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C^{∞}-approximation of continuous ovals of constant width

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§ 1. Introduction.

Let M be an oval (i. e., a closed convex curve) in a Euclidean 2-space E^2. For a point x of M a straight line l passing through x is called a supporting line at x if M is contained in one of the half planes determined by l. If M is a C^1-curve, then tangent lines are supporting lines. M is said to have constant width, if the distance between each pair of parallel supporting lines is constant. Examples of continuous ovals of constant width are Reuleaux triangles, Sallee constructions (cf. [7], and also B. B. Peterson [4]), and so on.

We prove C^{∞}-approximation theorem:

THEOREM A. *Let M be a continuous oval of constant width H in E^2. Then, for any positive number δ, we can construct a C^{∞}-oval $M^{\#}$ of constant width H in the δ-neighborhood of M in E^2.*

THEOREM B. *In Theorem A, if M is symmetric with respect to a straight line m in E^2, then $M^{\#}$ can be constructed so that $M^{\#}$ is symmetric with respect to m.*

A generalization of an oval of constant width to higher dimension is a hypersurface of constant width in a Euclidean n-space E^n. If M is a continuous oval of constant width in $E^2 \subset E^n$, which is symmetric with respect to the x^1-axis, then one gets a continuous hypersurface of constant width in E^n as its revolution hypersurface with respect to the x^1-axis in E^n.

By Theorem B we obtain

THEOREM C. *If a continuous hypersurface M of constant width H is a revolution hypersurface in E^n, then for any positive number δ, we can construct a revolution C^{∞}-hypersurface $M^{\#}$ of constant width H in the δ-neighborhood of M in E^n.*

In the last section we mension about twin hypersurfaces which are generalizations of hypersurface of constant width.

§ 2. Preliminaries.

Let E^2 be a Euclidean 2-space with the natural coordinates (x^1, x^2). Let $M=\{x(s)\}$ be a C^3-curve (i.e., continuously thrice differentiable curve) in E^2 with arc-length parameter s. $\xi_1(s)=dx(s)/ds$ is the unit tangent vector field on $\{x(s)\}$. Then $d\xi_1(s)/ds=k(s)\xi_2(s)$ holds, where $k(s)$ is the curvature of $x(s)$ and $(\xi_1(s), \xi_2(s))$ is the right handed orthonormal frame field on $\{x(s)\}$. A curve $^*M=\{^*y(s)\}$ defined by

$$^*y(s)=x(s)+\rho(s)\xi_2(s) \tag{2.1}$$

is called the evolute of $M=\{x(s)\}$, where $\rho(s)=1/k(s)$. The curvature $^*k$ of $^*y(s)$ is given by [if $\rho(s)$ is of class C^2 and $d\rho(s)/ds\neq 0$]

$$^*k(s)=\frac{1}{\rho(s)\left|\dfrac{d\rho(s)}{ds}\right|}. \tag{2.2}$$

Conversely, for a C^2-curve $M^*=\{y^*(s^*)\}$ with arc-length parameter s^*, the curve $M=\{x(s^*)\}$ defined by

$$x(s^*)=y^*(s^*)+(c-s^*)\xi_1^*(s^*) \tag{2.3}$$

for some constant c is called the involute of $M^*=\{y^*(s^*)\}$. The evolute and the involute are dual.

Now we define a $(+00-)$-model. Let $k(s)$ be a C^∞-function on an open interval (s_1, s_4) such that $dk(s)/ds>0$ for $s_1<s<s_2$, $dk(s)/ds=0$ for $s_2\leqq s\leqq s_3$, and $dk(s)/ds<0$ for $s_3<s<s_4$. Let D be the C^∞-curve with $k(s)$ as its curvature and with s as its arc-length parameter. Let $^*D$ be the evolute of D. We call $^*D$ a $(+00-)$-model, if $s_2\neq s_3$.

Putting $s_2=s_3$, we call $^*D$ a $(+0-)$-model. 00 means that $dk(s)/ds=0$ holds on some interval, and 0 means that $dk(s)/ds=0$ holds at a single point. Taking some part of a $(+00-)$-model or a $(+0-)$-model, we have a $(+00)$-model, or a $(+0)$-model. A $(00-)$-model or a $(0-)$-model is equivalent to a $(+00)$-model or $(+0)$-model. Notice that, for example in a $(+0-)$-model $^*D$, the point corresponding to $x(s)$ where $dk(s)/ds=0$ (and hence $d\rho(s)/ds=0$) is very complicated because of (2.2).

§ 3. Proofs of Theorems.

For a continuous oval $M=\{x(s)\}$ of constant width H in E^2, two points $x(s)$ and $x(s')$ of M are called pair points, if $|x(s)-x(s')|=H$, where $|\ |$ denotes the Euclidean length of vectors in E^2.

LEMMA 3.1. *Assume that* $x(s')$ *and* $x(s_1)$, *and,* $x(s')$ *and* $x(s_2)$, *are pair points such that* $s_1 < s_2$. *Then the subarc* $\{x(s): s_1 \leqq s \leqq s_2\}$ *is a piece of the circle of radius* H *with* $x(s')$ *as its center.*

PROOF. First we notice that every supporting line has only one point in common with M by constancy of width. Let l_1' and l_1 be the parallel supporting lines at $x(s')$ and $x(s_1)$, and let l_2' and l_2 be the parallel supporting lines at $x(s')$ and $x(s_2)$. Let $C[x(s_1)x(s_2)]$ be the part from $x(s_1)$ to $x(s_2)$ of the circle of radius H with $x(s')$ as its center. Considering supporting lines at $x(s')$ between l_1' and l_2', we see that $\{x(s): s_1 \leqq s \leqq s_2\} = C[x(s_1)x(s_2)]$. Q. E. D.

We call such a point $x(s')$ a corner point of M, and we call $C[x(s_1)x(s_2)]$ the subarc corresponding to $x(s')$, if it is maximal [that is, it is not a proper subset of a subarc of the circle in M]. If we pick up all corner points w_b, generally the set $\{w_b\}$ may be an infinite set. Let C_b be the subarc corresponding to w_b. By $\{w_\beta, C_\beta\}$ we mean the subset of $\{w_b, C_b\}$ such that the length of C_β is greater than $\varepsilon/4$, where ε is a sufficiently small positive number $< H/2$.

Let l_0 and l_0' be the parallel supporting lines at $x(0)$ and $x(s_0)$. We decompose the subarc $M_0 = \{x(s): 0 \leqq s \leqq s_0\}$ into

$$M_0 = \{w_\lambda\} \cup \{C_\mu \cap M_0\} \cup \{F_i\} ,$$

where $\{w_\lambda\}$ is the subset of $\{w_\beta\}$ such that $w_\lambda \in M_0$, $\{C_\mu\}$ is the subset of $\{C_\beta\}$ such that $C_\mu \cap M_0$ is non-empty, and $F_i = \{x(s): s_i \leqq s \leqq t_i\}$ such that

(i) $0 < |x(s_i) - x(t_i)| < \varepsilon$,

(ii) $\{x(s): s_i < s < t_i\}$ does not intersect with $\{w_\lambda\}$ nor $\{C_\mu\}$, nor F_j $(j \neq i)$,

(iii) for pair points $x(s_i)$ and $x(s_i')$, and, $x(t_i)$ and $x(t_i')$, $|x(s_i') - x(t_i')| < \varepsilon$ holds [in this case, if $x(s_i)$ is a corner point, we assume that the point $x(s_i')$ is a boundary point of a piece of a circle].

Since M is compact, $\{w_\lambda\}$ and $\{C_\mu\}$ are finite sets, and we can choose F_i so that $\{F_i\}$ is a finite set. We assume $i = 1, \cdots, h$. The possibility for (i), (ii) and (iii) comes from the fact that subarcs corresponding to corner points $\{w_b\}$ in $M - \{w_\beta, C_\beta\}$ have length $\leqq \varepsilon/4$.

Let F_i be any one of $\{F_i\}$, and let l_i, l_i' and $\bar{l}_i$, $\bar{l}_i'$ be the parallel supporting lines at $x(s_1)$, $x(s_i')$ and $x(t_i)$, $x(t_i')$, respectively. If we draw a convex curve from $x(s_i)$ to $x(t_i)$ in the triangle defined by l_i, $\bar{l}_i$ and the segment $[x(s_i)x(t_i)]$, then the curve is in the ε-neighborhood of M by virtue of (i), (iii) and $\varepsilon < H/2$. This is the same for $x(s_i')$ and $x(t_i')$.

LEMMA 3.2. *Each* $F_i = \{x(s): s_i \leqq s \leqq t_i\}$ *and the corresponding subarc* $\{x(s): s_i' \leqq s \leqq t_i'\}$ *can be replaced by a* C^∞*-curve* $(x(s_i)x(t_i))$ *and the corresponding subarc* $(x(s_i')x(t_i'))$ *so that the resulting oval is of constant width* H.

PROOF. If $|x(s_i') - x(t_i')| = 0$, then F_i is itself of class C^∞. So we consider the following two cases. Let Q be the intersection of the segments $[x(s_i)x(s_i')]$ and $[x(t_i)x(t_i')]$.

(I-1) If $|x(s_i)-Q|$ and $|x(t_i)-Q|$ are both equal to a real number R, we draw the circle with Q as its center and R as its radius. We replace F_i by the part between $x(s_i)$ and $x(t_i)$ of the circle. Similarly we replace the corresponding subarc of F_i by the part between $x(t_i')$ and $x(s_i')$ of the circle with Q as its center and $H-R$ as its radius. The parallel supporting lines at $x(s_i)$ and $x(t_i)$, $x(s_i')$ and $x(t_i')$ are the same with respect to M and with respect to the new oval. Therefore the resulting oval is of constant width H and is in the ε-neighborhood of M.

(I-2) If $|x(s_i)-Q|>|x(t_i)-Q|$, then $|x(t_i')-Q|>|x(s_i')-Q|$, since $|x(s_i)-x(s_i')|=H=|x(t_i)-x(t_i')|$. Hence, we obtain

$$H<|x(s_i)-Q|+|Q-x(t_i')|.$$

On the other hand, we have $|x(s_i)-x(t_i')|\leqq H$. If $|x(s_i)-x(t_i')|=H$, then F_i is of class C^∞, and hence we can assume that

$$|x(s_i)-x(t_i')|<H.$$

Then we can choose a point u of the segment $[x(s_i)Q]$ and a point v of the segment $[Qx(t_i')]$ and we can draw a concave C^∞-curve (uv) from u to v in the triangle $[x(s_i)Qx(t_i')]$ such that (uv) is tangent to two segments at u and v, and such that

$$|x(s_i)-u|+|(uv)|+|v-x(t_i')|=H,$$

where $|(uv)|$ means the arclength of (uv). Then the involute of (uv) with the initial vector $x(s_i)-u$ at u is a convex C^∞-curve from $x(s_i)$ to $x(t_i)$, because

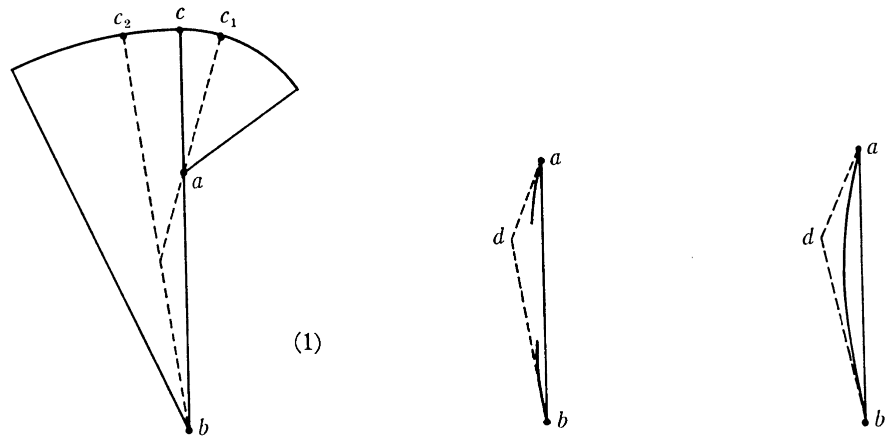
$$|x(s_i)-u|+|(uv)|=H-|x(t_i')-v|=|x(t_i)-v|.$$

Similarly we have the involute from $x(t_i')$ to $x(s_i')$. We replace F_i and the corresponding subarc by these involutes. The parallel supporting lines at $x(s_i)$ and $x(t_i)$, $x(s_i')$ and $x(t_i')$ are the same with respect to M and with respect to the new curves. Hence, the new oval is of constant width H and is in the ε-neighborhood of M. Q. E. D.

By Lemma 3.2 we obtain a piecewise C^∞-oval M_1 of constant width H in the ε-neighborhood of M.

LEMMA 3.3. *M_1 can be approximated by a C^∞-oval M_2 of constant width H_2 in the 2ε-neighborhood of M_1.*

PROOF. Let $M_1(\varepsilon)$ be the outer ε-parallel oval of M_1. At each corner point of M_1, its ε-parallel means a piece of the circle of radius ε with the corner point as its center. Since M_1 is a piecewise C^∞-oval, $M_1(\varepsilon)$ is a C^1-oval with piecewise C^∞-curves. $M_1(\varepsilon)$ is of constant width $H+2\varepsilon$. Let $*M_1(\varepsilon)$ be the evolute of $M_1(\varepsilon)$. $*M_1(\varepsilon)$ is completely contained in the interior of the domain determined by $M_1(\varepsilon)$, and $*M_1(\varepsilon)$ is composed of concave curves and isolated points. We construct a connected $*M_2(\varepsilon)$ from $*M_1(\varepsilon)$ so that its involute is a C^∞-oval of constant width. Let N be the number of parts where connecting process is required. It suffices to consider the following five cases.



(1)

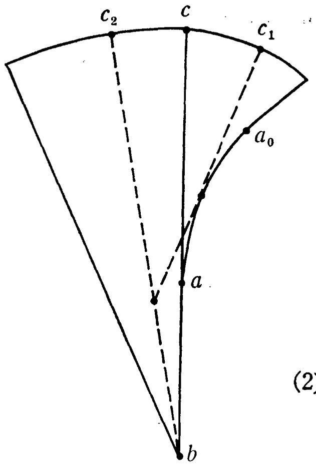
(II-1) Two points a, b in $*M_1(\varepsilon)$ appeared as centers of pieces of circles like (1) can be connected by the following way. Let $c \in M_1(\varepsilon) \cap [ab]$, where $[ab]$ denotes the segment or the straight line passing through a, b. Take c_1 and c_2 in $M_1(\varepsilon)$ which are very close to c. Let $d=[c_1a] \cap [c_2b]$. First we attach a (00—)-model to $[ad]$ at a and to $[bd]$ at b in the triangle $[abd]$. Here by "attaching a (00—)-model to $[ad]$ at a" we mean that the tangent lines to the attached (00—)-model converge to $[ad]$ at a. Next we draw a convex curve $(ab)^*$ from a to b such that

(i) $(ab)^*$ is of class C^∞ except for a and b,

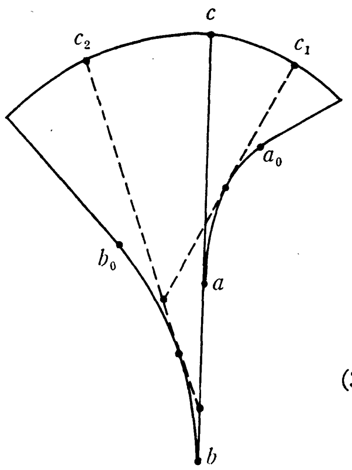
(ii) $(ab)^*$ coincides with some neighborhoods of a and b in the attached models.

In this case we can assume that $|a-d|+|d-b|-|a-b|<\varepsilon/2N$, and hence we can assume that

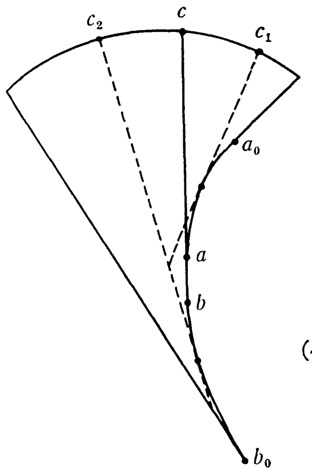
$$|(ab)^*|-|a-b|<\varepsilon/2N.$$



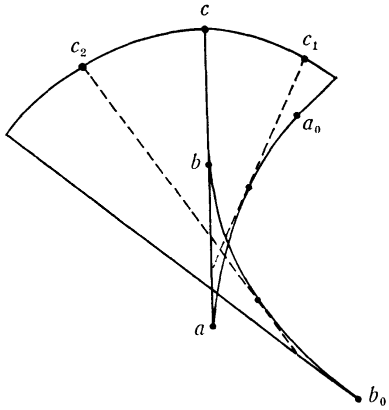
(2)



(3)



(4)



(5)

(II-2) If (2) is the case (where b is a center of a piece of a circle and (aa_0) is a curve), take c, c_1 ane c_2 as before. Let a_1 be the center of curvature at c_1 of $M_1(\varepsilon)$. Let $d=[c_1a_1]\cap[c_2b]$. In the triangle $[a_1db]$ we draw a curve $(a_1b)^*$ such that

(i) $(a_1b)^*$ is of class C^∞ except for b,

(ii) some neighborhood of b in $(a_1b)^*$ coincides with a (00—)-model attached to $[bd]$ at b,

(iii) some neighborhood of a_1 in $(a_1b)^*$ coincides with (aa_0).

In this case we can assume that

$$-\varepsilon/2N<|(a_0a_1)|+|(a_1b)^*|-|(a_0a)|-|a-b|<\varepsilon/2N.$$

(II-3) If (3) is the case (where (a_0a) and (b_0b) are curves), take c, c_1 and c_2 as before. Let a_1 and b_2 be the centers of curvature at c_1 and c_2, respectively. Let $d_1=[c_1a_1]\cap[ab]$, and $d_2=[c_2b_2]\cap[ab]$. We draw two convex curves $(a_1b)^*$ and $(bb_2)^*$ in the triangles $[a_1d_1b]$ and $[b_2d_2b]$ such that

(i) $(a_1b)^*$ and $(bb_2)^*$ are of class C^∞ except for b,

(ii) some neighborhood of b in $(a_1b)^*$ and $(bb_2)^*$ coincides with a $(+0-)$-model attached to $[ab]$ at b,

(iii) some neighborhood of a_1 in $(a_1b)^*$ coincides with (a_0a),

(iv) some neighborhood of b_2 in $(bb_2)^*$ coincides with (b_0b).

In this case we can assume that

$$-\varepsilon/2N<|(a_1a)|+|a-b|+|(bb_2)|-|(a_1b)^*|-|(bb_2)^*|<\varepsilon/2N.$$

(II-4) If (4) is the case, take c, c_1 and c_2 as before. Let a_1 and b_2 be the centers of curvature at c_1 and c_2, respectively. Let $d=[c_1a_1]\cap[c_2b_2]$. We draw a convex C^∞-curve $(a_1b_2)^*$ which coincides with some neighborhoods of a_1 in (a_0a) and of b_2 in (b_0b). In this case we can assume that

$$-\varepsilon/2N<|(a_1a)|+|a-b|+|(bb_2)|-|(a_1b_2)^*|<\varepsilon/2N.$$

(II-5) If (5) is the case, take c, c_1, c_2, a_1, and b_2 as before. Let $d_1=[c_1a_1]\cap[ab]$ and $d_2=[c_2b_2]\cap[ab]$. Let m_1 and m_2 be the middle points of $[a_1d_1]$ and $[b_2d_2]$, respectively. Let $d_3=[am_1]\cap[bm_2]$. We draw three convex curves $(a_1a)^*$, $(ab)^*$ and $(bb_2)^*$ in the triangles $[a_1m_1a]$, $[abd_3]$ and $[m_2bb_2]$ such that

(i) three curves are of class C^∞ except for a, b,

(ii) some neighborhood of a_1 in $(a_1a)^*$ coincides with (a_0a),

(iii) some neighborhood of a in $(a_1a)^*$ and $(ab)^*$ coincides with a $(+0-)$-model attached to $[ad_3]$ at a,

(iv) some neighborhood of b is similar to the case (iii),

(v) some neighborhood of b_2 in $(bb_2)^*$ coincides with (bb_0).

In this case we can assume that

$$-\varepsilon/2N<|(a_1a)|+|a-b|+|(bb_2)|-|(a_1a)^*|-|(ab)^*|-|(bb_2)^*|<\varepsilon/2N.$$

Applying (II-1∼5) we have $^*M_2(\varepsilon)$. We construct the involute M_2 with some initial vector, where we assume that the end point of the initial vector is in $M_1(\varepsilon)$. Then, by our construction we see that M_2 is of class C^∞ and, lies in the 2ε-neighborhood of M_1, and that it has constant width H_2, $H+\varepsilon<H_2<H+3\varepsilon$. Q. E. D.

PROOF OF THEOREM A. By a similar deformation of M_2, we have a C^∞-oval M_3 of constant width H. By taking ε sufficiently small, we see that M_3 can be constructed in the δ-neighborhood of M. This proves Theorem A.

Next we prove Theorem B. Let $M=\{x(s)\}$ be a continuous oval with constant width H, which is symmetric with respect to a straight line m in E^2. Let $M\cap m=\{x(0), x(s_0)\}$. Let $x(s_1)$ and $x(s_2)$ be the pair points in M such that $x(s_1)-x(s_2)$ is orthogonal to m, $s_1<s_2$. In this case the subarc

$$M_4=\{x(s): 0\leqq s\leqq s_1\}$$

is essential. The subarc corresponding to M_4 is $M_4'=\{x(s): s_0\leqq s\leqq s_2\}$. By SM_4 and SM_4' we denote the symmetries of M_4 and M_4' with respect to m. Clearly,

$$M=M_4\cup SM_4'\cup M_4'\cup SM_4\,.$$

Let l_0 and l_0' be the parallel supporting lines at $x(0)$ and $x(s_0)$, and let l_1 and l_1' be the parallel supporting lines at $x(s_1)$ and $x(s_2)$. The difference between proofs of Theorems A and B is in handling neighborhoods of $x(0)$ and $x(s_1)$.

By the way similar to the proof of Theorem A, we can replace M_4 and its corresponding subarc M_4' by a piecewise C^∞-curve M_5 and its corresponding subarc M_5' in the ε-neighborhood of M, where M_5 is a curve from $x(0)$ to $x(s_1)$ and M_5' is a curve from $x(s_0)$ to $x(s_2)$. Then

$$M_6=M_5\cup SM_5'\cup M_5'\cup SM_5$$

is a piecewise C^∞-oval of constant width H.

Let $M_6(\varepsilon)$ be the outer ε-parallel of M_6, and let $^*M_6(\varepsilon)$ be its evolute. $^*M_6(\varepsilon)$ is symmetric with respect to m. We construct a connected $^*M_7(\varepsilon)$ from $^*M_6(\varepsilon)$ so that

(i) its involute M_8 is a C^∞-oval of constant width,

(ii) M_8 is symmetric with respect to m, and

(iii) M_8 is in the 2ε-neighborhood of M_6.

Let $z(0)$, $z(s_0)\in m\cap M_6(\varepsilon)$ be the ε-parallel points of $x(0)$, $x(s_0)$, respectively.

(III-1) If $x(0)$ is a corner point of M_6, then some neighborhood of $z(0)$ in $M_6(\varepsilon)$ is a piece of the circle with $x(0)$ as its center, and hence it is of class C^∞.

(III-2) If $x(0)$ is not a corner point of M_6, we replace some neighborhood of $w(0)$ in $^*M_6(\varepsilon)$ by a $(+0-)$-model attached to m at $w(0)$, where $w(0)$ denotes the center of curvature at $z(0)$ of $M_6(\varepsilon)$. In this case this $(+0-)$-model can be chosen so that it is symmetric with respect to m.

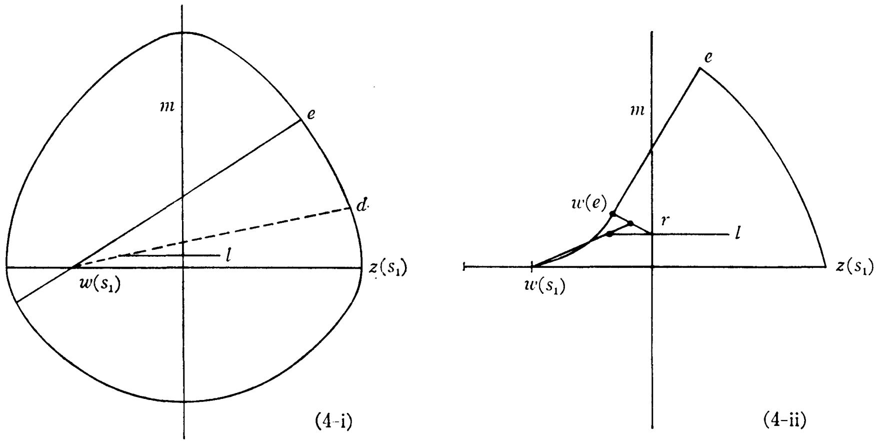
Next let $z(s_1)$, $z(s_2)\in[x(s_1)x(s_2)]\cap M_6(\varepsilon)$ be the ε-parallel points of $x(s_1)$, $x(s_2)$, respectively.

(III-3) Assume that the center $w(s_1)$ of curvature at $z(s_1)$ of $M_6(\varepsilon)$ is in m. If $^*M_6(\varepsilon)$ is of class C^∞ near $w(s_1)$, then no modification is necessary at this step.

If $M_6(\varepsilon)$ is a piece of a circle near $z(s_1)$, then no modification is necessary at this step.

If $^*M_6(\varepsilon)$ is not of class C^∞ at $w(s_1)$, we replace some neighborhood of $w(s_1)$ in $^*M_6(\varepsilon)$ by a piece of a circle with center in m, which is tangent to $[x(s_1)x(s_2)]$ at $w(s_1)$.

(III-4) Assume that the center $w(s_1)$ of curvature at $z(s_1)$ does not lie in m. In this case it suffices to consider the following two cases.



(4-i) (4-ii)

(III-4-i) Assume that the subarc from e to $z(s_1)$ of $M_6(\varepsilon)$ is a piece of the circle with $w(s_1)$ as its center. Take a point d in $M_6(\varepsilon)$ sufficiently near $z(s_1)$ like (4-i). Put $p=[w(s_1)d]\cap m$ and $q=[w(s_1)z(s_1)]\cap m$. Let r be the middle point of $[pq]$. Let l be a straight line passing through r and orthogonal to m. Put $u=l\cap[w(s_1)d]$. We draw a convex curve $(w(s_1)r)^*$ from $w(s_1)$ to r in the triangle $[w(s_1)ur]$ such that

(i) $(w(s_1)r)^*$ is of class C^∞ except for $w(s_1)$,

(ii) some neighborhood of $w(s_1)$ in $(w(s_1)r)^*$ is a (00$-$)-model attached to $[w(s_1)u]$ at $w(s_1)$,

(iii) some neighborhood of r in $(w(s_1)r)^*$ coincides with a piece of a circle which is tangent to l.

(III-4-ii) Assume that the subarc from $w(e)$ to $w(s_1)$ of $^*M_6(\varepsilon)$ is like (4-ii) of the figure. Let l be a straight line which is orthogonal to m and sufficiently near $[z(s_2)z(s_1)]$. Put $r=l\cap m$. Let v be the middle point of $[w(e)r]$. Put $k=[w(e)e]\cap[w(s_1)v]$ and $h=l\cap[w(s_1)v]$. We draw two convex curves $(w(e)w(s_1))^*$ and $(w(s_1)r)^*$ in the triangle $[w(e)w(s_1)k]$ and $[hw(s_1)r]$ such that

(i) they are of class C^∞ except for $w(s_1)$,

(ii) some neighborhood of $w(e)$ in $(w(e)w(s_1))^*$ coincides with $(w(e)w(s_1))$ of $^*M_6(\varepsilon)$,

(iii) some neighborhood of $w(s_1)$ in $(w(e)w(s_1))^*\cup(w(s_1)r)^*$ is a (+0$-$)-model attached to $[w(s_1)v]$ at $w(s_1)$,

(iv) some neighborhood of r in $(w(s_1)r)^*$ coincides with a piece of a circle which is tangent to l.

Therefore, combining what we have proved in the proof of Theorem A, we can construct $^*M_7(\varepsilon)$ such that

(1) its involute M_8 with some initial vector is a C^∞-oval of constant width H_8, $H+\varepsilon<H_8<H+3\varepsilon$,

(2) M_8 is symmetric with respect to m, and

(3) M_8 is in the 2ε-neighborhood of M_6, and hence in the 3ε-neighborhood

of M.

Consequently, if we take ε sufficiently small, we see that we can construct a C^{∞}-oval M_9 of constant width H, which is symmetric with respect to m and is in the δ-neighborhood of M. This proves Theorem B.

Theorem C follows from Theorem B.

§ 4. Remarks.

REMARK 1. Let M be a convex C^h-hypersurface ($h\geqq 4$) in a Euclidean $(n+1)$-space E^{n+1}. Assume that the origin 0 is inside M. Let S^n be the standard sphere in E^{n+1}. For a point $\xi\in S^n$, the distances between 0 and parallel supporting hyperplanes of M orthogonal to ξ are denoted by $h(\xi)$ and $h(-\xi)$, where $h(\xi)$ is one for the positive side of ξ. $h(\xi)$ is called the support function of M. M is of constant width H if and only if $h(\xi)+h(-\xi)=H$. Let $-\varphi(\xi)$ be the sum of the principal radii of curvature at the point of M having normal ξ. J. P. Fillmore [2] studied some relations between $h(\xi)$ and $\varphi(\xi)$.

Especially, applying Christoffel's theorem (cf. W. J. Firey [3]) and using spherical harmonics (of odd degree), one can construct various real analytic hypersurfaces of constant width in E^{n+1} (J. P. Fillmore [2]).

REMARK 2. For E^2 and S^1 we put $\theta=\arg\xi$. For each equilateral $(2r+1)$-polygon ($r\geqq 1$), there corresponds a Reuleaux polygon as a continuous oval of constant width. The corresponding real analytic oval of constant width is given by

$$h(\theta)=a+b\cos(2r+1)\theta\,, \qquad \text{or}$$

$$1/k=a-4r(r+1)b\cos(2r+1)\theta\,,$$

where a and b are constant such that $a>4r(r+1)b$, and k denotes the curvature at the point corresponding to θ.

Notice that $h(\theta)+h(\theta+\pi)=2a$ and $h(\theta)=h(-\theta)$. If we imbed E^2 in E^{n+1} and rotate such ovals with respect to the x^1-axis (defined by $\theta=0$), we obtain real analytic hypersurfaces of constant width $2a$.

§ 5. Twin hypersurfaces.

S. A. Robertson [5], [6] and J. Bolton [1] studied some generalization of hypersurfaces of constant width (transnormal hypersurfaces imbeded in E^m).

As another generalization of hypersurfaces of constant width we define twin hypersurfaces.

DEFINITION. Let (M, g) be an n-dimensional C^{∞}-Riemannian manifold with metric tensor g. Let f_1 and f_2 be isometric C^{∞}-immersions of (M, g) into E^{n+1}. Assume that

(i) (M, g) is orientable and complete,

(ii) there exists a diffeomorphism ϕ of M such $f_1(x)-f_2(\phi x)$ is normal to $f_1(M)$ at $f_1(x)$ and to $f_2(M)$ at $f_2(x)$, for each x of M,

(iii) $f_1(x)-f_2(\phi x)$ is of constant length for $x \in M$.

Then we call this triplet $((M, g), f_1, f_2)$ a twin C^∞-hypersurface.

A C^∞-hypersurface of constant width in E^{n+1} is a special example such that

(1) $f_1=f_2$,

(2) ϕ is the antipodal diffeomorphism [i.e., for pair points x, y, $\phi x=y$].

EXAMPLE. Let P be a closed curve in E^2, with two vertices v_1 and v_2, and with two convex curves (v_1v_2) and (v_2v_1) such that

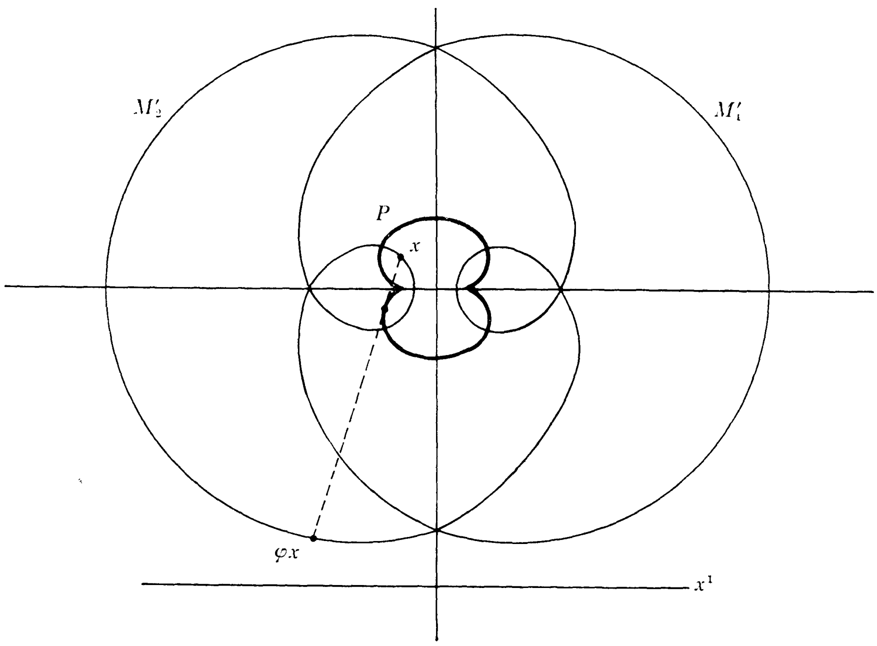
(1) P is symmetric with respect to $[v_1v_2]$,

(2) P is symmetric with respect to the x^2-axis which is orthogonal to $[v_1v_2]$,

(3) P is of class C^∞ except for v_1 and v_2,

(4) some neighborhoods of v_1 and v_2 are $(+0-)$-models attached to $[v_1v_2]$ at v_1 and v_2.

Let M_1' be an involute of P and let M_2' be its symmetry with respect to the x^2-axis. M_2' is also an involute of P. By our construction of P, M_1' and M_2' are closed, of class C^∞, and there exist a constant q and a transformation $\varphi: M_1' \to M_2'$ such that $x-\varphi x$ is normal to M_1' at x and to M_2' at φx and $|x-\varphi x| =q$ for all x of M_1'.



Take the x^1-axis so that it does not meet M_1'. We imbed E^2 into E^{n+1}.

By rotating M_1' and M_2' with respect to the x^1-axis, we obtain two hypersurfaces M_1 and M_2. Let $M_1=(M, g)$ be a Riemannian manifold with the induced metric from the Euclidean metric of E^{n+1}. Let f_1 be the inclusion map of M_1, $f_1:(M, g)\rightarrow M_1\subset E^{n+1}$. Let $S: M_1'\leftrightarrow M_2'$ be the symmetric transformation with respect to the x^2-axis in E^2 and let $f_2=S\circ f_1:(M, g)\rightarrow M_2\subset E^{n+1}$, where S denotes also its extension: $M_1\leftrightarrow M_2$. We extend the diffeomorphism $\varphi: M_1'\rightarrow M_2'$ naturally to the diffeomorphism $\varphi: M_1\rightarrow M_2$, denoted by the same letter φ. We define a diffeomorphism ϕ of (M, g) by $\phi=f_1^{-1}\circ S\circ\varphi\circ f_1$. Then we get

$$f_1x-f_2\circ\phi x=f_1x-\varphi\circ f_1x$$

for all x of (M, g). Since f_1x is identified with x, $((M, g), f_1, f_2)$ is a twin hypersurface.

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