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Tightness of compact Hausdorff spaces and normality of product spaces

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Let X be a topological space. The *tightness* of X, denoted by $t(X)$, is less than or equal to the cardinal number τ if $M\subset X$ and $x\in Cl(M)$ imply the existence of $N\subset M$ such that the cardinality of N is $\leqq\tau$ and $x\in Cl(N)$. This important notion was introduced by A. V. Arhangel'skiĭ [**1**] and has been investigated by many mathematicians. Arhangel'skiĭ also gave a characterization of the tightness of compact spaces (see Lemma 4 below). In this paper we give another characterization. We show that the tightness of a compact space X is characterized by the normality of the product space of X with a space of ordinal numbers.

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§ 1. Preliminaries.

The symbol τ denotes an infinite cardinal number and τ^{+} denotes the smallest cardinal number greater than τ. Ordinal numbers are denoted by α, β and γ. The initial ordinal number of τ is denoted by $\omega(\tau)$. Let X be a topological space. Then the set $\{x_{\alpha}: x_{\alpha}\in X$ and $\alpha<\omega(\tau)\}$ is called a free sequence in X of length τ if $Cl(\{x_{\alpha}:\alpha<\beta\})\cap Cl(\{x_{\alpha}:\alpha\geqq\beta\})=\emptyset$ for any $\beta<\omega(\tau)$. A space is called strongly τ-compact if the closure of any subset of cardinality $\leqq\tau$ is compact ([**4**], p. 762).

In the arguments below, all spaces are assumed to be completely regular and T_1.

LEMMA 1 ([**3**], Theorem 4). *Let X be a compact space and Y be a pseudo-compact space. Then $X\times Y$ is pseudo-compact.*

LEMMA 2 ([**3**], Theorem 1). *Let $X\times Y$ be pseudo-compact. Then $\beta(X\times Y)=\beta X\times\beta Y$, where βX is the Stone-Čech compactification of X.*

LEMMA 3. *Let M and N be disjoint closed sets in a normal space X. Then $Cl_{\beta X}(M)\cap Cl_{\beta X}(N)=\emptyset$.*

LEMMA 4 ([**2**], Theorem 1). *For any infinite compact space X*, $\mathrm{t}(X)=\sup\{\tau:$ *there exists a free sequence in X of length τ*$\}$.

LEMMA 5 ([**5**], Theorem 1.4). *Let X be a paracompact space with $t(X)\leqq\tau$*

and Y be a normal strongly τ-compact space. Then $X\times Y$ is collectionwise normal.

§ 2. Theorem.

THEOREM. *Let X be an infinite compact space. Then the following conditions are equivalent.*

(1) $\mathrm{t}(X)\leqq\tau$.

(2) *$X\times Y$ is normal for any normal strongly τ-compact space Y.*

(3) *$X\times[0, \omega(\tau^{+}))$ is normal.*

PROOF. (1)→(2). This case is a direct corollary of Lemma 5.

(2)→(3) Since $[0, \omega(\tau^{+}))$ is normal and strongly τ-compact, this case is clearly proved.

(3)→(1). We assume $t(X)\geqq\tau^{+}$. Put $Z=X\times[0, \omega(\tau^{+}))$. Since $[0, \omega(\tau^{+}))$ is pseudo-compact, then by Lemmas 1 and 2, $\beta Z=X\times[0, \omega(\tau^{+})]$. The inequality $t(X)\geqq\tau^{+}$ implies the existence of a free sequence in X of length τ^{+} (Lemma 4). Let $F=\{x_{\alpha}: \alpha<\omega(\tau^{+})\}$ be such a free sequence. Put $F_{\beta}=\{x_{\alpha}: \alpha>\beta\}$. Then $\cap\{Cl(F_{\beta}): \beta<\omega(\tau^{+})\}\neq\emptyset$ since X is compact. Let $p\in\cap\{Cl(F_{\beta}): \beta<\omega(\tau^{+})\}$. Put

$$M=Cl_{Z}(\{(x_{\alpha}, \alpha): \alpha<\omega(\tau^{+})\}),$$

$$N=\{(p, \alpha): \alpha<\omega(\tau^{+})\}.$$

Then M and N are closed sets in Z.

We show $M\cap N=\emptyset$. If $(p, \alpha)\in M\cap N$, then $(p, \alpha)\in Cl_{Z}(\{(x_{\beta}, \beta): \beta\leqq\alpha\})$ or $(p, \alpha)\in Cl_{Z}(\{(x_{\beta}, \beta): \beta>\alpha\})$. If $(p, \alpha)\in Cl_{Z}(\{(x_{\beta}, \beta): \beta\leqq\alpha\})$, then $p\in Cl_{X}(\{x_{\beta}: \beta\leqq\alpha\})$. But by definition of p, $p\in Cl_{X}(\{x_{\beta}: \beta\geqq\alpha+1\})$. This contradicts the definition of the free sequence. The second case is impossible since $X\times[0, \alpha]$ is an open neighborhood of (p, α) which does not intersect $\{(x_{\beta}, \beta): \beta>\alpha\}$.

Next we show $(p, \omega(\tau^{+}))\in Cl_{\beta Z}(M)\cap Cl_{\beta Z}(N)$. Let $U\times(\alpha, \omega(\tau^{+})]$ be an arbitrary neighborhood of $(p, \omega(\tau^{+}))$. Then $U\cap\{x_{\beta}: \beta>\alpha\}\neq\emptyset$. Therefore there exists $\gamma>\alpha$ such that $x_{\gamma}\in U$. This shows that $(x_{\gamma}, \gamma)\in U\times(\alpha, \omega(\tau^{+})]$. Thus $M\cap(U\times(\alpha, \omega(\tau^{+})])\neq\emptyset$. On the other hand, $(p, \gamma)\in N\cap(U\times(\alpha, \omega(\tau^{+})])$. Thus we obtain $(p, \omega(\tau^{+}))\in Cl_{\beta Z}(M)\cap Cl_{\beta Z}(N)$. By Lemma 3, those are impossible since $X\times[0, \omega(\tau^{+}))$ is normal. The proof is finished.

The author does not know whether the implication (3)→(1) in this theorem is still true for the cases when X is merely paracompact.

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