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Two remarks on irreducible characters of finite general linear groups

By Takuro SHINTANI

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Introduction.

0-1. Let k be a finite field and K be a finite extension of k. It is well-known (and is easily verified) that any character of $K^{\times}=GL(1, K)$, invariant under the action of the Galois group of K with respect to k, is a composition of the norm homomorphism from $K^{\times}$ onto $k^{\times}$ and a suitable character of $k^{\times}$. In this paper, we prove an analogous result for irreducible characters of finite general linear groups $GL_n(k)$. In more detail, let σ be the Frobenius automorphism of K with respect to k. Then σ acts naturally on $GL_n(K)$ as an automorphism with the fixed points set $GL_n(k)$. An irreducible representation R of $GL_n(K)$ is said to be *σ-invariant* if the representation $R^{\sigma}=R\circ\sigma$ is equivalent to R. If R is σ-invariant, there exists a linear transformation I_{σ} of the representation space V of R which satisfies

$$R(g)I_{\sigma}=I_{\sigma}R(g^{\sigma}) \qquad (\forall g\in GL_n(K))$$

(I_{σ} is unique up to a constant scalar factor). We extend any class function χ on $GL_n(k)$ to a class function on $GL_n(K)$ by setting

$$\chi(x)=\begin{cases}\chi(x') & \text{if there exists an } x'\in GL_n(k) \text{ which is conjugate to } x \text{ in } GL_n(K)\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

This is possible since two elements in $GL_n(k)$ are conjugate if and only if they are conjugate in $GL_n(K)$.

Now, we have:

THEOREM 1. *Let notations be as above. For a suitable normalization of I_{σ}, there exists an irreducible character χ_R of $GL_n(k)$ which satisfies* trace $I_{\sigma}R(g)$ $=\chi_R(\mathrm{Norm}_{K/k}(g))$ ($\forall g\in GL_n(K)$), *where*

$$\mathrm{Norm}_{K/k}(g)=g^{\sigma^{m-1}}g^{\sigma^{m-2}}\cdots g^{\sigma}g \qquad (m=\deg K/k)\,.$$

Moreover, the mapping $R\mapsto\chi_R$ establishes the bijection from the set of equivalence classes of σ-invariant irreducible representations of $GL_n(K)$ onto the set *of irreducible characters of $GL_n(k)$.*

If $[K, k]$ is prime to the cardinality of $GL(n, K)$, this result is an immediate consequence of Theorem 3 of Glauberman [8]. (See also [9].)

0-2. Applying Theorem 1, we derive a formula for irreducible characters of $GL_n(k)$ which belong to the "discrete series"[1]. Assume $m=\deg(K/k)=n$. A character ξ of $K^\times$ is said to be *regular* if $\xi^{\sigma^k}\neq\xi$ for $k=1, 2, \cdots, n-1$. Denote by B_n (resp. U_n) the group of $n\times n$ upper triangular (resp. unipotent upper triangular) matrices in GL_n. Let $\omega^{-1}=\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}\in GL_n$ be a standard cyclic permutation matrix of degree n. For a character ξ of $K^\times$, we denote by ψ_ξ a function on $GL_n(K)$ given as follows:

$$\psi_\xi(g)=\begin{cases} \prod_{i=1}^{n}\xi(b_{ii}^{\sigma^{i-1}}), & \text{if } g=u\omega b\in U_n(K)\omega B_n(K) \\ 0 & \text{otherwise}, \end{cases}$$

where b_{ii} is the i-th diagonal entry of b.

Our second result is the following

THEOREM 2. *Let notations be as above. If ξ is a regular character of $K^\times$, there exists an irreducible character χ_ξ of $GL_n(k)$ (which belong to the discrete series)*[1] *such that*

$$|B_n(K)|^{-1}q^{-n(n-1)/2}\sum_{x\in GL_n(K)}\psi_\xi(x^\sigma gx^{-1})=\zeta_\xi\chi_\xi(\mathrm{Norm}_{K/k}g),$$

where $|B_n(K)|$ (resp. q) is the cardinality of $B_n(K)$ (resp. k) and ζ_ξ is a root of unity in $\boldsymbol{Q}\left(\exp\frac{2\pi i}{n}\right)$, independent of g. Moreover any irreducible character of $GL_n(k)$ in the discrete series is equal to χ_ξ for a suitable regular character ξ of $K^\times$.

0-3. This paper consists of four sections. In §1, we recall some preliminary results in character theory of finite groups. The second (resp. third section) is devoted to the proof of Theorem 1 (resp. Theorem 2). In the last section 4, we give more detailed results for GL_2.

Notations.

For a group G and an element x of G, $Z_G(x)$ (resp. x^G) denotes the centralizers of x in G (resp. the conjugacy class of x in G). If G is a linear algebraic group defined over a finite field k, $G(k)$ denotes the finite group of

1) The definition of "discrete series" is given in 4.3 of [5].

its k-rational elements. For a finite set S, $|S|$ is the cardinality of S. A generalized character of a finite group G is an integral linear combinations of characters of G.

§ 1.

1°. Let G be a finite group and σ be an automorphism of G. Denote by m the order of σ and by $\langle\sigma\rangle$ the group of automorphisms of G generated by σ. Let $\tilde{G}$ be the semi-direct product of G with $\langle\sigma\rangle$. Namely, $\tilde{G}$ is the group with the underlying set $\langle\sigma\rangle\times G$ whose composition rule is given by

$$(\tau, g)(\tau', g')=(\tau\tau', g^{\tau'}g') \ (\tau, \tau'\in\langle\sigma\rangle, g, g'\in G).$$

We identify G with a normal subgroup of $\tilde{G}$ via the imbedding: $g\mapsto(1, g)$.

DEFINITION 1-1. A complex irreducible representation of $\tilde{G}$ is said to be *of the first* (resp. *second*) *kind* if its restriction to G is still irreducible (resp. reducible).

We denote by $X(\langle\sigma\rangle)$ the character group of $\langle\sigma\rangle$. For a representation R of $\tilde{G}$, ξR ($\xi\in X(\langle\sigma\rangle)$) is a representation of $\tilde{G}$ given by $(\xi R)(\tau, g)=\xi(\tau)R(\tau, g)$.

LEMMA 1-1. *Let R be an irreducible representation of $\tilde{G}$ and χ_R be its character.*

(i) *If R is of the second kind, χ_R vanishes on the subset $\sigma\times G$ of $\tilde{G}$.*

(ii) *If R is of the first kind,*

$$|G|^{-1}\sum_{x\in G}|\chi_R(\sigma^l, x)|^2=1 \quad \textit{for} \quad l=0, 1, \cdots, m-1. \tag{1.1}$$

PROOF. The first part is well-known. To prove the second part, we note that, if R is of the first kind, R and ξR ($\xi\in X(\langle\sigma\rangle)$) are equivalent, if and only if $\xi=1$. Thus we have, by the orthogonality relations between the irreducible characters of a finite group,

$$|G|^{-1}\sum_{l=0}^{m-1}\zeta_m^{kl}\sum_{x\in G}|\chi_R(\sigma^l, x)|^2=m \quad \text{or} \quad 0$$

according as $k\equiv 0 \bmod m$ or not $\left(\zeta_m=\exp\frac{2\pi i}{m}\right)$. Hence we obtain (1.1).

The proof of the next lemma is similar to that of the previous lemma.

LEMMA 1-2. *Let R_1 and R_2 be irreducible representations of the first kind of $\tilde{G}$ whose restrictions to G are inequivalent. Then*

$$\sum_{x\in G}\chi_{R_1}(\sigma^l, x)\overline{\chi_{R_2}(\sigma^l, x)}=0 \qquad (l=0, 1, \cdots, m-1),$$

where χ_{R_1} (resp. χ_{R_2}) is the character of R_1 (resp. R_2).

DEFINITION 1-2. A representation R of G is said to be σ-invariant if the representation R^σ given by $g \mapsto R(g^\sigma)$ is equivalent to R.

The following lemma is immediate to see.

LEMMA 1-3. *For a given σ-invariant irreducible representation R of G, there are exactly m mutually inequivalent irreducible representations of $\tilde{G}$ whose restrictions to G are equivalent to R. If $\tilde{R}$ is one of them, any other one is equivalent to $\xi\tilde{R}$ for a suitable character ξ of $\langle\sigma\rangle$.*

LEMMA 1-4. *Let R be a representation of G. Then, there exists a representation ρ of $\tilde{G}$ which satisfies*

$$\mathrm{trace}\,\rho(\sigma, x) = \mathrm{trace}\,R(x^{\sigma^{m-1}} x^{\sigma^{m-2}} \cdots x^\sigma x) \qquad (\forall x \in G)\,.$$

PROOF. Denote by V the representation space of R. Let I_σ be a linear transformation of $\underbrace{V \otimes V \otimes \cdots \otimes V}_{m \text{ times}}$ given by

$$I_\sigma(V_1 \otimes V_2 \otimes \cdots \otimes V_m) = V_2 \otimes V_3 \otimes \cdots \otimes V_m \otimes V_1\,.$$

Set

$$\rho(\sigma^l, x) = I_\sigma^l \cdot R(x^{\sigma^{m-1}}) \otimes R(x^{\sigma^{m-2}}) \otimes \cdots \otimes R(x^\sigma) \otimes R(x)\,.$$

Then it is easy to see that ρ is a representation of $\tilde{G}$ on $V \otimes \cdots \otimes V$ and that

$$\begin{aligned}\mathrm{trace}\,\rho(\sigma, x) &= \mathrm{trace}\,R(x^{\sigma^{m-1}}) R(x^{\sigma^{m-2}}) \cdots R(x) \\ &= \mathrm{trace}\,R(x^{\sigma^{m-1}} x^{\sigma^{m-2}} \cdots x^\sigma x)\,.\end{aligned}$$

q. e. d.

§ 2.

1°. Let G be a linear algebraic group defined over a finite field k. Let K be the extension of k of degree m. Then, the Frobenius automorphism σ of K with respect to k defines an automorphism: $g \mapsto g^\sigma$ of $G(K)$ in a natural manner. The group $G(k)$ is the subgroup of $G(K)$ formed by σ-fixed points in $G(K)$.

DEFINITION 2-1.[2] Two elements x_1 and x_2 of $G(K)$ are said to be *σ-twistedly conjugate* in $G(K)$ if there exists a $g \in G(K)$ which satisfies $x_2 = g^\sigma x_1 g^{-1}$.

For an $x \in G(K)$, we denote by $x^{G(K),\sigma}$ the subset of $G(K)$ consisting of all elements which are σ-twistedly conjugate to x in $G(K)$.

$$x^{G(K),\sigma} = \{g^\sigma x g^{-1}\,;\ g \in G(K)\}\,.$$

We call the subset $x^{G(K),\sigma}$ *the σ-twisted conjugacy class of x in $G(K)$*. Further, set

$$Z_{G(K),\sigma}(x) = \{g \in G(K)\,;\ g^\sigma x g^{-1} = x\}\,.$$

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2) Cf. §3 of [**3**].

Then, the mapping $g \mapsto g^\sigma x g^{-1}$ establishes the bijection from $G(K)/Z_{G(K),\sigma}(x)$ onto $x^{G(K),\sigma}$.

DEFINITION 2-2. A σ-twisted class function on $G(K)$ is a complex valued function on $G(K)$ which satisfies $f(g^\sigma x g^{-1})=f(x)$ for any x, $g\in G(K)$.

We identify a σ-twisted class function with a function on the set of σ-twisted conjugacy classes by setting $f(x)=f(x^{G(K),\sigma})$.

DEFINITION 2-3. The *norm* of $x\in G(K)$ with respect to $G(k)$ (which we denote by $N_{K/k}(x)$) is given by $N_{K/k}(x)=x^{\sigma^{m-1}}\cdot x^{\sigma^{m-2}}\cdots x^\sigma x$. If there is no fear of confusion, we write $N(x)$ instead of $N_{K/k}(x)$.

It is easy to see that

$$y=g^\sigma x g^{-1} \Rightarrow N(y)=gN(x)g^{-1}. \tag{2.1}$$

If G is abelian, the norm mapping $x\mapsto N(x)$ is a homomorphism of $G(K)$ into $G(k)$.

LEMMA 2-1. *If G is a connected linear abelian algebraic group defined over a finite field k, the mapping $x\mapsto N_{K/k}(x)$ is a surjective homomorphism from $G(K)$ onto $G(k)$ (K is the extension of k of degree m).*

PROOF. Denote by $\bar{k}$ the algebraic closure of k. Since G is connected, for each $x\in G(K)$, there exists a $y\in G(\bar{k})$ which satisfies $x=(y^\sigma)y^{-1}$ (see Theorem I, 2.2 of [7]). If $N_{K/k}(x)=1$, we have $y^{\sigma^m}y^{-1}=1$ and $y\in G(K)$. Thus,

$$\{x\in G(K);\ N_{K/k}(x)=1\}=\{y^\sigma y^{-1};\ y\in G(K)\}.$$

Hence $|\{x\in G(K); N_{K/k}(x)=1\}|=|G(K)|/|G(k)|$. Therefore, we have $|\{N_{K/k}(x); x\in G(K)\}|=|G(k)|$. As $\{N_{K/k}(x); x\in G(K)\}$ is a subset of $G(k)$, we have

$$\{N(x);\ x\in G(K)\}=G(k).$$

q. e. d.

LEMMA 2-2 (see Remark III, 3-23 of [7]). *The group of invertible elements of any associative algebra with identity is connected.*

LEMMA 2-3 (see I, 3-4 of [7]). *Let G be a linear algebraic group and assume that Z_x, the group of centralizers of x in G is connected for any x. Then two elements x, y of $G(k)$ are conjugate in $G(k)$ if and only if they are conjugate in $G(K)$.*

2°. Let GL_n be the general linear group considered as a linear algebraic group defined over k. For an unordered partition $\mu=(n_1,\cdots,n_r)$ $(0\leqq n_1,\cdots,n_r\in\boldsymbol{Z};\ n_1+\cdots n_r=n)$, let P_μ be the subgroup of GL_n consisting of all elements

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ 0 & A_{22} & & A_{2r} \\ & & & \\ 0 & 0 & & A_{rr} \end{bmatrix} \in GL_n$$

for which $A_{ii}\in GL_{n_i}$. Then P_μ is a linear algebraic subgroup of GL_n defined over k. If $\mu=(n)$. $P_\mu=GL_n$.

The following lemma follows immediately from Lemma 2-2 and Lemma 2-3.

LEMMA 2-4. *Two elements in $P_\mu(k)$ are conjugate in $P_\mu(k)$ if and only if they are conjugate in $P_\mu(K)$.*

By Lemma 2-4, any class function f on $P_\mu(k)$ is extended to a class function on $P_\mu(K)$ by setting

$$f(x)=\begin{cases} f(y) & \text{if } y\in x^{P_\mu(K)}\cap P_\mu(k)\,, \\ 0 & \text{if } x^{P_\mu(K)}\cap P_\mu(k)=\emptyset\,. \end{cases}$$

In the following, we always understand that a class function on $P_\mu(k)$ is extended to a class function on $P_\mu(K)$ in this manner. For any $x\in GL_n$, let A_x be the subalgebra of $M(n)$ generated by x and let $A_x^\times$ be the group of invertible elements of A_x. Then $A_x^\times$ is a connected abelian linear algebraic subgroup of GL_n defined over k. It is the centre of the centralizers of x in GL_n. If $x\in P_\mu$, $A_x^\times$ is also a linear algebraic subgroup of P_μ defined over k. The following lemma is a special case of Lemma 2-1.

LEMMA 2-5. *For each $x\in GL_n(k)$, the mapping: $y\mapsto N_{K/k}(y)=y^{\sigma^{m-1}}y^{\sigma^{m-2}}\cdots y^\sigma y$ gives a surjective homomorphism from $A_x^\times(K)$ onto $A_x^\times(k)$.*

For an $x\in P_\mu(K)$ set

$$N_{K/k}(x^{P_\mu(K),\sigma})=N_{K/k}(x)^{P_\mu(K)}\cap P_\mu(k)\,.$$

By Lemma 2-4, $N_{K/k}(x^{P_\mu(K),\sigma})$ is either the empty set or a single conjugacy class in $P_\mu(k)$ (in the next lemma we will show that it is never empty). By (2.1), $N_{K/k}(x^{P_\mu(K),\sigma})$ depends only upon the σ-twisted conjugacy class of x in $P_\mu(K)$.

LEMMA 2-6. *Set $G=P_\mu$ (μ is an arbitrary unordered partition of n).*

(i) *The mapping $x^{G(K),\sigma}\mapsto N_{K/k}(x^{G(K),\sigma})=N_{K/k}(x)^{G(K)}\cap G(k)$ establishes a bijection from the set of σ-twisted conjugacy classes in $G(K)$ onto the set of conjugacy classes in $G(k)$.*

(ii) $|x^{G(K),\sigma}|\,|G(K)|^{-1}=|N_{K/k}(x^{G(K),\sigma})|\,|G(k)|^{-1}$ $(\forall x\in G(K))$.

PROOF. Let $\{x_1, x_2, \cdots, x_c\}$ be a complete system of representatives for conjugate classes in $G(k)$. Then we have $x_1, \cdots, x_c\in G(k)$ and $G(k)=\bigcup_{i=1}^{c} x_i^{G(k)}$ (disjoint union). For each x_i, take an $x_i^*\in A_{x_i}^\times(K)\subset G(K)$ which satisfies $x_i=N(x_i^*)$. This is possible by Lemma 2-5, since $x_i\in A_{x_i}^\times(k)$. We have $x_i^{G(k)}=N_{K/k}(x_i^{*G(K),\sigma})$. By Lemma 2-4 and by (2.1), $x_1^{*G(K),\sigma}$, $x_2^{*G(K),\sigma}$, $\cdots$ and $x_c^{*G(K),\sigma}$ are *mutually disjoint*. Next, we will show that

$$(2.2)\qquad Z_{G(K),\sigma}(x_i^*)=Z_{G(k)}(x_i)\qquad (i=1, \cdots, c)\,.$$

In fact, if $g\in Z_{G(K),\sigma}(x_i^*)$, $g^\sigma x_i^* g^{-1}=x_i^*$. By (2.1), $gx_ig^{-1}=x_i$ and $g\in Z_{G(K)}(x_i)$.

Since $A_{x_i}^{\times}(K)$ is in the centre of $Z_{G(K)}(x_i)$, g commutes with $x_i^*\in A_{x_i}^*(K)$. Thus, we have $g=g^\sigma$ and $g\in Z_{G(k)}(x_i)$. Hence $Z_{G(K),\sigma}(x_i^*)\subset Z_{G(k)}(x_i)$. As the inverse inclusion relation is obvious, we obtain (2.2). Thus we have

$$|x_i{}^{*G(K),\sigma}|=|G(K)||Z_{G(k)}(x_i)|^{-1}=|G(K)||G(k)|^{-1}|x_i{}^{G(k)}|.$$

Hence

$$\sum_{i=1}^{c}|x_i{}^{*G(K),\sigma}|=|G(K)||G(k)|^{-1}\sum_{i=1}^{c}|x_i{}^{G(k)}|=|G(K)|.$$

Since $x_1{}^{*G(K),\sigma},\cdots,x_c{}^{*G(K),\sigma}$ are disjoint subsets of $G(K)$ we obtain

$$G(K)=\bigcup_{i=1}^{c}x_i{}^{*G(K),\sigma}\qquad\text{(disjoint union)}.$$

COROLLARY TO LEMMA 2-6. (i) *For any* $x\in P_\mu(K)$, x *and* x^σ *are* σ*-twistedly conjugate in* $P_\mu(K)$.

(ii) *Let* f_1 *and* f_2 *be class functions on* $P_\mu(k)$ *and let* $f_1{}^*$ *and* $f_2{}^*$ *be* σ*-twisted class functions on* $P_\mu(K)$ *which satisfy*

$$f_i{}^*(x)=f_i(N_{K/k}x)\qquad(i=1,2,\ \forall x\in P_\mu(K)).$$

Then

$$|P_\mu(K)|^{-1}\sum_{x\in P_\mu(K)}f_1{}^*(x)f_2{}^*(x)=|P_\mu(k)|^{-1}\sum_{x\in P_\mu(K)}f_1(x)f_2(x).$$

3°. For an unordered partition μ of n, denote by $\tilde{P}_\mu(K)$ the semi-direct product of $P_\mu(K)$ with the group of automorphisms of $P_\mu(K)$ generated by the Frobenius automorphism σ of K over k (see § 1). For $\mu=\{n\}$, we write $\tilde{P}_\mu(K)=\widetilde{GL}_n(K)$. Each $\tilde{P}_\mu(K)$ is a subgroup of $\widetilde{GL}_n(K)$. We note that if $\tilde{f}$ is a class function on $\tilde{P}_\mu(K)$, the function $x\mapsto\tilde{f}(\sigma,x)$ is a σ-twisted class function on $P_\mu(K)$.

LEMMA 2-7.[3] *The number of mutually inequivalent irreducible* σ*-invariant* (cf. Definition 1-2) *representations of* $GL_n(K)$ *is equal to the number of conjugacy classes in* $GL_n(k)$.

PROOF. The dimension of the space spanned by restrictions of irreducible characters of $\widetilde{GL}_n(K)$ to the subset $\sigma\times GL_n(K)$ is equal to the number of conjugacy classes of $\widetilde{GL}_n(K)$ contained in $\sigma\times GL_n(K)$. By (i) of Lemma 1-1, Lemma 1-2 and by Lemma 1-3, the dimension is equal to the number of inequivalent, σ-invariant irreducible representations of $GL_n(K)$. On the other hand, (σ,x) and (σ,y) are conjugate in $\widetilde{GL}_n(K)$ if and only if x and $y^{\sigma l}$ are σ-twistedly conjugate in $GL_n(K)$ for some integer l. However y and $y^{\sigma l}$ are always σ-twistedly conjugate. Hence, the number of $\widetilde{GL}_n(K)$-conjugacy classes

3) The lemma is an immediate consequence of a general theorem stated at page 1473 of [8].

contained in $\sigma\times GL_n(K)$ is equal to the number of σ-twisted conjugacy classes in $GL_n(K)$. The lemma now follows from Lemma 2-6.

Let G be a finite group and let H be a subgroup of G. For a class function f on H, we denote by $i[f|H\mapsto G]$ the class function on G induced from the class function f on H:

$$i[f|H\mapsto G](x)=|H|^{-1}|Z_x(G)|\sum_{y\in x^G\cap H}f(y).$$

If f is a character of H, then $i[f|H\mapsto G]$ is also a character of G.

LEMMA 2-8. *Let $\tilde{f}$ be a class function on $\tilde{P}_\mu(K)$ and f be a class function on $P_\mu(k)$. If, $\tilde{f}(\sigma, x)=f(N_{K/k}(x))$ $(\forall x\in P_\mu(K))$, then*

$$\begin{aligned}&i[\tilde{f}|\tilde{P}_\mu(K)\longmapsto \widetilde{GL}_n(K)](\sigma, x)\\ &=i[f|P_\mu(k)\longmapsto GL_n(k)](N_{K/k}x) \qquad (\forall x\in GL_n(K)).\end{aligned}$$

PROOF. We put $P_\mu=P$ and $GL_n=G$. We extend $\tilde{f}$ to a function in $\tilde{G}$ which vanishes outside $\tilde{P}$. Take a $g\in G(K)$ which satisfies $N_{K/k}(g)\in G(k)$. We have

$$\begin{aligned}&i[\tilde{f}|\tilde{P}(K)\longmapsto \tilde{G}(K)](\sigma, g)\\ &=m^{-1}|P(K)|^{-1}\sum_{l=0}^{m-1}\sum_{x\in G(K)}\tilde{f}(\sigma, \{x^\sigma g x^{-1}\}^{\sigma^l})\\ &=|P(K)|^{-1}\sum_{x\in G(K)}\tilde{f}(\sigma, x^\sigma g x^{-1})\\ &=|P(K)|^{-1}|Z_{G(K),\sigma}(g)|\sum_y \tilde{f}(\sigma, y),\end{aligned}$$

where the summation is over $g^{G(K),\sigma}\cap P(K)$. Now, the set $g^{G(K),\sigma}\cap P(K)$ is a disjoint union of a finite number of σ-twisted conjugacy classes in $P(K)$. Set $g^{G(K),\sigma}\cap P(K)=\bigcup_{j=1}^{c}y_j^{P(K),\sigma}$ (disjoint union), where $y_j\in P(K)$. We may assume that $N_{K/k}(y_j)\in P(k)$ $(j=1,\cdots,c)$. By the first part of Lemma 2-6, we have

$$N_{K/k}(g)^{G(k)}\cap P(k)=\bigcup_{j=1}^{c}N_{K/k}(y_j)^{P(k)} \qquad \text{(disjoint union).}$$

Since $\tilde{f}(\sigma, x)=f(N_{K/k}(x))$ $(\forall x\in\tilde{P}(K))$, we have, by the second part of Lemma 2-6,

$$\begin{aligned}&|P(K)|^{-1}\sum_x \tilde{f}(\sigma, x) \quad (\text{the summation is over } g^{G(K),\sigma}\cap P(K))\\ &=|P(K)|^{-1}\sum_{j=1}^{c}|y_j^{P(K),\sigma}|f(N_{K/k}(y_j))=|P(k)|^{-1}\sum_{j=1}^{c}|N_{K/k}(y_j)^{P(k)}|f(N_{K/k}(y_j))\\ &=|P(k)|^{-1}\sum_x f(x) \quad (\text{the summation is over } N_{K/k}(g)^{G(k)}\cap P(k)).\end{aligned}$$

Since $|Z_{G(K),\sigma}(g)|=|Z_{G(k)}(N_{K/k}(g))|$, we have

$$i[\tilde{f}\,|\,\tilde{P}_\mu(K)\longmapsto\tilde{G}(K)](\sigma, g)$$
$$=i[f\,|\,P_\mu(k)\longmapsto G(k)](N_{K/k}(g))\,.$$

q. e. d.

REMARK. The above lemma still holds if the subgroup P_μ is replaced by a linear algebraic subgroup H of GL_n defined over k for which Lemma 2-4 and Lemma 2-6 hold. Any connected abelian algebraic subgroup of GL_n defined over k provides an example of such H.

DEFINITION 2-4. For an unordered partition $\mu=(n_1, \cdots, n_r)$ of n and for class functions f_i (resp. $\tilde{f}_i$) of $GL_{n_i}(k)$ (resp. $\widetilde{GL}_{n_i}(K)$) ($i=1, 2, \cdots, r$), we set

$$f_1\circ\cdots\circ f_r=i[f\,|\,P_\mu(k)\longmapsto GL_n(k)]$$
$$(\text{resp. } \tilde{f}_1\circ\cdots\circ\tilde{f}_r=i[\tilde{f}\,|\,\tilde{P}_\mu(K)\longmapsto\widetilde{GL}_n(K)])\,,$$

where f (resp. $\tilde{f}$) is the class function on $P_\mu(k)$ (resp. $\tilde{P}_\mu(K)$) given by $f(A)=f_1(A_{11})\cdots f_r(A_{rr})$ (resp. $\tilde{f}((\tau, A))=\tilde{f}_1((\tau, A_{11})\cdots\tilde{f}_r((\tau, A_{rr})))$ for

$$A=\begin{bmatrix} A_{11} & & & * \\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{rr}\end{bmatrix}.$$

We note that if the class function f_i (resp. $\tilde{f}_i$) is a character of $GL_{n_i}(k)$ (resp. $\widetilde{GL}_{n_i}(K)$) for $i=1, 2, \cdots, r$, then $f_1\circ\cdots\circ f_r$ (resp. $\tilde{f}_1\circ\cdots\circ\tilde{f}_r$) is also a character of $GL_n(k)$ (resp. $\widetilde{GL}_n(K)$).

The next Lemma 2-9 is an immediate consequence of Lemma 2-8.

LEMMA 2-9. *Let $\mu=(n_1, \cdots, n_r)$ be a partition of n and let f_i (resp. $\tilde{f}_i$) be a class function of $GL_{n_i}(k)$ (resp. $\widetilde{GL}_{n_i}(K)$) ($1\leqq i\leqq r$). If $\tilde{f}_i(\sigma, x)=f_i(N_{K/k}x)$ for $\forall x\in GL_{n_i}(K)$ and for $i=1, 2, \cdots, r$, then $\tilde{f}_1\circ\cdots\circ\tilde{f}_r(\sigma, x)=f_1\circ\cdots\circ f_r(N_{K/k}(x))$ ($\forall x\in GL_n(K)$).*

Let $\bar{k}$ be the algebraic closure of k. Take a character $\theta:\bar{k}^\times\to\boldsymbol{C}^\times$ of the multiplicative group of $\bar{k}$ so that, for any finite extension L of k, the restriction of θ to $L^\times$ is a generator of the character group of $L^\times$.

DEFINITION 2-5. Denote by Σ_r^l ($1\leqq r\leqq n$, $l\in\boldsymbol{Z}$) the function on $GL_n(\bar{k})$ given by $\Sigma_r^l(x)=\sum_{1\leqq i_1<i_2<\cdots<i_r<n}\theta^l(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r})$, where $\lambda_1, \cdots, \lambda_n$ are latent roots of x.

By Theorem 1 of [2], for any finite extension L of k, the restriction of Σ_r^l to $GL_n(L)$ is a generalized character of $GL_n(L)$.

For the proof of the next very deep result, see Theorem 5 (with its proof), Theorem 12 and Theorem 13 of Green [2].

LEMMA 2-10 (J. A. Green). *Each irreducible character of $GL_n(k)$ is a suitable integral linear combinations of Σ_r^l ($1\leqq r\leqq n$, $l\in\boldsymbol{Z}$) and characters of the* *form* $f_1 \circ f_2 \circ \cdots \circ f_r$, *where* f_i *is a character of* $GL(n_i)$ $(1 \leqq i \leqq r,\ n = n_1 + \cdots + n_r,\ 0 < n_i < n)$ *and* $\mu = (n_1, \cdots, n_r)$ *is a partition of* n *different from* (n).

LEMMA 2-11. *For each irreducible character* χ *of* $GL_n(k)$, *there exists an irreducible character* $\tilde{\chi}$ *of* $\widetilde{GL_n(K)}$ *which satisfies* $\varepsilon\tilde{\chi}(\sigma, x) = \chi(N_{K/k}(x))$ $(\forall x \in GL_n(K))$, *where* $\varepsilon = \pm 1$ *and is independent of* x.

PROOF. We use induction with respect to n. For $n=1$, the lemma is well-known. We assume that the lemma has been established for smaller values of n. By Lemma 2-10, any irreducible character of $GL_n(k)$ is a suitable *integral* linear combination of generalized characters Σ_r^l $(l \in \boldsymbol{Z},\ 1 \leqq r \leqq n)$ and characters of the form $f_1 \circ \cdots \circ f_r$, where f_i is a character of $GL_{n_i}(k)$ and $\mu = (n_1, \cdots, n_r)$ $(0 < n_1, n_2, \cdots, n_r < n)$. Since $n_i < n$ $(1 \leqq i \leqq r)$, there exist, by the induction hypothesis, an irreducible character $\tilde{f}_i$ of $\widetilde{GL_{n_i}(K)}$ and $\varepsilon_i = \pm 1$ which satisfy $\varepsilon_i \tilde{f}_i(\sigma, x) = f_i(N_{K/k}(x))$ $(x \in GL_{n_i}(K))$. Hence, by Lemma 2-9, $\varepsilon_1 \cdots \varepsilon_r \tilde{f}_1 \circ \tilde{f}_2 \circ \cdots \circ \tilde{f}_r(\sigma, x) = f_1 \circ \cdots \circ f_r(N_{K/k}x)$. By Theorem 1 of [2] and by Lemma 1-4, there exists a generalized character X_r^l of $\widetilde{GL}_n(K)$ which satisfies $X_r^l(\sigma, x) = \Sigma_r^l(N_{K/k}(x))$ $(\forall x \in GL_n(K))$. Since χ is an integral linear combinations of Σ_r^l and $f_1 \circ f_2 \circ \cdots \circ f_r$, there exists a generalized character $\tilde{\chi}$ of $\widetilde{GL}_n(K)$ which satisfies $\tilde{\chi}(\sigma, x) = \chi(N_{K/k}(x))$ $(\forall x \in GL_n(K))$. Let $\{R_1, \cdots, R_s\}$ be the set of all the σ-invariant mutually inequivalent irreducible representations of $GL_n(K)$. For each i, take an extension $\tilde{R}_i$ of R_i to a representation of $\widetilde{GL}_n(K)$ and set $\xi_i(x) = \text{trace}\ \tilde{R}_i(\sigma, x)$. By Lemma 1-1 and Lemma 1-3, there exist $c_1, \cdots, c_s \in \boldsymbol{Z}[\zeta_m]$ $\left(\zeta_m = \exp \frac{2\pi\sqrt{-1}}{m}\right)$ which satisfy $\tilde{\chi}(\sigma, x) = \sum_{i=1}^{s} c_i \xi_i(x)$. By Lemma 1-1 and Lemma 1-2, we have

$$|GL_n(K)|^{-1} \sum_{x \in GL_n(K)} \xi_i(x) \bar{\xi}_j(x) = \delta_{ij}\,.$$

On the other hand, it follows from Corollary to Lemma 2-6 that,

$$\begin{aligned} |GL_n(K)|^{-1} \sum_{x \in GL_n(K)} |\tilde{\chi}(\sigma, x)|^2 &= |GL_n(K)|^{-1} \sum_{x \in GL_n(K)} |\chi(N_{K/k}(x))|^2 \\ &= |GL_n(k)|^{-1} \sum_{x \in GL_n(k)} |\chi(x)|^2 = 1\,. \end{aligned}$$

Thus $1 = \sum_{i=1}^{s} |c_i|^2$. Denote by $\mathfrak{G}$ the Galois group of $\boldsymbol{Q}(\zeta_m)$ with respect to $\boldsymbol{Q}$. Since the complex conjugation is an element of $\mathfrak{G}$ and since $\mathfrak{G}$ is abelian, we have $1 = \sum_{i=1}^{s} c_i^{\tau} \bar{c}_i^{\tau}$ $(\forall \tau \in \mathfrak{G})$. Setting $d = |\mathfrak{G}|$, we have

$$d = \sum_{i=1}^{s} \sum_{\tau \in \mathfrak{G}} c_i^{\tau} \bar{c}_i^{\tau}\,.$$

Since $c_i \in \boldsymbol{Z}[\zeta_m]$, if $c_i \neq 0$,

$$\sum_{\tau \in \mathfrak{G}} c_i^{\tau} \bar{c}_i^{\tau} \geqq d \sqrt[d]{|\prod_{\tau \in \mathfrak{G}} c_i^{\tau}|^2} \geqq d$$

and the equality holds if and only if $|c_i^{\tau}|=1$ for arbitrary $\tau\in\mathfrak{G}$. Hence $c_i=0$ except for a single index $i=i_0$ and c_{i_0} is a root of unity in $\boldsymbol{Z}[\zeta_m]$. Thus, $\tilde{\chi}(\sigma, x)=\pm\zeta\xi_{i_0}(x)=\pm\zeta\,\mathrm{trace}\,R_{i_0}(\sigma, x)$ for a suitable m-th root of unity ζ (if m is even, we may remove the sign $\pm$). By Lemma 1-3, there exists an irreducible representation $\tilde{R}'$ of $\widetilde{GL}_n(K)$ which satisfies $\mathrm{trace}\,\tilde{R}'(\sigma, x)=\zeta\,\mathrm{trace}\,\tilde{R}_{i_0}(\sigma, x)$. The proof of Lemma 2-12 is now complete. q. e. d.

The following theorem is an immediate consequence of Lemma 2-11 and Lemma 2-7.

THEOREM 1. *For a given σ-invariant irreducible representation R of $GL_n(K)$, there exist a linear transformation I_σ of the representation space of R and an irreducible character χ_R of $GL_n(k)$ which satisfy*

$$\begin{cases} R(g^\sigma)=I_\sigma^{-1}R(g)I_\sigma\,, \\ \mathrm{trace}\,I_\sigma R(g)=\chi_R(N_{K/k}(g)) \qquad (\forall g\in GL_n(K)) \end{cases}$$

(I_σ and χ_R are determined uniquely by R, $I_\sigma^m=\pm1$). Furthermore the mapping $R\mapsto\chi_R$ is a bijection from the set of equivalence classes of σ-invariant irreducible representations of $GL_n(K)$ onto the set of irreducible characters of $GL_n(k)$.

REMARK. Under the assumptions of Theorem 1, *if m is relatively prime* to $|GL_n(K)|$, Theorem 3 of Glauberman [8] implies that there exist a linear transformation J of the representation space of R and an irreducible character λ of $GL_n(k)$ which satisfy the following equalities:

$$\begin{cases} R(g^\sigma)=J^{-1}R(g)J \qquad (\forall g\in GL_n(K))\,, \\ \mathrm{trace}\,JR(g)=\lambda(g) \qquad (\forall g\in GL_n(k))\,. \end{cases}$$

Since m is relatively prime to $|GL_n(k)|$, there exists an irreducible character χ_R of $GL_n(k)$ which satisfies

$$\lambda(g)=\chi_R(g^m)=\chi_R(N_{K/k}(g)) \qquad (\forall g\in GL_n(k))\,.$$

Thus, *if m is relatively prime to* $|GL_n(K)|$, Theorem 1 is an immediate consequence of the Glauberman theorem.

§ 3.

1°. We denote by B_n the group of $n\times n$ upper triangular matrices. Namely, $B=P_\mu$ for $\mu=(1^n)=(\underbrace{1, 1, \cdots, 1}_{n})$ (for notation see § 2.2).

Let D_n (resp. U_n) be the group of diagonal (resp. upper triangular unipotent) matrices in GL_n. Then both D_n and U_n are algebraic subgroups of B_n defined over k. Moreover, $B=D_nU_n$ (semidirect product). We denote by

$X(D_n(K))$ the character group of $D_n(K)$. For each $\chi \in X(D(K))$, there are characters $\chi_1, \cdots, \chi_n$ of $K^\times = GL_1(K)$ such that $\chi\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = \chi_1(t_1)\chi_2(t_2)\cdots\chi_n(t_n)$ $\left(\forall \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \in D_n(K)\right)$. We write $\chi = (\chi_1, \chi_2, \cdots, \chi_n)$.

Let W_n be the group of $n \times n$ permutation matrices. The group W_n normalizes D_n and acts on $X(D_n(K))$ in a natural manner. For $w \in W_n$ and $\chi \in X(D_n(K))$, we denote by χ^w the character of $D_n(K)$ given by $\chi^w(t) = \chi(wtw^{-1})$ $(\forall t \in D_n(K))$. A character χ of $D_n(K)$ is said to be *regular* if $\chi^w \neq \chi$ for any $1 \neq w \in W_n$. Any character of $D_n(K)$ is uniquely extended to a linear character of $B_n(K)$. The extension (which we denote by the same letter χ) is given by $\chi(b) = \prod_{i=1}^{n} \chi_i(b_{ii})$, where b_{ii} is the i-th diagonal entry of b. For a character χ of $D_n(K)$, we denote by R_χ the representation of $GL_n(K)$ induced from the linear character χ of $B_n(K)$. The representation space V_χ of R_χ is the space of complex valued functions f on $GL_n(K)$ which satisfies $f(bx) = \chi(b)f(x)$ $(\forall b \in B_n(K))$. The representation R_χ is given by $\{R_\chi(g)f\}(x) = f(xg)$. For $\chi = (\chi_1, \cdots, \chi_n)$, trace $R_\chi(g) = \chi_1 \circ \chi_2 \circ \cdots \circ \chi_n(g)$ (for notations, see Definition 2-4).

The following lemma is well-known (see e.g. Theorem 4.7 of [**5**]).

LEMMA 3-1. (i) *Notations being as above, the representation R_χ is irreducible if and only if the character χ of $D_n(K)$ is regular.*

(ii) *For two regular characters χ_1 and χ_2 of $D_n(K)$, the representations R_{χ_1} and R_{χ_2} are equivalent if and only if $\chi_2 = \chi_1^w$ for some $w \in W_n$.*

Denote by U^- the group of $n \times n$ unipotent lower triangular matrices. The group U^- is an algebraic subgroup of GL_n defined over k. For each $w \in W_n$, set

$$U_w^+ = U \cap w^{-1}Uw \quad \text{and} \quad U_w^- = U \cap w^{-1}U^-w\,.$$

Both U_w^+ and U_w^- are algebraic subgroups of U. Moreover, the following formulas hold (see Proposition 3.3 of [**4**]):

$$U(K) = U_w^-(K)U_w^+(K), \quad U_w^-(K) \cap U_w^+(K) = 1, \quad G(K) = \bigcup_{w \in W} U_w^-(K)w^{-1}B(K)$$

(disjoint union).

For $w \in W_n$, we denote by $I_w(\chi)$ the linear mapping from V_χ into V_{χ^w} given as follows:

$$(I_w(\chi)f)(g) = \sum_{u \in U_w^-} f(wug) \qquad (f \in V_\chi)\,.$$

It is obvious that $I_w(\chi)$ commutes with the action of $GL_n(K)$. If w is of order r, $I_w(\chi^{w^{r-1}}) \cdots I_w(\chi)$ is a linear transformation of V_χ.

LEMMA 3-2. *Let notations be as above. If χ is a regular character,*

$$I_w(\chi^{w^{r-1}})I_w(\chi^{w^{r-2}}) \cdots I_w(\chi) = c \cdot 1\,, \tag{3-1}$$

where

$$c=\prod_{i=1}^{r}|(wU_{w}^{-}w^{-1}\cap w^{i}U_{w^{i}}^{-}w^{-i})(K)|.$$

PROOF. The left hand side of (3-1) is a linear transformation of V_χ which commutes with the action of $GL_n(K)$ through the representation R_χ. Hence it is a scalar multiplication, as R_χ is irreducible by Lemma 3-1 (i). Denote by f_z^χ $(z\in W_n)$ a function on $GL_n(K)$ given as follows:

$$f_z^\chi(g)=\begin{cases}\chi^z(b) & \text{if } g=uzb\ (u\in U,\ b\in B)\\ 0 & \text{if } g\notin UzB.\end{cases}$$

Then f_z^χ is an element of V^χ and satisfies $R_\chi(b)f_z^\chi=\chi^z(b)f_z^\chi$ $(^\forall b\in B(K))$. Up to a constant factor, f_z^χ is characterized by this property if χ is regular. Hence, $I_w(\chi)f_z^\chi$ coincides with $f_{w^{-1}z}^{\chi^w}$ up to a constant factor. On the other hand,

$$(I_w(\chi)f_z^\chi)(w^{-1}z)=\sum_{u\in U_w^-}f_z^\chi(wuw^{-1}z).$$

Now it is easy to see that $wuw^{-1}z\in UzB$ if and only if $wuw^{-1}\in zU_z^-z^{-1}$ and that $f_z^\chi(wuw^{-1}z)=1$ if $wuw^{-1}\in zU_z^-z^{-1}$. Thus,

$$(I_w(\chi)f_z^\chi)=|(wU_w^-w^{-1}\cap zU_z^-z^{-1})(K)|f_{w^{-1}z}^{\chi^w}.$$

Hence we conclude that

$$\prod_{i=0}^{r-1}|(wU_w^-w^{-1}\cap w^izU_{w^iz}^-z^{-1}w^{-i})(K)|$$

is independent of $z\in W_n$ and is equal to c in the right hand side of (3.2).

q. e. d.

2°. In the remaining part of this section, we assume that the degree of K over k $(=m)$ is a multiple of n. We denote by k_n the field extension of k of degree n. Then k_n is a subfield of K. A character ξ of the group $k_n^\times=GL_1(k_n)$ is said to be *regular* if $\xi\neq\xi^{\sigma l}$ for $l=1,\cdots,n-1$ (we denote by x^σ the character of $k_n^\times$ given by $x\to\chi(x^\sigma)$). For a character ξ of $k_n^\times$, we denote by $\tilde{\xi}$ a character of $K^\times$ given by $\tilde{\xi}(x)=\xi(N_{K/k_n}(x))$, where N_{K/k_n} is the norm map from K to k_n. Let χ_ξ be a character of $B_n(K)$ given by $\chi_\xi(b)=\prod_{i=1}^{n}\tilde{\xi}^{\sigma i-1}(b_{ii})$, where b_{ii} is the i-th diagonal entry of b. If ξ is a regular character of $k_n^\times$, χ_ξ is a regular character of $D_n(K)$. Set

$$\omega^{-1}=\begin{pmatrix}0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1\\ 1 & & & & 0\end{pmatrix}\in W_n.$$

Denote by ψ_ξ a function on $GL_n(K)$ given as follows:

$$\psi_\xi(g)=\begin{cases} \chi_\xi(b) & \text{if } g=u\omega b\in U(K)\omega B(K) \\ 0 & \text{otherwise}. \end{cases}$$

An irreducible character of $GL_n(k)$ is said to be *cuspidal* if it is orthogonal to any character of the form $f_1\circ f_2$, where f_1 (resp. f_2) is a character of $GL_l(k)$ (resp. $GL_{n-l}(k)$) ($l=1, \cdots, n-1$). (For notations, see Definition 2-4.)

THEOREM 2. *Let notations be as above.*

(i) *For a regular character* ξ *of* $k_n^\times$, *there exists a root of unity* ζ_ξ *in* $\boldsymbol{Q}\left(\exp\frac{2\pi i}{m}\right)$ *and a cuspidal irreducible character* X_ξ *of* $GL_n(k)$ *such that* $q^{-m(n-1)/2}|B_n(K)|^{-1}\sum_{x\in GL_n(K)}\psi_\xi(x^\sigma gx^{-1})=\zeta_\xi X_\xi(N_{K/k}(g))$ ($^\forall g\in GL_n(K)$) ($q=|k|$, $m=\deg(K/k)$, $n|m$).

(ii) *For two regular characters* ξ_1 *and* ξ_2 *of* $k_n^\times$ $X_{\xi_1}=X_{\xi_2}$ *if and only if* $\xi_1=\xi_2{}^{\sigma^l}$ *for some* $l\in\boldsymbol{Z}$. *Moreover, any cuspidal irreducible character of* $GL_n(k)$ *is equal to* X_ξ *for a suitable regular character* ξ *of* $k_n^\times$.

PROOF. Denote by R_ξ the representation of $GL_n(K)$ induced from the character χ_ξ of $B_n(K)$. If ξ is regular, R_ξ is irreducible by Lemma 3-1 (i). The representation space V of R_ξ is the space of all the complex valued functions on $GL_n(K)$ which satisfy $f(bx)=\chi_\xi(b)f(x)$ ($^\forall b\in B_n(K)$). Set

$$(I_\sigma f)(x)=q^{-m(n-1)/2}\sum_{u\in U_{\omega^{-1}}^-}f(\omega^{-1}ux^\sigma).$$

Then, I_σ is a linear transformation of V which satisfies $R_\xi(g)I_\sigma=I_\sigma R_\xi(g^\sigma)$ ($^\forall g\in GL_n(K)$). Moreover, by Lemma 3-2, $I_\sigma^m=1$. Thus R_ξ is a σ-invariant irreducible representation of $GL_n(K)$. Furthermore, the mapping $(\sigma^l, g)\mapsto I_\sigma^l R_\xi(g)$ is an irreducible representation $\tilde{R}_\xi$ of $GL_n(K)$ (for notations, see 2-3°) on V whose restriction to $GL_n(K)$ coincides with R_ξ. It is easy to see that trace $I_\sigma R_\xi(g)=q^{-m(n-1)/2}|B_n(K)|^{-1}\sum_{x\in GL_n(K)}\psi_\xi(x^\sigma gx^{-1})$. By Theorem 1, there exists a root of unity ζ_ξ in $\boldsymbol{Q}\left(\exp\frac{2\pi i}{m}\right)$ and an irreducible character X_ξ of $GL_n(k)$ such that trace $I_\sigma R_\xi(g)=\zeta_\xi X_\xi(N_{K/k}(g))$ ($^\forall g\in GL_n(K)$). We will show that X_ξ is orthogonal to any character of $GL_n(k)$ of the form $f_1\circ f_2$ (see Definition 2-4) where f_i is an irreducible character of $GL_{n_i}(k)$ ($i=1, 2, n_1+n_2=n, 0<n_1, n_2$). By Theorem 1, there exists an irreducible representation $\tilde{R}_i$ of $GL_{n_i}(K)$ on V_i which satisfies ε_i trace $\tilde{R}_i(\sigma, x)=f_i(N_{K/k}(x))$ ($^\forall x\in GL_{n_i}(K)$ $\varepsilon_i=\pm1$, $i=1, 2$). The restriction R_i of $\tilde{R}_i$ to $GL_{n_i}(K)$ is a *σ-invariant* irreducible representation of $GL_{n_i}(K)$ on V_i ($i=1, 2$). Set $\mu=(n_1, n_2)$. Denote by $R_1\circ R_2$ (resp. $\tilde{R}_1\circ\tilde{R}_2$) the representation of $GL_n(K)$ (resp. $\widetilde{GL}_n(K)$) induced from the representation of $P_\mu(K)$ (resp. $\tilde{P}_\mu(K)$) on $V_1\otimes V_2$ given by

$$\begin{pmatrix} A_{11} & A_{12} \\ & A_{22} \end{pmatrix} \longmapsto R(A_{11}) \otimes R(A_{22})$$

$$(\text{resp. } (\tau, \begin{pmatrix} A_{11} & A_{12} \\ & A_{22} \end{pmatrix}) \longmapsto \tilde{R}_1(\tau, A_{11}) \otimes \tilde{R}_2(\tau, A_{22}))\,.$$

It is easy to see that the restriction of $\tilde{R}_1 \circ \tilde{R}_2$ to $GL_n(K)$ is equivalent to $R_1 \circ R_2$. It follows from Lemma 2-9 that $\varepsilon_1\varepsilon_2$ trace $\tilde{R}_1 \circ \tilde{R}_2((\sigma, g)) = f_1 \circ f_2(N_{K/k}(g))$ $({}^{\forall}g \in GL_n(K))$.

By Corollary to Lemma 2-6 and Lemma 1-2, to prove that the character X_ξ is orthogonal to $f_1 \circ f_2$, it is sufficient to show that the representation R_ξ is not among the irreducible components of the representation $R_1 \circ R_2$. Assume that R_ξ were an irreducible component of $R_1 \circ R_2$. Then it is easy to see that, for suitable subsequence $\{i_1, i_2, \cdots, i_{n_1}\}$ of $\{0, 1, \cdots, n-1\}$, R_1 would be equivalent to the representation of $GL_{n_1}(K)$ induced from the character $b \to \prod_{k=1}^{n_1} \tilde{\xi}^{\sigma^{i_k}}(b_{kk})$ of $B_{n_1}(K)$ (b_{kk} is the k-th diagonal entry of b). But this is impossible, since the latter representation is never σ-invariant (see Lemma 3-1) while R_1 is σ-invariant. Thus, X_ξ is a cuspidal irreducible character of $GL_n(k)$. It follows immediately from Lemma 1-2 and Lemma 3-1 that $X_{\xi_1} = X_{\xi_2}$ if and only if $\xi_1 = \xi_2^{\sigma^l}$ for a suitable $l \in \boldsymbol{Z}$. On the other hand, it is known (see Theorem 8-6 of [6]) that the number of cuspidal irreducible characters of $GL_n(k)$ is equal to the number of orbits of the Galois group of k_n with respect to k in the set of regular characters of $k_n^\times$. The proof of our theorem is now complete.

§ 4.

1°. We recall the explicit description of irreducible representations of $GL_2(k)$. For a character χ of $k^\times$, we denote by L_χ the one dimensional representation of $GL_2(k)$ given by $L_\chi(g) = \chi(\det g)$. For a pair (χ_1, χ_2) of characters of $k^\times$, we denote by $R_{(\chi_1,\chi_2)}$ the representation of $GL_2(k)$ induced from the one dimensional character $\begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$ of $B_2(k)$, the group of upper triangular matrices. The representation space $V_{(\chi_1,\chi_2)}$ is the space of complex valued functions on $GL_2(k)$ which satisfy

$$f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} x\right) = \chi_1(a)\chi_2(d) f(x) \qquad \left({}^{\forall}\begin{pmatrix} a & b \\ & d \end{pmatrix} \in B_2(k)\right).$$

The representation $R_{(\chi_1,\chi_2)}$ is given by $\{R_{(\chi_1,\chi_2)}(g)f\}(x) = f(xg)$. Let $V'_{(\chi,\chi)}$ be the subspace of $V_{(\chi,\chi)}$ given by

$$V'_{(\chi,\chi)}=\{f\in V_{(\chi,\chi)},\ \sum_{x\in G}\chi^{-1}(\det x)f(x)=0\}.$$

Then $V'_{(\chi,\chi)}$ is invariant under the action of $R_{(\chi,\chi)}(g)$. Denote by S_χ the subrepresentation of $R_{(\chi,\chi)}$ with the representation space $V'_{(\chi,\chi)}$. Let ξ be a character of $k_2^\times$ (k_2 is the quadratic extension of k). We assume that $\xi^q\neq\xi$ ($q=|k|$). We choose a non-trivial character χ of the additive group k. We denote by ρ_ξ the representation of $GL_2(k)$ on the space of complex valued functions on $k^\times$ given by

$$(\rho_\xi(g)f)(t)=\sum_{u\in k^\times}K_g(t,u)f(u),$$

where

$$K_g(t,u)=\begin{cases}\xi(d)\chi\Big(\dfrac{b}{d}u\Big)\delta\Big(t-\dfrac{a}{d}u\Big), & \text{for } g=\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\\[2ex] -\dfrac{1}{q}\chi\Big(\dfrac{at+du}{c}\Big)\displaystyle\sum_{xx^q=tu(ad-bc)}\chi\Big(-\dfrac{x+x^q}{c}\Big)\xi(xu^{-1}), & \\ \qquad\qquad\text{for } g=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\ c\neq 0, & \end{cases}$$

where we put $\delta(x)=\begin{cases}0 & x\neq 0\\ 1 & x=0\end{cases}$ (see Chap. 2, § 4, 1 of [1]). The representations L_χ and S_χ are always irreducible. By Lemma 3-1 the representation $R_{(\chi_1,\chi_2)}$ is irreducible if and only if $\chi_1\neq\chi_2$. Two representations $R_{(\chi_1,\chi_2)}$ and $R_{(\chi_1',\chi_2')}$ are equivalent if and only if $(\chi_1,\chi_2)=(\chi_1',\chi_2')$ or $(\chi_1,\chi_2)=(\chi_2',\chi_1')$.

The representation ρ_ξ $(\xi\neq\xi^q)$ is always irreducible. Two representations ρ_{ξ_1} and ρ_{ξ_2} are equivalent if and only if $\xi_1=\xi_2$ or $\xi_1^q=\xi_2$. Moreover, any irreducible representation of $GL_2(k)$ is equivalent to some of L_χ, S_χ, $R_{(\chi_1,\chi_2)}$ and ρ_ξ. Further, any irreducible representation of $GL(2,k)$ with a cuspidal character is equivalent to some of ρ_ξ.

2°. Let K be the field extension of k of degree m and let σ be the Frobenius automorphism of K with respect to k. Let λ be a character of $K^\times$. The one-dimensional representation L_λ of $GL_2(K)$ is σ-invariant (cf. Definition 1-2) if and only if the character λ is σ-invariant. Assume λ is σ-invariant. There exists a character χ of $k^\times$ which satisfies $\lambda(x)=\chi(N_{K/k}(x))=\chi(xx^\sigma\cdots x^{\sigma^{m-1}})$ $(^\forall x\in K^\times)$. It is obvious that $L_\lambda(x)=L_\chi(N_{K/k}(x))$ $(^\forall x\in GL_2(K))$. For a character λ of $K^\times$, the representation S_λ of $GL_2(K)$ is σ-invariant if and only if λ is σ-invariant. Assume λ is σ-invariant and set $\lambda(x)=\chi(N_{K/k}(x))$ (χ is a character of $k^\times$). Denote by I_σ the linear transformation of $V'_{(\lambda,\lambda)}$ given by $(I_\sigma f)(x)=f(x^\sigma)$. Then it is easy to see that

$$S_\lambda(g)I_\sigma=I_\sigma S_\lambda(g^\sigma)\qquad(^\forall g\in GL_2(K))$$

and that

$$\operatorname{trace} I_\sigma S_\lambda(g)=\operatorname{trace} S_\chi(N_{K/k}(g)).$$

Let λ_1 and λ_2 be two mutually distinct characters of $K^\times$. If both λ_1 and λ_2 are σ-invariant, the representation $R_{(\lambda_1,\lambda_2)}$ is σ-invariant. Set $\lambda_1(x)=\chi_1(N_{K/k}(x))$ and $\lambda_2=\chi_2(N_{K/k}(x))$, where χ_1 and χ_2 are characters of $k^\times$. Denote by I_σ the linear transformation of $V_{(\lambda_1,\lambda_2)}$ given by

$$(I_\sigma f)(x)=f(x^\sigma) \qquad (f\in V_{(\lambda_1,\lambda_2)},\ x\in GL_2(K)).$$

It is easy to see that

$$R_{(\lambda_1,\lambda_2)}(g)I_\sigma=I_\sigma R_{(\lambda_1,\lambda_2)}(g^\sigma) \qquad (^\forall g\in GL_2(K)).$$

It follows from Lemma 2-9 that

$$\text{trace}\ I_\sigma R_{(\lambda_1,\lambda_2)}(g)=\text{trace}\ R_{(\chi_1,\chi_2)}(N_{K/k}(g)) \qquad (^\forall g\in GL_2(K)).$$

3°. If m, the degree of K with respect to k is odd, the representation $R_{(\lambda_1,\lambda_2)}$ is σ-invariant only if both λ_1, λ_2 are σ-invariant. However if m is even, $R_{(\lambda_1,\lambda_2)}$ ($\lambda_1\neq\lambda_2$; λ_1 and λ_2 are not σ-invariant) is σ-invariant if $(\lambda_1^\sigma, \lambda_2^\sigma)=(\lambda_2, \lambda_1)$. If $(\lambda_1^\sigma, \lambda_2^\sigma)=(\lambda_2, \lambda_1)$ there exists a character ξ of $k_2^\times$ (k_2 is the quadratic extension of k) which satisfies

$$\text{(4.1)} \qquad \begin{cases} \lambda_1(x)=\xi(N_{K/k_2}(x)) \\ \lambda_2(x)=\xi(N_{K/k_2}(x^\sigma)). \end{cases} \qquad (^\forall x\in K^\times)$$

Denote by I_σ the linear transformation of $V_{(\lambda_1,\lambda_2)}$ given by

$$(I_\sigma f)(x)=q^{-m/2}\sum_{u\in K} f\left(\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix} 1 & u\\ & 1\end{pmatrix}x^\sigma\right).$$

By Theorem 2, $R_{(\lambda_1,\lambda_2)}(g)I_\sigma=I_\sigma R_{(\lambda_1,\lambda_2)}(g^\sigma)$.

PROPOSITION 4-1. *Let notations be as above. We have*

$$\text{trace}\ I_\sigma R_{(\lambda_1,\lambda_2)}(g)=\zeta_\xi\ \text{trace}\ \rho_\xi(N_{K/k}(g)),$$

where

$$\zeta_\xi=\xi(-1)^{m/2}q^{1-m/2}(q-1)^{-1}\sum_y(\xi/\xi^\sigma)(N_{K/k_2}(y))$$

(*the summation is over the set* $\{y\in K^\times;\ \text{trace}_{K/k}y=0\}$).

PROOF. Denote by U_ξ the space of complex valued functions F on K^2 which satisfy

$$F(tx_1, t^{-1}x_2)=(\lambda_1^{-1}\lambda_2)(t)f(x_1, x_2) \qquad (^\forall t\in K^\times).$$

Take a non-trivial character χ of the additive group k and set

$$\tilde\chi(x)=\chi(\text{trace}_{K/k}x) \qquad \text{for}\quad x\in K\ (\text{trace}_{K/k}x=x+x^\sigma+\cdots+x^{\sigma^{m-1}}).$$

Then $\tilde\chi$ is a non-trivial σ-invariant additive character of K.

Let $\pi_{(\lambda_1,\lambda_2)}$ be the representation of $GL_2(K)$ on U_ξ given as follows:

$$\{\pi_{(\lambda_1,\lambda_2)}(g)F\}(x_1, x_2) \qquad \left(g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)$$

$$=\begin{cases} \lambda_1(ad)\tilde{\chi}\left(\frac{b}{d}x_1x_2\right)F(x_1a, x_2d^{-1}) \quad \text{for} \quad c=0, \\ \lambda_1(\det g)q^{-m}\sum_{(y_1,y_2)\in K^2}\tilde{\chi}\left(\frac{ax_1x_2-(\det g x_1y_2+x_2y_1)+dy_1y_2}{c}\right) \\ \qquad\qquad \times F(y_1, y_2) \quad \text{for} \quad c\neq 0. \end{cases}$$

Let T be a linear mapping from U_ξ into $V_{(\lambda_1,\lambda_2)}$ given by

$$(TF)(g)=\lambda_1(\det g)\sum_{y\in K}F(c, y)\tilde{\chi}(yd) \qquad \text{for} \quad g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

It is easy to see (and is well-known) that T is a linear isomorphism which satisfies

$$T\pi_{(\lambda_1,\lambda_2)}(g)=R_{(\lambda_1,\lambda_2)}(g)T \qquad (^\forall g\in GL(2, K)).$$

Further set

$$(J_\sigma F)(x_1, x_2)=F(x_2^\sigma, x_1^\sigma) \qquad \text{for} \quad F\in U_\xi.$$

It follows from (4.1) that J_σ is a linear isomorphism of U_ξ which satisfies

$$\pi_{(\lambda_1,\lambda_2)}(g)J_\sigma=J_\sigma\pi_{(\lambda_1,\lambda_2)}(g^\sigma) \qquad (^\forall g\in GL_2(K)).$$

Hence, it is easy to see that $I_\sigma T=cTJ_\sigma$, where

$$c=\lambda_1(-1)q^{-m/2}\sum_{y\in K^\times}(\lambda_1\lambda_2^{-1})(y)\tilde{\chi}(y)=\zeta_\xi.$$

Thus trace $I_\sigma R_{(\lambda_1,\lambda_2)}(g)=\zeta_\xi$ trace $J_\sigma\pi_{(\lambda_1,\lambda_2)}(g)$.

By easy computations, we have

$$\operatorname{trace} J_\sigma\pi_{(\lambda_1,\lambda_2)}(g)=\begin{cases} \xi(N_{K/k}a)(q-1) \quad \text{for} \quad g=\begin{pmatrix} a & \\ & a\end{pmatrix}, \\ 0 \quad \text{for} \quad g=\begin{pmatrix} a & \\ & d\end{pmatrix}, \quad N_{K/k}(a/d)\neq 1, \\ -\xi(N_{K/k}a) \quad \text{for} \quad g=\begin{pmatrix} a & b \\ & a\end{pmatrix}, \quad tr_{K/k}(b/a)\neq 0. \end{cases}$$

If $g=\begin{pmatrix} & b \\ 1 & \end{pmatrix}$ $(N_{K/k_2}(b)\notin k)$,

$$\operatorname{trace} J_\sigma\pi_{(\lambda_1,\lambda_2)}(g)=\frac{\lambda_1(-b)}{q^m(q^m-1)}\sum_{\substack{(y_1,y_2)\in K^2 \\ t\in K^\times}}\tilde{\chi}(by_2y_2^\sigma t^{-1}-y_1y_1^\sigma t)\lambda_1(t^{-1}t^\sigma).$$

It is easy to verify that, for $t\in K^\times$,

$$\sum_{x\in K}\tilde{\chi}(txx^\sigma)=\begin{cases} (-q)^{m/2} \quad \text{if} \quad N_{K/k_2}(-t/t^\sigma)\neq 1, \\ (-q)^{m/2-1}q^2 \quad \text{if} \quad N_{K/k_2}(-t/t^\sigma)=1. \end{cases}$$

Thus

$$\operatorname{trace} J_\sigma \pi_{(\lambda_1,\lambda_2)}(g) = -\xi(N_{K/k_2}(b)) - \xi(N_{K/k_2}(b^\sigma)) \qquad \text{for} \quad g = \begin{pmatrix} & b \\ 1 & \end{pmatrix}.$$

Hence $\operatorname{trace} J_\sigma \pi_{(\lambda_1,\lambda_2)}(g) = \operatorname{trace} \rho_\xi(N_{K/k}(g))$.

4°. Let K_2 be the quadratic extension of K and let η be a character of $K_2^\times$ ($\eta^{\sigma^m} \neq \eta$). If $m = \deg(K/k)$ is even, the irreducible representation ρ_η of $GL_2(K)$ is never σ-invariant. If m is odd, ρ_η is σ-invariant if and only if there exists a character ξ of $k_2^\times$ ($\xi^\sigma \neq \xi$) which satisfies

$$\eta(x) = \xi(N_{K_2/k_2}(x)) \qquad ({}^\forall x \in K_2).$$

In this case ($\eta = \xi \circ N_{K_2/k_2}$) denote by I_σ the linear transformation of the space of complex valued functions on $K^\times$ given by $(I_\sigma f)(x) = f(x^\sigma)$. Then it is easy to see that $\rho_\eta(g) I_\sigma = I_\sigma \rho_\eta(g^\sigma)$.

PROPOSITION 4-2. *Notations being as above,*

$$\operatorname{trace} I_\sigma \rho_\eta(g) = \operatorname{trace} \rho_\xi(N_{K/k}(g)).$$

PROOF. Omitted.

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Takuro SHINTANI
Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Bunkyo-ku
Tokyo, Japan