t)|^2$ for each $s\in S^{\#}$, we see that $|X|=|C_X(t)|^2$. Thus we may assume that t normalizes no Sylow 3-subgroup of G. Then we have that G is isomorphic to $PGL(2, 9)$ and X is of order 2^8. Note that t inverts a Sylow 5-subgroup P of G and $N_G(P)$ is dihedral of order 20. It will suffice to show that P acts fixed-point-free on X. Suppose false. Then $|C_X(P)|=16$. Since G has 36 Sylow 5-subgroups, we have $|C_X(P)^{\#}|\cdot|G:N_G(P)|>|V^{\#}|$. This means that there is $v\in V^{\#}$ which is centralized by two distinct Sylow 5-subgroups, and so $C_G(v)$ is not 5-closed. Thus $C_G(v)$ contains a subgroup A isomorphic to A_5. We may assume that A contains P. As $N_{G'}(P)\leqq A$, there is an involution u in $N_A(P)\cap C(t)$. We have $|C_V(P)\cap C(u)|=4$. Since ut and t are conjugate and $[V, P]\leqq C_V(tu)$, we have $|C_V(t)|=|C_V(tu)|=2^6$, and so $C_V(P)\leqq C_V(t)$. Thus $t\in C_G(v)$. But since $N_G(A)\leqq G'$, we have that $G=\langle A, t\rangle\leqq C_G(v)$, contrary to (2).

The following lemma and the proof are by Goldschmidt [**5**], Corollary 4.

LEMMA 2.2. *Let X be an elementary abelian 2-subgroup of a finite group G and T a Sylow 2-subgroup of $N_G(A)$. Assume that for each element t of $T-X$, $m(X)>m(T/X)+m(C_X(t))$. Then T is a Sylow 2-subgroup of G and A is strongly closed in T.*

PROOF. Let Y be a subgraup of T conjugate to X in G. If $X\neq Y$, then for $y\in Y-X$, $C_X(y)\geqq X\cap Y$. Thus $m(T/X)+m(C_X(y))\geqq m(XY/X)+m(X\cap Y)$. Since $XY/X\cong Y/X\cap Y$, we have that $m(T/X)+m(C_X(y))\geqq m(Y)=m(X)$, a contradiction. Hence $X=Y$, and so X is weakly closed in T. In particular, T is a Sylow 2-subgroup of G. Next, suppose X is not strongly closed in T. Let t be an involution of $T-A$ conjugate to an element of X. Among all $g\in G$ such that $t^g\in X$, choose g in such a way that $|X\cap X^{g^{-1}}|$ is maximal. Set $Y_0=X\cap X^{g^{-1}}$, $Y_1=\langle Y_0, t\rangle$ and $Y_2=C_X(t \bmod Y_0)$. Clearly, $Y_0<Y_2\leqq N_G(Y_1)$. Since $\langle X^{g^{-1}}, Y_2\rangle\leqq N_G(Y_1)$, the weak closure of X implies that $Y_2{}^a\leqq N_G(X^{g^{-1}})$ for some $a\in N_G(Y_1)$. Thus $Y_2{}^{agn}\leqq T$ for some $n\in N_G(X)$. We have that $\langle Y_0{}^{agn}, t^{agn}\rangle=Y_1{}^{agn}=Y_1{}^{gn}\leqq X$, and so $Y_0{}^{agn}\leqq X\cap X^{agn}$. Thus the maximality of $|Y_0|$ implies that $Y_0{}^{agn}=X\cap X^{agn}$. By a change of notation, we may sssume

that $Y_2^g \leqq T$. We have $Y_0^g = X \cap X^g \geqq X \cap Y_2^g \geqq Y_0^g$, and so $Y_0^g = Y_2^g \cap X$. Thus $Y_2/Y_0 \cong Y_2^g/Y_0^g \cong Y_2^g X/X \leqq T/X$, and so $m(Y_2/Y_0) \leqq m(T/X)$. Set $Y = C_X(t)$. Since X is elementary, $[X, t] \leqq Y$. Furthermore, $X/Y_2 \cong [X/Y_0, t] \cong [X, t]Y_0/Y_0 \leqq Y/Y_0$, and so $m(X/Y_2) \leqq m(Y/Y_0)$. Hence

$$\begin{aligned} m(X) &= m(Y) + m(Y_2/Y) + m(X/Y_2) \\ &\leqq m(Y) + m(Y_2/Y) + m(Y/Y_0) \\ &= m(Y) + m(Y_2/Y_0) \\ &\leqq m(Y) + m(T/X), \end{aligned}$$

contrary to the assumption. The lemma is proved.

LEMMA 2.3 (Fletcher [**3**]). *$C\theta\theta$-groups with elementary abelian Sylow 3-subgroup of order 9 are 3-closed or isomorphic to one of A_6, M_{10}, $PGL(2, 9)$, $L_3(4)$, where M_{10} is a subgroup of the Mathieu group M_{11} of index 11.*

LEMMA 2.4 (Harada [**9**]). *Let G be a simple group which contains an elementary abelian subgroup of order 16 such that A is a Sylow 2-subgroup of $C_G(A)$ and $N_G(A)/C_G(A)$ is isomorphic to A_6 or A_7. Then G is of sectional 2-rank 4. In particular, G is isomorphic to M_{22}, M_{23}, McL, Ly, $L_4(q)$, $q \equiv 5 (\mathrm{mod}\, 8)$, or $U_4(q)$, $q \equiv 3 (\mathrm{mod}\, 8)$.*

LEMMA 2.5 (Smith-Tayler [**11**]). *Let G be a finite group with noncyclic abelian Sylow p-subgroup P. Assume that $|N_G(P) : PC_G(P)| = 2$. Then $O^p(G) < G$ or G is p-solvable.*

§ 3. The proof of the theorem.

Throughout the remainder of this paper, G denotes a simple group satisfying the assumption of our theorem, that is, $C_G(s)$ is isomorphic to Z_3^2 or $Z_3 \times A_4$ for each element s of G of order 3. If G is a $C\theta\theta$-group, then it follows from Fletcher theorem that G is isomorphic to A_6 or $L_3(4)$. Thus we may assume that $C_G(s) \cong Z_3 \times A_4$ for an element s of G of order 3. Let S be a Sylow 3-subgroup of G. Then clearly $S \cong Z_3^2$ and $C_G(S) = S$. We argue by induction on $|G|$.

LEMMA 3.1. *The following hold:*

(1) *$N_G(S)$ is a Frobenius group such that $|N_G(S) : S| = 4$ or 8.*

(2) *For any element s of order 3, $C_G^*(s) \cong S_3$ or S_4.*

PROOF. Since G is simple and S is abelian, it follows from Burnside's transfer theorem and Smith-Tayler's theorem that $|N_G(S) : S| \geqq 4$. Since $C_G(s) \cap N_G(S) = S$ for any $s \in S^\#$, $N_G(S)$ is a Frobenius group, proving (1). Since an involution of $N_G(S)$ inverts S, (2) follows easily.

LEMMA 3.2. *The following hold:*

(1) *Let* $s\in S^{\#}$, $V=O_2(C_G(s))\neq 1$. *Then* $C_G(V)=O_2(C_G(V))\langle s\rangle$.

(2) *Let* $X\in \mathcal{U}^*(S)$. *Then* X *is a* 2-*group and* $N_G(X)/X$ *is a* $C\theta\theta$-*group. Furthermore*, $|X|=4$ *or* $C_G(X)\leqq X$.

(3) *Let* $X\in \mathcal{U}(S)$ *and* $\langle s_i\rangle$, $1\leqq i\leqq 4$, *be the four subgroups of* S *of order* 3. *Then*

$$X=C_X(s_1)C_X(s_2)C_X(s_3)C_X(s_4),$$

and

$$|X|=\prod_i |C_X(s_i)|.$$

(4) *Let* $X\in \mathcal{U}(S)$ *and let* t *be an involution of* $N_G(S)$. *Then* t *normalizes* X *and* $|X|=|C_X(t)|^2$.

PROOF. (1) By Lemma 3.1, we have that $\langle s\rangle\in Syl_3(C_G(V))$ and $C_G^*(s)\cap C_G(V)=\langle s\rangle\times V$. Thus $C_G(V)$ has a normal 3-complement by Burnside's transfer theorem. Since $O_{3'}(C_G(V))$ is normalized by S, (1) holds.

(2) Set $N=N_G(X)$ and $\bar{N}=N/X$. $C_{\bar{N}}(s)=\overline{C_N(s)}$ for any $s\in S^{\#}$. Since $C_N(s)$ has a normal 3-complement, if follows from the maximality of X that $\overline{C_N(s)}$ is a 3-group, proving (2).

(3) This is wellknown (See [**6**], Theorem 5.3.16).

(4) Since t normalizes $O_2(C_G(s))$ for any $s\in S^{\#}$, we have that $t\in N_G(X)$. Since t inverts S, it follows from (3) that $|X|=|C_X(t)|^2$. The lemma is proved.

LEMMA 3.3. *Let* $X\in \mathcal{U}^*(S)$. *Assume that* X *is abelian. Then one of the following holds:*

(1) X *is a four-group;*

(2) X *is strongly closed in a Sylow* 2-*subgroup of* G;

(3) $X\cong Z_2^4$ *and* $N_G(X)/X\cong A_6$.

PROOF. Set $N=N_G(X)$ and $\bar{N}=N/X$. Let T be a Sylow 2-subgroup of N. We may assume that $|X|>4$. By Lemma 2.1(1), Lemma 3.2(2) and Fletcher's theorem, we have that $\bar{N}$ is 3-closed or isomorphic to A_6, $PGL(2, 9)$ or M_{10}. By Lemma 2.2(3), for any involution t of $T-X$, $m(X)=2m(C_X(t))\geqq 4$. Thus if $m(X)\geqq 6$ or $m(T/X)=1$, then $m(X)>m(T/X)+m(C_X(t))$. By Lemma 2.2, (2) holds. Assume $|X|=16$. Then $\bar{N}\cong A_6$ by Lemma 2.1(2), and so (3) holds. The lemma is proved.

LEMMA 3.4. *Let* $X\in \mathcal{U}^*(S)$. *Assume that* X *is not abelian. Then there is a subgroup* Y *of* X *of order* 16 *such that* $C_G(Y)=Y$ *and* $N_G(Y)/Y\cong A_7$.

PROOF. Let u be an element of $N_G(S)$ of order 4, $t=u^2$, and $\langle s_i\rangle$, $1\leqq i\leqq 4$, be the four subgroups of S of order 3. Set $V_i=O_2(C_G(s_i))$ for each i. Then $t\in N_G(V_i)-C_G(V_i)$ for each i. We may assume that $V_1^u=V_2$ and $V_3^u=V_4$. Since $X'\neq 1$, $|X|=2^6$ or 2^8. We may assume that $V_3\leqq X'\cap Z(X)$.

We shall first assume that X is of order 2^6. Then $X'=V_3$. By Lemma 3.2, $N=N(V_3)=XS\langle t\rangle$. Since $V_3^u=V_4\neq V_3$ and u normalizes $[V_1, V_2]$, we see that $[V_1, V_2]\neq V_3$, and so $X\neq V_1V_2V_3$. Thus $V_4\leqq X$. Set $Y=V_3V_4$, $L=N_G(Y)$ and $\bar{L}=L/Y$. Clearly $\langle X, S, u\rangle\leqq L$ and $N_{\bar{L}}(\bar{S})=\bar{S}\langle\bar{u}\rangle$. Since u does not normalize X, we have that $O_{3'}(\bar{L})=O_3(\bar{L})=1$. Let $\bar{L}_0$ be a minimal normal subgroup of $\bar{L}$. Then $\bar{L}_0$ is simple and satisfies the assumption of our theorem. Since $C_G(Y)=Y$, we have that $\bar{L}=\bar{L}_0\cong A_7$, as required.

Next assume that X is of order 2^8. Since $X=\langle O_2(C_G(s))\mid s\in S^{\#}\rangle$, we have that $N_G(S)\leqq N_G(X)$, and so $X'\cap Z(X)\geqq V_3V_4$. Since X is not abelian, we have that $X'=Z(X)=\Phi(X)=V_3V_4$ and $N_G(S)=S\langle u\rangle$. There are $v_1\in V_1$ and $v_2\in V_2$ such that $[v_1, v_2]\neq 1$. Acting S on the relation, we see that any element of $V_1^{\#}$ is commute with no element of $V_2^{\#}$. Thus involutions of X are contained in $X'V_1\cup X'V_2$. Since $|C_{X/X'}(t)|=|C_{X'}(t)|=4$, any involution of $T-X$ is conjugate to t in T. Since $|C_X(t)|=16$, elementary abelian subgroup of T of order 2^6 are only V_1X' and V_2X'. In particular, X is characteristic in T, and so T is a Sylow 2-subgroup of G. We shall show that if two elements of X are conjugate in G, then they are conjugate in $N_G(X)$. Assume $a, b\in X$, $b=a^g$, $g\in G$. We will show that $a\sim b$ in $N(X)$. We may assume that $a, b\in V_1X'$ and $C_T(a)^g\leqq C_T(b)\in Syl_2(C_T(b))$. Since $(V_1X')^g=V_1X'$ or V_2X', we have that $g\in N_G(V_1X')T$. Set $L=N_G(V_1X')$. Then $N_L(S)=S\langle t\rangle$. By Smith-Tayler's theorem, L is 3-solvable. Thus $L\leqq N_G(X)$, and so $g\in N(X)$. We proved that if $a\sim b$ in G for involutions a, b in X, then $a\sim b$ in $N_G(X)$. In particular, if x is an involution of $X-X'$, then, $|C_G(x)|_2\leqq 2^7$ and $m(C_T(x))=6$. By Harada's transfer theorem ([8], Lemma 16), t is conjugate to an element of X. Take an element g of G such that $t^g=x\in X$ and $C_T(t)^g\leqq T$. Let v_i, $1\leqq i\leqq 4$, be involutions of $V_i\cap C_G(t)$. Then $C=C_T(t)=\langle u, v_1, v_2, v_3, v_4\rangle$. Since $v_1{}^u=v_2$, we have that $[v_1, v_2]=v_3v_4$. Thus $|C|=2^6$ and $m(C)=4$. Hence $x\in X'$. In particular, $t\not\sim v_1$. Since $v_1{}^g\in X-X'$ and $t\sim tv_1$, $t\sim(tv_1)^g=xv_1{}^g\in X-X'$. This is a contradiction. The lemma is proved.

We can now establish our theorem. Let $X\in \mathit{И}^*(S)$. If X is of order 4, then since $X=Syl_2(C_G(X))$ by Lemma 3.2(1), we have that a Sylow 2-subgroup of G is dihedral or semi-dihedral. Such simple groups are known by [1] and [7]. From this, we can earsily show that $G\cong A_7$, $L_3(7)$ or $U_3(5)$. By Goldschmidt theorem [5], there is no strongly closed abelian subgroup in a Sylow 2-subgroup of G. Thus if $|X|>4$, then there is an elementary abelian subgroup Y such that $A\in Syl_2(C_G(A))$ and $N_G(A)/A\cong A_6$ or A_7 by Lemma 3.3 and 3.4 and so Harada's theorem [9] implies that $G\cong M_{22}$. Adding the $C\theta\theta$-cases, G is isomorphic to one of the groups in the conclusion of our theorem.

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References

[1] J. Alperin, R. Brauer, and D. Gorenstein, Finite groups with quasi-dihedral and wreathed Sylow 2-subgroups, Trans. Amer. Math. Soc., **151** (1970), 1-261.
[2] J. H. Conway, Three lectures on exceptional groups, Finite simple groups, edited by M. B. Powell and G. Higman, Academic Press, 1971, 215-247.
[3] L. R. Fletcher, A characterization of $L_3(4)$, J. Algebra, **19** (1971), 274-281.
[4] L. R. Fletcher, A transfer theorem for $C\theta\theta$-groups, Quart. J. Math. Oxford (2), **22** (1971), 503-533.
[5] D. M. Goldschmidt, 2-fusion in finite groups, Ann. of Math., **99** (1974), 70-117.
[6] D. Gorenstein, Finite groups, Harper and Row, 1968.
[7] D. Gorenstein and J. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups, I, II, III, J. Algebra, **2** (1965), 85-151, 218-270, 364-393.
[8] K. Harada, Finite simple groups with short chains of subgroups, J. Math. Soc. Japan, (1968), 655-672.
[9] K. Harada, On finite groups having self-centralizing 2-subgroups of small order, Amer. Math. Soc. Notices, 1973, Jan. A-98.
[10] G. Higman, Odd characterization of finite simple groups, Lecture notes, Univ. Michigan, 1868.
[11] S. D. Smith and A. P. Tayler, On finite groups with a certain Sylow-normalizer, II, J. Algebra, **26** (1973), 366-367.
[12] J. Tits, Groups finis simples sporadiques, Seminaire Bourbaki, No. 375 (1970), 187-211.

Tomoyuki YOSHIDA
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo, Japan