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On projective normality of abelian varieties

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Fix an algebraically closed field k of characteristic p. All abelian varieties we will talk about are always defined over k, and in particular, X will denote an abelian variety of dimension g throughout the paper. Recently, S. Koizumi [1] discovered a very useful fact, which he calls the "rank theorem", and using it he proved projective normality of the model of X embedded in $\boldsymbol{P}(\Gamma(L^a))$ in the usual way, in the case of $a\geqq 3$ and any ample invertible sheaf L on X. He has, however, restricted his considerations only to the case of characteristic $p=0$. In the present paper, mainly following his ideas in [1], we generalize his main results to almost all characteristic cases.

After recalling some fundamental properties of theta groups in Section 0, we shall prove the "rank theorem" in Section 1 in the following style:

RANK THEOREM (Theorem 1.4). *Let L be a principal invertible sheaf on X; and a, b be positive integers prime to each other with $a<b$ and $p\nmid ab(a+b)$. Let θ be a suitable section of $\Gamma(L^{ab})$ such that $\{U_\lambda\theta\}_{\lambda\in H(ab)^*}$ is a basis of $\Gamma(L^{ab})$, where $H(ab)^*$ is a lifting in the theta group $\mathcal{G}(L^{ab})$ of a maximal isotropic direct summand of X_{ab} with respect to $e^{L^{ab}}$ and U is the natural action of $\mathcal{G}(L^{ab})$ on $\Gamma(L^{ab})$. Moreover we denote by $H(a)^*$ and $H(b)^*$ the subgroups of $H(ab)^*$ consisting of elements of order dividing a and b respectively. Then the matrix*

$$(U_{\lambda+\mu}\theta(0))_{(\lambda,\mu)\in H(a)^*\times H(b)^*}$$

is of rank a^g.

In the last section 2, we shall consider the canonical map:

$$\Gamma(L^a)\otimes\Gamma(L^b)\longrightarrow\Gamma(L^{a+b})$$

where L is an ample invertible sheaf on X, and show the surjectivity of the map for $a\geqq 2$ and $b\geqq 3$ in the case of characteristic $p\neq 2, 3, 5$.

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TERMINOLOGY AND NOTATION. For any integer n and any abelian variety X,

$n_X: X\longrightarrow X$ the homomorphism defined by $x\longmapsto nx$

$X_n=\ker n_X$

$\hat{X}$ the dual abelian variety of X.

For any invertible sheaf L on X,

$\phi_L : X \longrightarrow \hat{X}$ the homomorphism defined by $x \longmapsto T_x^* L \otimes L^{-1}$

$$K(L) = \ker \phi_L$$

$e^L : K(L) \times K(L) \longrightarrow G_m$ the canonical pairing defined by L (cf. Mumford [3], p. 227)

$\mathcal{G}(L)$ the theta group of L

$$\Gamma(L) = \Gamma(X, L).$$

For a vector space V and its elements $y_1, \cdots, y_n$,

$\langle y_1, \cdots, y_n \rangle$ the subspace spanned by $y_1, \cdots, y_n$.

For a group G operating on a vector space V, we say a subspace W of V is G-stable (resp. G-invariant), if $\sigma(W) = W$ for any $\sigma \in G$ (resp. $\sigma(x) = x$ for any $\sigma \in G$ and $x \in W$). Moreover we denote V^G the subset of V consisting of G-invariant elements.

0. Let L be an ample invertible sheaf on X of separable type; i.e., an invertible sheaf which is ample and $p \nmid$ degree ϕ_L. Then there exist subgroups $H(L)_1$ and $H(L)_2$ of $K(L)$ such that $K(L) = H(L)_1 \oplus H(L)_2$ and $e^L|_{H(L)_i \times H(L)_i} \equiv 1$ $(i = 1, 2)$. In the paper we call such a subgroup $H(L)_i$ a maximal isotropic direct summand of $K(L)$. We have an exact sequence containing the theta group $\mathcal{G}(L)$ as one of its members:

$$1 \longrightarrow k^* \longrightarrow \mathcal{G}(L) \xrightarrow{j(L)} K(L) \longrightarrow 0$$

and $\mathcal{G}(L)$ has a unique irreducible representation $\Gamma(L)$ in which k^* acts by its natural character. The action U of $\mathcal{G}(L)$ on $\Gamma(L)$ is given as follows:

$$U_z : \Gamma(L) \xrightarrow{T_x^*} \Gamma(T_x^* L) \xrightarrow{\phi^{-1}} \Gamma(L)$$

for $z = (x, \phi) \in \mathcal{G}(L)$ with $x \in K(L)$ and $\phi : L \simeq T_x^* L$. For the details on these facts one can see Mumford [2], § 1, [3] or [4]. We mean by a level subgroup K^* in $\mathcal{G}(L)$ a subgroup such that $k^* \cap K^* = \{1\}$. Then there is a 1-1 correspondence between level subgroups K^* in $\mathcal{G}(L)$ and pairs (π, α):

$$\begin{cases} \pi : X \longrightarrow Y = X/K \text{ the canonical map} \\ \alpha : \pi^* M \xrightarrow{\sim} L \text{ an isomorphism for some invertible sheaf } M \text{ on } Y \end{cases}$$

where $K=j(L)(K^*)$ (cf. Mumford [2], § 1, Proposition 1).

Two theta groups which arise from two deta (X, L) and (Y, M) related by an isogeny have following relations:

PROPOSITION 0.1. *Let $f: X\to Y$ be a separable isogeny of abelian varieties with $K=\ker f$. Let L and M be invertible sheaves on X and Y respectively, such that there exists an isomorphism $\alpha: f^*M\simeq L$. Let K^* be the level subgroup of $\mathcal{G}(L)$ defined by the isomorphism α, and we put $j=j(L)$. Then we have*

(i) $f^{-1}(K(M))\subset K(L)$,

(ii) $\{$*centralizer of* K^* *in* $\mathcal{G}(L)\}=j^{-1}(f^{-1}(K(M))$, *which we denote by* $\mathcal{G}(M)^*$,

(iii) $\mathcal{G}(M)\cong\mathcal{G}(M)^*/K^*$ *canonically*

(cf. Mumford [2], § 1, Proposition 2).

PROPOSITION 0.2. *Under the same assumptions as in Proposition* 0.1, *for any element z in $\mathcal{G}(M)^*$, we denote by $\bar{z}$ its canonical image in $\mathcal{G}(M)$. Let $f^*: \Gamma(Y, M)\to\Gamma(X, L)$ be the injection defined by the pair (f, α). Then we have the commutative diagram:*

$$\begin{array}{ccc} \Gamma(Y, M) & \xrightarrow{f^*} & \Gamma(X, L) \\ U_{\bar{z}}\downarrow & & \downarrow U_z \\ \Gamma(Y, M) & \xrightarrow{f^*} & \Gamma(X, L) \end{array}$$

(cf. Mumford [2], § 1).

Concerning products of two abelian varieties, we have

PROPOSITION 0.3. *Let X and Y be two abelian varieties, and let L and M be ample invertible sheaves of separable type on X and Y. Let $p_1: X\times Y\to X$ and $p_2: X\times Y\to Y$ be the projections. Then we have the canonical isomorphism:*

$$\mathcal{G}(p_1^*L\otimes p_2^*M)\cong\mathcal{G}(L)\times\mathcal{G}(M)/\{(\lambda, \lambda^{-1})\mid\lambda\in k^*\}$$

(cf. Mumford [2], § 3, Lemma 1).

The section will end with two easy remarks which will be used later.

PROPOSITION 0.4. *Let L be a principal invertible sheaf on X (i.e., L is ample and $\chi(L)=1$), and let m, n be positive integers which are prime to each other and $p\nmid mn$. Let $j=j(L^{mn})$. Then $j^{-1}(X_n)$ $(\subset\mathcal{G}(L^{mn}))$ is isomorphic to $\mathcal{G}(L^n)$. Therefore if M is a $j^{-1}(X_n)$-stable non-trivial subspace in $\Gamma(L^{mn})$, we have* $\dim M=rn^g$ *for some $r\geqq 1$.*

PROOF. If we take a maximal isotropic direct summand $H(mn)$ of $K(L^{mn})=X_{mn}$, then $H(n)=\{mx\mid x\in H(mn)\}$ becomes a maximal one of $K(L^n)=X_n$ and we have isomorphisms $K(L^{mn})\cong H(mn)\times\hat{H}(mn)$ and $K(L^n)\cong H(n)\times\hat{H}(n)$, where $\hat{H}$ indicates the dual group of a group H. Here we denote by i the canonical inclusion $H(n)\to H(mn)$. Moreover theta groups $\mathcal{G}(L^{mn})$ and $\mathcal{G}(L^n)$ are isomorphic to Heisenberg groups $K(mn)=k^*\times H(mn)\times\hat{H}(mn)$ and $K(n)=k^*\times H(n)\times\hat{H}(n)$ respectively. Now choosing a positive integer m' such that $mm'\equiv 1 \bmod n$, we embed $\hat{H}(n)$ into $\hat{H}(mn)$ by $\iota: l(\ )\mapsto l(m'm\cdot(\ ))$ for any $l\in\hat{H}(n)$. Then obviously $K(n)=k^*\times H(n)\times \hat{H}(n) \xrightarrow{1_{k^*}\times i\times \iota} K(mn)=k^*\times H(mn)\times \hat{H}(mn)$ is an injective homomorphism and its image corresponds to $j^{-1}(X_n)$, which implies our assertion. Q. E. D.

Hereafter we denote by P_X, or simply by P, the Poincaré sheaf on $X\times\hat{X}$, and for any $\alpha\in\hat{X}$ we mean by P_α the restricted sheaf $P|_{X\times\{\alpha\}}$.

LEMMA 0.5. *Let L be a principal invertible sheaf on X, and let m, n be two positive integers such that $p\nmid mn$. For a closed point $\alpha\in\hat{X}$, we put $j=j(L^{mn}\otimes P_\alpha)$. Then $j^{-1}(X_n)$ is contained in the centralizer of $j^{-1}(X_m)$ in $\mathcal{G}(L^{mn}\otimes P_\alpha)$.*

PROOF. Since $e^{L^{mn}\otimes P_\alpha}=e^{L^{mn}}$, we have only to show that $e^{L^{mn}}(x, y)=1$ for any $x\in X_m$ and $y\in X_n$. In fact, since $x\in K(L^m)$ and $y\in n_X^{-1}K(L^m)$, we have $e^{L^{mn}}(x, y)=e^{L^m}(x, 0)=1$. Therefore we obtain our assertion. Q. E. D.

1. First of all we give an easy lemma which makes the first step of the "rank theorem".

LEMMA 1.1. *Let M be a principal invertible sheaf on an abelian variety Y of* dim g. *Let n be a positive integer prime to p. Then there exists a triplet (X, π, L):*

$$\begin{cases} X: \text{an abelian variety} \\ \pi: X \longrightarrow Y \text{ an isogeny of degree } n^g \\ L: \text{a principal symmetric invertible sheaf on } X \end{cases}$$

such that $\pi^*M\cong L^n\otimes P_\gamma$ for some $\gamma\in\hat{X}$ and $\ker\pi$ *is a maximal isotropic direct summand of $K(L^n)=X_n$.*

PROOF. We put $\hat{M}=(\phi_M^{-1})^*M$, and we take a maximal isotropic direct summand $\hat{H}$ of $K(\hat{M}^n)$. Moreover we put $\hat{X}=\hat{Y}/\hat{H}$ and we denote by $\hat{\pi}$ the canonical projection $\hat{Y}\to\hat{X}$. Then there exists a principal invertible sheaf $\hat{L}$ on $\hat{X}$ such that $\hat{\pi}^*\hat{L}\cong\hat{M}^n$. Hence we have $n\phi_M^{-1}=\pi\circ\phi_{\hat{L}}\circ\hat{\pi}$ or $n_Y=\pi\circ\phi_{\hat{L}}\circ\hat{\pi}\circ\phi_M$, where $\pi: X\to Y$ is the dual map of $\hat{\pi}$. On the other hand, $\phi_{\pi^*M}=\hat{\pi}\circ\phi_M\circ\pi$. Therefore we have $n_Y\circ\pi=\pi\circ\phi_{\hat{L}}\circ\hat{\pi}\circ\phi_M\circ\pi=\pi\circ\phi_{\hat{L}}\circ\phi_{\pi^*M}$, i. e., $n_X=\phi_{\hat{L}}\circ\phi_{\pi^*M}$, which implies $K(\pi^*M)=X_n$, because $\phi_{\hat{L}}$ is isomorphic. Hence there exists a principal invertible sheaf L' on X such that $\pi^*M\cong L'^n$. Moreover it is an easy fact that every invertible sheaf is algebraically equivalent to a symmetric invertible sheaf. Therefore we can see the existence of such an L in the proposition. Furthermore from the way of the choice of $\hat{H}$, $\ker\pi$ has the required property. Q.E.D.

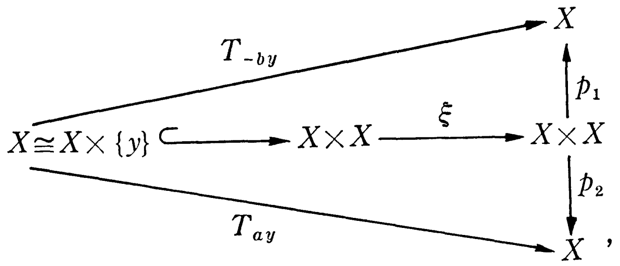
The next proposition is a translation of "generalized addition formulas" in [1] into the abstract case, which also play an essential role in the proof of our "rank theorem".

PROPOSITION 1.2. *Let a, b be positive integers, and we define a homomorphism $\xi: X\times X\to X\times X$ by $(x, y)\mapsto(x-by, x+ay)$. Let L be a symmetric invertible* *sheaf on* X. *Then we have*

$$\xi^*(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta))\cong p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$$

for any α, $\beta\in\hat{X}$, *where* $p_i: X\times X\to X$ *denotes the projection to the i-th component for* $i=1, 2$.

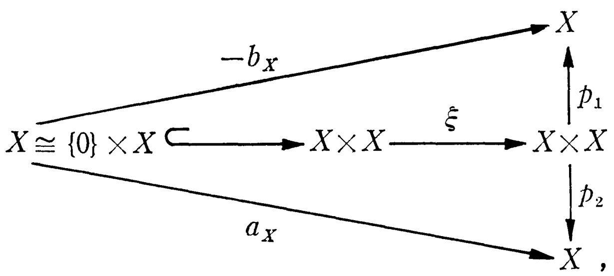
PROOF. Let y be any closed point of X. First of all we notice that $T_{ny}^*L\cong T_y^*L^n\otimes L^{1-n}$ for any integer n. From this notice and the following commutative diagram:



we have

$$\begin{aligned}\xi^*(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta))|_{X\times\{y\}} &\cong (T_{-by}^*L^a\otimes P_\alpha)\otimes(T_{ay}^*L^b\otimes P_\beta)\\ &\cong (T_y^*L^{-ab}\otimes L^{a(1+b)}\otimes P_\alpha)\otimes(T_y^*L^{ab}\otimes L^{b(1-a)}\otimes P_\beta)\\ &\cong L^{a+b}\otimes P_{\alpha+\beta}.\end{aligned}$$

On the other hand, from the symmetricity of L and the commutative diagram:



we have

$$\begin{aligned}\xi^*(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta))|_{\{0\}\times X} &\cong (-b_X)^*(L^a\otimes P_\alpha)\otimes(a_X)^*(L^b\otimes P_\beta)\\ &\cong (L^{ab^2}\otimes P_{-b\alpha})\otimes(L^{a^2b}\otimes P_{a\beta})\cong L^{ab(a+b)}\otimes P_{a\beta-b\alpha}.\end{aligned}$$

Therefore by Seesaw theorem, we obtain

$$\xi^*(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta))\cong p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}).$$

Q. E. D.

REMARK. The homomorphism ξ in the above proposition is separable if and only if $p \nmid a+b$. As for deg ξ, we have the explicit equality deg $\xi=(a+b)^{2g}$.

Throughout the rest of the section, L denotes a principal symmetric invertible sheaf on X, and a, b are positive integers such that $(a, b)=1$ and $p \nmid ab(a+b)$. We mean by ξ the homomorphism defined in Proposition 1.2. Then by the proposition we have an isomorphism

$$\xi^*(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta)) \xrightarrow[\sim]{\phi} p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$$

and the injection

$$\Gamma(L^a\otimes P_\alpha)\otimes\Gamma(L^b\otimes P_\beta) \xrightarrow{\xi^*} \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})\,.$$

Once for all ϕ is fixed and both sides in the former relation will be identified in the rest of the paper. Now we take non-zero elements u and v in $\Gamma(L^a\otimes P_\alpha)$ and $\Gamma(L^b\otimes P_\beta)$ respectively, and fix them. Let $\{s_1, \cdots, s_l\}$ and $\{t_1, \cdots, t_m\}$ be basis of $\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$ and $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ respectively, where $l=(a+b)^g$ and $m=\{ab(a+b)\}^g$. Then we obtain an equation,

$$(*)\qquad u(x-by)v(x+ay)=\sum_{\substack{1\leq\mu\leq l\\ 1\leq\nu\leq m}} c_{\mu\nu}s_\mu(x)t_\nu(y)$$

for some $c_{\mu\nu}\in k$. The isomorphism ϕ defines a lifting of the group $K=\ker\xi$:

$$\begin{array}{ccccccc} 1 \longrightarrow k^* \longrightarrow & \mathcal{G}(p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})) & \xrightarrow{j} & X_{a+b}\times X_{ab(a+b)} \longrightarrow 0 \\ & \cup & & \cup \\ & K^* & \xrightarrow{\sim} & K\,. \end{array}$$

We denote by $\mathcal{G}^*$ the centralizer of K^*. Then since $K=\{(by, y)\,|\,y\in X_{a+b}\}$, we have

$$(1)\qquad \mathcal{G}^*\supset j^{-1}(\{0\}\times X_{ab})\,,$$

from Proposition 0.3 and Lemma 0.5. For any decomposition $K(L^a)=H(a)_1\oplus H(a)_2$ and $K(L^b)=H(b)_1\oplus H(b)_2$ into maximal isotropic subgroups, there exists a decomposition $K(L^{ab(a+b)})=H(ab(a+b))_1\oplus H(ab(a+b))_2$ into maximal ones such that $H(ab(a+b))_i\supset H(a)_i$, $H(b)_i$ for $i=1, 2$. Let $H(ab(a+b))_i^*$ be a level subgroup in $\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ of $H(ab(a+b))_i$ for each $i=1, 2$. Then $H(a)_i$ and $H(b)_i$ are also simultaneously lifted up to subgroups $H(a)_i^*$ and $H(b)_i^*$ in $H(ab(a+b))_i^*$ respectively. The image of the subgroup $\{1\}\times H(ab(a+b))_i^*$ by the canonical map:

$$\begin{aligned} &\mathcal{G}(L^{a+b}\otimes P_{\alpha+\beta})\times\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) \\ &\quad\longrightarrow \mathcal{G}(L^{a+b}\otimes P_{\alpha+\beta})\times\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})/\{(\lambda, \lambda^{-1})\,|\,\lambda\in k^*\} \\ &\qquad\cong\mathcal{G}(p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))\,, \end{aligned}$$

which we also denote by $H(ab(a+b))_i^*$, is a level subgroup of $\{0\}\times H(ab(a+b))_i$ for each $i=1, 2$. Therefore the subgroups $H(a)_i^*$ and $H(b)_i^*$ in $H(ab(a+b))_i^*$ also can be identified with level subgroups of $\{0\}\times H(a)_i$ and $\{0\}\times H(b)_i$ respectively. From the above inclusion relation (1),

$$\mathcal{G}^*\supset H(a)_i^*,\ H(b)_i^*$$

for $i=1, 2$. Since $(ab, a+b)=1$, we have

$$H(a)_i^*\cap K^*=\{1\}\quad \text{and}\quad H(b)_i^*\cap K^*=\{1\}.$$

Therefore the subgroups $H(a)_i^*$ and $H(b)_i^*$ are canonically isomorphic to subgroups of

$$\begin{aligned}\mathcal{G}^*/K^* &\cong \mathcal{G}(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta))\\ &\cong \mathcal{G}(L^a\otimes P_\alpha)\times \mathcal{G}(L^b\otimes P_\beta)/\{(\lambda, \lambda^{-1})\,|\,\lambda\in k^*\}\end{aligned}$$

(cf. Proposition 0.1 and Proposition 0.3), which we denote by $\bar{H}(a)_i^*$ and $\bar{H}(b)_i^*$ respectively. Moreover $\bar{H}(a)_i^*$ and $\bar{H}(b)_i^*$ are canonically identified with subgroups of $\mathcal{G}(L^a\otimes P_\alpha)$ and $\mathcal{G}(L^b\otimes P_\beta)$ respectively, because $(a, b)=1$. For any element $z\in H(a)_i^*\cup H(b)_i^*$, we denote by $\bar{z}$ its canonical image in $\bar{H}(a)_i^*\cup \bar{H}(b)_i^*$. Under these notations we have the key proposition.

PROPOSITION 1.3. *Let* $j'=j(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$.

(0) *We have*

$$\operatorname{rank}(c_{\mu\nu})=l,\quad i.e.,\ =(a+b)^g \qquad for\quad c_{\mu\nu}\text{'s in } (*)$$

and

$$u(x-by)v(x+ay)\in \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes W_0$$

where W_0 *is a* $j'^{-1}(X_{a+b})$*-stable subspace of* $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ *of* $\dim l$.

Moreover if we put $i_0=1$ *or* 2, *then we have the following three statements.*

(i) *If* v *is* $\bar{H}(b)_{i_0}^*$*-invariant,* W_0 *is not only* $j'^{-1}(X_{a+b})$*-stable, but* $H(b)_{i_0}^*$*-invariant.*

(ii) *If* $\{U_{\bar{\lambda}}u\}_{\lambda\in H(a)_{i_0}^*}$ *is a basis of* $\Gamma(L^a\otimes P_\alpha)$ *and we put* $W=\sum_{\lambda\in H(a)_{i_0}^*}U_\lambda W_0$ *in* $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$, *then* W *is the direct sum of* $U_\lambda W_0$*'s.*

(iii) *If* $\{U_{\bar{\lambda}}u\}_{\lambda\in H(a)_{i_0}^*}$ *and* $\{U_{\bar{\lambda}}v\}_{\lambda\in H(b)_{i_0}^*}$ *are basis of* $\Gamma(L^a\otimes P_\alpha)$ *and* $\Gamma(L^b\otimes P_\beta)$ *respectively, then*

$$\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})=\bigoplus_{(\lambda,\mu)\in H(a)_{i_0}^*\times H(b)_{i_0}^*}U_{\lambda+\mu}W_0\,.$$

PROOF. Since

$$\xi^*(\Gamma(L^a\otimes P_\alpha)\otimes\Gamma(L^b\otimes P_\beta))=(\Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))^{K^*},$$

$u(x-by)v(x+ay)$ is invariant under the action of K^*. If $r=\operatorname{rank}(c_{\mu\nu})<(a+b)^g$, there exist non-degenerate matrices P and Q such that

$$u(x-by)v(x+ay)={}^t(s_\mu(x))P^{-1}\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}Q^{-1}(t_\nu(y)),$$

where $(s_\mu(x))$ and $(t_\nu(y))$ mean column vectors. Now we put ${}^t(s_\mu)P^{-1}={}^t(s'_\mu)$ and $Q^{-1}(t_\nu)=(t'_\nu)$. Then

$$u(x-by)v(x+ay)=\sum_{i=1}^{r} s'_i(x)t'_i(y).$$

On the other hand, since $\mathcal{G}(L^{a+b}\otimes P_{\alpha+\beta})$ operates irreducibly on $\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$, the subspace $\langle s'_1, \cdots, s'_r\rangle$ must be bijectively mapped to a distinct subspace of $\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$ by a suitable element of $j''^{-1}(X_{a+b})$, where $j''=j(L^{a+b}\otimes P_{\alpha+\beta})$. So $u(x-by)v(x+ay)$ can not be invariant under the action of K^*, which contradicts our first notice. Therefore $r=\operatorname{rank}(c_{\mu\nu})$ must be equal to $l=(a+b)^g$. After choosing a suitable basis, we may assume that

$$u(x-by)v(x+ay)=\sum_{i=1}^{l} s_i(x)t_i(y). \tag{2}$$

If we put $W_0=\langle t_1, \cdots, t_l\rangle$, then it becomes stable under the action of $j'^{-1}(X_{a+b})$ $(\subset\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))$, because $\Gamma(L^{a+b}\otimes P_{\alpha+\beta})=\langle s_1, \cdots, s_l\rangle$. Hence we obtain our first assertion (0). For the rest of our assertions, we may assume, without loss of generality, that $i_0=1$. By Proposition 0.2, for each $z\in H(a)_1^*\cup H(a)_2^*\cup H(b)_1^*\cup H(b)_2^*\subset\mathcal{G}(p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))$ we have a commutative diagram:

$$\begin{array}{ccc} \Gamma(L^a\otimes P_\alpha)\otimes\Gamma(L^b\otimes P_\beta) & \xrightarrow{\xi^*} & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) \\ U_{\bar z}\downarrow & & \downarrow U_z \\ \Gamma(L^a\otimes P_\alpha)\otimes\Gamma(L^b\otimes P_\beta) & \xrightarrow{\xi^*} & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}). \end{array} \tag{3}$$

Applying this diagram to the equation (2), we obtain

$$(U_{\bar\lambda}u)(x-by)v(x+ay)=\sum_{i=1}^{l} s_i(x)U_\lambda t_i(y) \qquad \text{for } \lambda\in H(a)_1^*\cup H(a)_2^* \tag{4}$$

and

$$u(x-by)(U_{\bar\lambda'}v)(x+ay)=\sum_{i=1}^{l} s_i(x)U_{\lambda'}t_i(y) \qquad \text{for } \lambda'\in H(b)_1^*\cup H(b)_2^*. \tag{5}$$

Therefore if v is $H(b)_1^*$-invariant, the latter equation implies that

$$\sum_{i=1}^{l} s_i(x)U_{\lambda'}t_i(y)=\sum_{i=1}^{l} s_i(x)t_i(y) \qquad \text{for } \lambda'\in H(b)_1^*,$$

i. e.,

$$U_{\lambda'}t_i(y)=t_i(y) \quad (i=1, \cdots, l) \qquad \text{for any } \lambda'\in H(b)_1^*.$$

Hence (i) has been proved. As for the assertion (ii), we first assume that

$\{U_{\bar\lambda}u\}_{\lambda\in H(a)_1^*}$ is a basis of $\Gamma(L^a\otimes P_\alpha)$. Then the equation (4) leads us to

$$(U_{\bar\lambda_2}U_{\bar\lambda_1}u)(x-by)v(x+ay)=\sum_{i=1}^{l}s_i(x)(U_{\lambda_2}U_{\lambda_1}t_i)(y)$$

for any $\lambda_1\in H(a)_1^*$ and $\lambda_2\in H(a)_2^*$. Since $\{U_{\bar\lambda}u\}_{\lambda\in H(a)_2^*}$ is a basis of $\Gamma(L^a\otimes P_\alpha)$, $(U_{\bar\lambda_2}U_{\bar\lambda_1}u)(x-by)$ can be expressed as a linear combination of $(U_{\bar\lambda}u)(x-by)$'s. Therefore $(U_{\bar\lambda_2}U_{\bar\lambda_1}u)(x-by)v(x+ay)$ is also expressed as a linear combination of $\{(U_{\bar\lambda}u)(x-by)v(x+ay)\}_{\lambda\in H(a)_1^*}$; i. e., $U_{\lambda_2}U_{\lambda_1}t_i$'s are expressed as linear combinations of $\{U_\lambda t_i\}_{\lambda\in H(a)_1^*,\ i=1,\cdots,l}$. This implies that W is $j'^{-1}(X_{a(a+b)})(\subset\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))$-stable. Therefore by Proposition 0.4, we obtain

$$\dim W\geqq\{a(a+b)\}^g,$$

which implies the equality $W=\bigoplus_{\lambda\in H(a)_1^*}U_\lambda W_0$. The last assertion (iii) in the proposition is proved in the same manner as (ii) is. Q. E. D.

THEOREM 1.4 (The rank theorem; cf. [1], Theorem 2.5). *Let Y be any abelian variety of* $\dim g$; *let M be any principal invertible sheaf on Y; and let a, b_0 be positive integers such that $b=b_0-a>0$, $(a,b_0)=1$ and $p\nmid abb_0$. Let $K(M^{ab_0})=H(ab_0)_1\oplus H(ab_0)_2$, $K(M^a)=H(a)_1\oplus H(a)_2$ and $K(M^{b_0})=H(b_0)_1\oplus H(b_0)_2$ are decompositions into maximal isotropic subgroups, such that $H(ab_0)_i\supset H(a)_i$, $H(b_0)_i$. Then these maximal isotropic subgroups are lifted up to level subgroups:*

$$\begin{array}{ccccccccc}1&\longrightarrow&k^*&\longrightarrow&\mathcal{G}(M^{ab_0})&\longrightarrow&Y_{ab_0}&\longrightarrow&0,\\&&&&\cup&&\cup&&\\&&&&H(ab_0)_i^{**}&\cong&H(ab_0)_i&&\\&&&&H(a)_i^{**}&\cong&H(a)_i&&\\&&&&H(b_0)_i^{**}&\cong&H(b_0)_i&&\end{array}$$

for $i=1,2$. Let $\theta\in\Gamma(M^{ab_0})$ be a section such that $\{U_z\theta\}_{z\in H(ab_0)_1^{**}}$ is a basis of $\Gamma(M^{ab_0})$ and that $\langle\{U_\lambda\theta\}_{\lambda\in H(a)_1^{**}}\rangle$ is $H(a)_2^{**}$-stable or $\langle\{U_\mu\theta\}_{\mu\in H(b_0)_1^{**}}\rangle$ is $H(b_0)_2^{**}$-stable. Then for any closed point $y\in Y$, we have the equality

$$\operatorname{rank}(U_{\lambda+\mu}\theta(y))_{(\lambda,\mu)\in H(a)_1^{**}\times H(b_0)_1^{**}}=a^g.$$

PROOF. By Lemma 1.1, there exist an abelian variety X, an isogeny $\pi:X\to Y$ and a principal symmetric invertible sheaf L on X such that

$$\pi^*(M^{ab_0})\cong L^{abb_0}\otimes P_\gamma\qquad\text{for some}\quad\gamma\in\hat X$$

and $\ker\pi$ is a maximal isotropic direct summand $H(b)_1$ of $K(L^b)=X_b$. Now we take a solution α, $\beta\in\hat X$ of the equation $a\beta-b\alpha=\gamma$. Then a fixed isomorphism $\pi^*(M^{ab_0})\cong L^{ab(a+b)}\otimes P_{a\beta-b\alpha}$ defines a lifting of the group $H(b)_1$:

$$1 \longrightarrow k^* \longrightarrow \mathcal{G}(L^{ab(a+b)} \otimes P_{a\beta-b\alpha}) \xrightarrow{j'} X_{ab(a+b)} \longrightarrow 0 .$$
$$H(b)_1^* \quad \cong \quad H(b)_1$$

Moreover if we denote by $\mathcal{G}(M^{ab_0})^*$ the centralizer of $H(b)_1^*$, we have a canonical isomorphism $\mathcal{G}(M^{ab_0}) \cong \mathcal{G}(M^{ab_0})^*/H(b)_1^*$. Since $H(b)_1^*$ is contained in the center of $\mathcal{G}(M^{ab_0})^*$ and $(ab_0, b)=1$, the given level subgroups $H(ab_0)_i^{**}$, $H(a)_i^{**}$ and $H(b_0)_i^{**}$ in $\mathcal{G}(M^{ab_0})$ are naturally isomorphic to subgroups $H(ab_0)_i^*$, $H(a)_i^*$ and $H(b_0)_i^*$ of $\mathcal{G}(M^{ab_0})^*$ respectively. Moreover we have the isomorphism defined by π^* from $\Gamma(M^{ab_0})$ to the $H(b)_1^*$-invariant subspace $\Gamma(L^{abb_0} \otimes P_{a\beta-b\alpha})^{H(b)_1^*}$, which is compatible with the actions of $\mathcal{G}(M^{ab_0})$ and $\mathcal{G}(M^{ab_0})^*$. Therefore we have been able to reduce our assertion to the equality

$$\operatorname{rank} (U_{\lambda+\mu}\theta(y))_{(\lambda,\mu)\in H(a)_1^* \times H(b_0)_1^*} = a^g ,$$

for any $y \in X$ and a section $\theta \in \Gamma(L^{abb_0} \otimes P_{a\beta-b\alpha})^{H(b)_1^*}$ such that $\{U_z\theta\}_{z\in H(ab_0)_1^*}$ is a basis of $\Gamma(L^{abb_0} \otimes P_{a\beta-b\alpha})^{H(b)_1^*}$ and that $\langle\{U_\lambda\theta\}_{\lambda\in H(a)_1^*}\rangle$ is $j'^{-1}(X_a)$-stable or $\langle\{U_\mu\theta\}_{\mu\in H(b_0)_1^*}\rangle$ is $j'^{-1}(X_{a+b})$-stable. Under the notation in Proposition 1.3, we take an $\bar{H}(a)_2^*$ (resp. $\bar{H}(b)_1^*$)-invariant non-zero element u (resp. v). Then $\{U_{\bar{\lambda}}u\}_{\lambda\in H(a)_1^*}$ becomes a basis of $\Gamma(L^a \otimes P_\alpha)$. Moreover, according to Proposition 1.3, we have

$$u(x-by)v(x+ay) \in \Gamma(L^{a+b} \otimes P_{\alpha+\beta}) \otimes W_0 \subset \Gamma(L^{a+b} \otimes P_{\alpha+\beta}) \otimes \Gamma(L^{ab(a+b)} \otimes P_{a\beta-b\alpha}) ,$$

where W_0 is $j'^{-1}(X_{a+b})$-stable and invariant under the actions of $H(b)_1^*$ and $H(a)_2^*$, and

$$\Gamma(L^{ab(a+b)} \otimes P_{a\beta-b\alpha}) \supset W = \bigoplus_{\lambda\in H(a)_1^*} U_\lambda W_0 .$$

Since W is $H(b)_1^*$-invariant and of $\dim (ab_0)^g$, we have

$$W = \bigoplus_{\lambda\in H(a)_1^*} U_\lambda W_0 = \Gamma(P^{abb_0} \otimes P_{a\beta-b\alpha})^{H(b)_1^*} .$$

If we take an $H(b_0)_2^*$-invariant θ' in W_0, $\{U_\mu\theta'\}_{\mu\in H(ab_0)_1^*}$ becomes a basis of W. Moreover, from the equation (4), we obtain

$$v(x+ay)(U_\lambda u(x-by))_{\lambda\in H(a)_1^*}$$
$$= (U_{\lambda+\mu}\theta'(y))_{(\lambda,\mu)\in H(a)_1^* \times H(b_0)_1^*}(c_{\mu i})(s_i(x))_{1\leq i\leq l} .$$

Since $\{U_\lambda u(x-by)\}_{\lambda\in H(a)_1^*}$ are linearly independent for any fixed y, we obtain

$$(*) \qquad \operatorname{rank} (U_{\lambda+\mu}\theta'(y))_{(\lambda,\mu)\in H(a)_1^* \times H(a+b)_1^*} = a^g .$$

If θ is an element of W such that $\{U_z\theta\}_{z\in H(ab_0)_1^*}$ is a basis of W and $W'=$

$\langle\{U_\lambda\theta\}_{\lambda\in H(a)_1^*}\rangle$ is $H(a)_2^*$-stable, then there exists a non-trivial $H(a)_2^*$-invariant element θ'' in W', and $W'=\langle\{U_\lambda\theta''\}_{\lambda\in H(a)_1^*}\rangle$. Therefore there exists a non-singular $a^g\times a^g$-matrix A such that

$$(U_\lambda\theta)_{\lambda\in H(a)_1^*}=A(U_\lambda\theta'')_{\lambda\in H(a)_1^*},$$

i. e.,

$$(U_{\lambda+\mu}\theta)_{(\lambda,\mu)\in H(a)_1^*\times H(a+b)_1^*}=A(U_{\lambda+\mu}\theta'')_{(\lambda+\mu)\in H(a)_1^*\times H(a+b)_1^*},$$

which implies the equality

$$(**)\qquad \operatorname{rank}(U_{\lambda+\mu}\theta(y))_{(\lambda,\mu)\in H(a)_1^*\times H(a+b)_1^*} = \operatorname{rank}(U_{\lambda+\mu}\theta''(y))_{(\lambda,\mu)\in H(a)_1^*\times H(a+b)_1^*}.$$

Moreover since W_0 is $H(a)_2^*$-invariant and of $\dim(a+b)^g$, $\{U_\mu\theta'\}_{\mu\in H(a+b)_1^*}$ and $\{U_\mu\theta''\}_{\mu\in H(a+b)_1^*}$ are basis of W_0. Therefore for some non-singular $(a+b)^g\times(a+b)^g$-matrix B, we have

$${}^t(U_\mu\theta'')_{\mu\in H(a+b)_1^*}={}^t(U_\mu\theta')_{\mu\in H(a+b)_1^*}B,$$

i. e.,

$$(U_{\lambda+\mu}\theta'')_{(\lambda,\mu)\in H(a)_1^*\times H(a+b)_1^*}=(U_{\lambda+\mu}\theta')_{(\lambda,\mu)\in H(a)_1^*\times H(a+b)_1^*}B,$$

which implies the equality

$$(***)\qquad \operatorname{rank}(U_{\lambda+\mu}\theta''(y))_{(\lambda,\mu)\in H(a)_1^*\times H(a+b)_1^*} = \operatorname{rank}(U_{\lambda+\mu}\theta'(y))_{(\lambda,\mu)\in H(a)_1^*\times H(a+b)_1^*}.$$

Hence from (*), (**) and (***), we obtain our required equality

$$\operatorname{rank}(U_{\lambda+\mu}\theta(y))_{(\lambda,\mu)\in H(a)_1^*\times H(a+b)_1^*}=a^g.$$

If we assume that θ is an element of W such that $\{U_z\theta\}_{z\in H(ab_0)_1^*}$ is a basis of W and $W''=\langle\{U_\mu\theta\}_{\mu\in H(b_0)_1^*}\rangle$ is $H(b_0)_2^*$-stable, then there also exists a non-trivial $H(b_0)_2^*$-invariant element θ''' in W'', and $W''=\langle\{U_\mu\theta'''\}_{\mu\in H(b_0)_1^*}\rangle$. Therefore by the same argument as in above, we also obtain our assertion in the case. Q.E.D.

2. In the section, a, b, d denote positive integers such that $(ad, a+b)=1$, $abd>a+b$ and $p\nmid abd(a+b)$. As in Proposition 1.2, we define a homomorphism $\xi: X\times X\to X\times X$ by $(x, y)\mapsto(x-by, x+ay)$.

PROPOSITION 2.1 (cf. [1], Proposition 3.2). *Let L be a symmetric principal invertible sheaf on X, and let α, β be two closed points on $\hat{X}$. Let $\hat{H}(abd)$ be a maximal isotropic direct summand of $K((\phi_L^{-1})L^{abd})=\hat{X}_{abd}$ and we put $\hat{H}(d)=$*

$\hat{X}_d \cap \hat{H}(abd)$. *Then*

$$\sum_{\gamma\in\hat{H}(d)} \Gamma(L^a\otimes P_{\alpha-\gamma})\otimes\Gamma(L^b\otimes P_{\beta+\gamma}) \longrightarrow \Gamma(L^{a+b}\otimes P_{\alpha+\beta})$$

is surjective.

PROOF. Let $\hat{Y}=\hat{X}/\hat{H}(d)$ and $\hat{\pi}:\hat{X}\to\hat{Y}$ be the canonical projection; furthermore let $\pi: Y\to X$ be the dualized map of $\hat{\pi}$. Then by Lemma 1.1, there exists a principal invertible sheaf M on Y such that $\pi^*L\cong M^d$ and $\ker\pi$ is a maximal isotropic direct summand of $K(M^d)$, which we put K. From the way of the choice of $\hat{H}(abd)$, there exists a maximal isotropic direct summand $H(abd(a+b))$ of $K(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})$ such that $H(abd(a+b))\cap K=\{0\}$. Here we put $H(abd)=H(abd(a+b))\cap Y_{abd}$, $H(a+b)=H(abd(a+b))\cap Y_{a+b}$, $H(a)=H(abd(a+b))\cap Y_a$ and $H(b)=H(abd(a+b))\cap Y_b$. We denote by adding **-symbol to them the level subgroups in $\mathcal{G}(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})$ such that $H(abd(a+b))^{**}\supset H(abd)^{**}$, $H(a+b)^{**}$; $H(abd)^{**}\supset H(a)^{**}$, $H(b)^{**}$; and K^{**} corresponds to the isomorphism $M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}\cong\pi^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$. If we denote by $\mathcal{G}^*$ the centralizer of K^{**} in $\mathcal{G}(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})$, we have the canonical isomorphism

$$\mathcal{G}^*/K^{**}\cong\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}).$$

By Lemma 0.5, $H(a+b)^{**}$, $H(a)^{**}$ and $H(b)^{**}$ are contained in $\mathcal{G}^*$. Therefore by the assumption $(d, a+b)=1$ and the fact $(H(a)^{**}\cup H(b)^{**})\cap K^{**}=\{0\}$, $H(a+b)^{**}$, $H(a)^{**}$ and $H(b)^{**}$ are canonically isomorphic to subgroups of $\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$, which we denote by $H(a+b)^*$, $H(a)^*$ and $H(b)^*$ respectively. Hence by Proposition 0.2, for any $\lambda\in H(a+b)^{**}\cup H(a)^{**}\cup H(b)^{**}$, we obtain a commutative diagram:

$$\begin{array}{ccc}
\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) & \xrightarrow{\ \pi^*\ } & \Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}) \\
\Big\downarrow U_{\lambda'} & & \Big\downarrow U_{\lambda} \\
\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) & \xrightarrow{\ \pi^*\ } & \Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}),
\end{array}$$

where λ' is the canonical image of λ in $H(a+b)^*\cup H(a)^*\cup H(b)^*$. Moreover the relation $\pi_*(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})\cong\pi_*(\pi^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}))\cong\sum_{\gamma\in H(d)} L^{ab(a+b)}\otimes P_{a\beta-b\alpha+\gamma}$ leads us to the decomposition:

$$(*)\qquad \sum_{\gamma\in\hat{H}(a)}\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha+\gamma}) \xrightarrow[\sim]{\ \pi^*\ } \Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}).$$

Furthermore, for any $\mu=(y,\phi)\in H(abd)^{**}$, we obtain a commutative diagram:

$$
(2)\quad U'_\mu \;
\begin{array}{ccc}
\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) & \xrightarrow{\ \pi^*\ } & \Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}) \\
\downarrow T^*_{\pi y} & & \downarrow T^*_y \\
\Gamma(T^*_{\pi y}L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) & \xrightarrow{\ \pi^*\ } & \Gamma(T^*_y M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha}) \\
\downarrow \wr & & \wr \downarrow \phi^{-1} \\
\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha+ab(a+b)\phi_L(\pi y)}) & \xrightarrow{\ \pi^*\ } & \Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})\,.
\end{array}
\; U_\mu
$$

Here we denote by U'_μ the composite of the left vertical arrows. On the other hand, for any $x\in X$, the diagram:

$$
\begin{array}{ccc}
X\times X & \xrightarrow{\ \xi\ } & X\times X \\
\downarrow T_{(0,x)} & & \downarrow T_{(-bx,ax)} \\
X\times X & \xrightarrow{\ \xi\ } & X\times X
\end{array}
$$

commutes. Hence we have an isomorphism

$$
\xi^*(T^*_{(-bx,ax)}(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta)))\cong T^*_{(0,x)}\xi^*(p_1^*(L^a\otimes P_\alpha)\otimes p_2^*(L^b\otimes P_\beta))\,,
$$

i. e.,

$$
\begin{aligned}
&\xi^*(p_1^*(L^a\otimes P_{\alpha-ab\phi_L(x)})\otimes p_2^*(L^b\otimes P_{\beta+ab\phi_L(x)})) \\
&\quad\cong p_1^*(L^{a+b}\otimes P_{\alpha+\beta})\otimes p_2^*(L^{ab(a+b)}\otimes P_{a\beta-b\alpha+ab(a+b)\phi_L(x)})\,.
\end{aligned}
$$

Therefore we obtain a commutative diagram:

$$
(3)\quad U''_\mu \;
\begin{array}{ccc}
\Gamma(L^a\otimes P_\alpha)\otimes\Gamma(L^b\otimes P_\beta) & \xrightarrow{\ \xi^*\ } & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) \\
\downarrow T^*_{(-b\pi y,a\pi y)} & & \downarrow 1\otimes T^*_{\pi y} \\
\Gamma(T^*_{-b\pi y}L^a\otimes P_\alpha)\otimes\Gamma(T^*_{a\pi y}L^b\otimes P_\beta) & \xrightarrow{\ \xi^*\ } & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(T^*_y L^{ab(a+b)}\otimes P_{a\beta-b\alpha}) \\
\downarrow & & \downarrow \\
\Gamma(L^a\otimes P_{\alpha-ab\phi_L(\pi y)})\otimes\Gamma(L^b\otimes P_{\beta+ab\phi_L(\pi y)}) & \xrightarrow{\ \xi^*\ } & \Gamma(L^{a+b}\otimes P_{\alpha+\beta})\otimes\Gamma(^{ab(a+b)} \\
 & & \otimes P_{a\beta-b\alpha+ab(a+b)\phi_L(\pi y)})\,.
\end{array}
\; U'_\mu
$$

Similarly, we denote by U''_μ the composite of the left vertical arrows. Once more we notice that the subgroups $H(a)^*$ and $H(b)^*$ of $\mathcal{G}(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ are canonically isomorphic to subgroups $\bar{H}(a)^*$ and $\bar{H}(b)^*$ of $\mathcal{G}(L^a\otimes P_\alpha)$ and $\mathcal{G}(L^b\otimes P_\beta)$ respectively (cf. § 1), and we denote by $\bar{z}$ the canonical image in $\bar{H}(a)^*\cup\bar{H}(b)^*$ of an element $z\in H(a)^*\cup H(b)^*$. Now we take sections u and v from $\Gamma(L^a\otimes P_\alpha)$ and $\Gamma(L^b\otimes P_\beta)$ such that $\{U_{\bar{\lambda}}u\}_{\lambda\in H(a)^*}$ and $\{U_{\bar{\mu}}v\}_{\mu\in H(b)^*}$ are basis of $\Gamma(L^a\otimes P_\alpha)$ and $\Gamma(L^b\otimes P_\beta)$ respectively. Then from Proposition 1.3, (iii), for a suitable basis $\{s_1, \cdots, s_l\}$ and a section $\theta\in\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ such that $\{U_{\lambda+\mu+\nu}\theta\}_{(\lambda,\mu,\nu)\in H(a+b)^*\times H(a)^*\times H(b)^*}$ becomes a basis of $\Gamma(L^{ab(a+b)}\otimes P_{a\beta-b\alpha})$ and the subspace $\langle\{U_\lambda\theta\}_{\lambda\in H(a+b)^*}\rangle$ is $j'^{-1}(X_{a+b})$-stable, we have

$$\xi^*(uv)={}^t(s_i)_{1\leq i\leq l}(U_\lambda\theta)_{\lambda\in H(a+b)^*}\,.$$

Applying the diagram (1) to this equality, we obtain

$$(1\otimes\pi^*)\xi^*(uv)={}^t(s_i)_{1\leq i\leq l}(U_\lambda\pi^*\theta)_{\lambda\in H(a+b)^{**}}\,.$$

Therefore from the commutative diagram (2) and (3), we have

$$(**)\qquad {}^t(((1\otimes\pi^*)\xi^*(U''_\mu(uv)))(x, y))_{\mu\in H(abd)^{**}}$$
$$={}^t(s_i(x))_{1\leq i\leq l}((U_{\lambda+\mu}(\pi^*\theta))(y))_{(\lambda,\mu)\in H(a+b)^{**}\times H(abd)^{**}}\,.$$

On the other hand, the subspace $\langle\{U_\lambda\pi^*\theta\}_{\lambda\in H(a+b)^{**}}\rangle$ is stable under the action of $j(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})^{-1}(Y_{a+b})$, and $\{U_{\lambda+\mu}\pi^*\theta\}_{(\lambda,\mu)\in H(a+b)^{**}\times H(abd)^{**}}$ becomes a basis of $\Gamma(M^{abd(a+b)}\otimes P_{a\hat{\pi}\beta-b\hat{\pi}\alpha})$. Therefore by the rank theorem, we obtain the equality

$$\operatorname{rank}(U_{\lambda+\mu}\pi^*\theta(0))_{(\lambda,\mu)\in H(a+b)^{**}\times H(abd)^{**}}=(a+b)^g\,.$$

Hence we obtain our assertion, putting $y=0$ in (**). Q. E. D.

REMARK. The assertion of Proposition 2.1 is still true without assuming the symmetricity of L, because every invertible sheaf is algebraically equivalent to a symmetric invertible sheaf.

THEOREM 2.2 (cf. [1], Theorem 4.2). *Let L be an ample invertible sheaf of separable type on X, and let α, β be two closed points on $\hat{X}$. Let $H(L^{abd})$ be a maximal isotropic direct summand of $K(L^{abd})$, and we put $H(L^d)=H(L^{abd})\cap K(L^d)$ and $\hat{H}(d)=\phi_L(H(L^d))$. Then*

$$\sum_{\gamma\in\hat{H}(d)}\Gamma(L^a\otimes P_{\alpha-\gamma})\otimes\Gamma(L^b\otimes P_{\beta+\gamma})\longrightarrow\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$$

is surjective.

PROOF. If we put $H=K(L)\cap H(L^d)$, then it is a maximal isotropic direct summand of $K(L)$. Let $\pi: X\to X/H$ be the canonical projection, and M be a principal invertible sheaf on X/H such that $\pi^*M\cong L$. Moreover we put $K=\phi_M(\pi(H(L^d)))$ and $\hat{H}=\ker\hat{\pi}$. Obviously, K is isomorphic to $\hat{H}(d)$ by $\hat{\pi}$. Now we take two points α', β' from $\hat{\pi}^{-1}(\alpha)$ and $\hat{\pi}^{-1}(\beta)$ respectively. Then for any

$\gamma'\in K$, we have

$$\begin{cases}\pi^*(M^a\otimes P_{\alpha'-\gamma'})\cong L^a\otimes P_{\alpha-\hat{\pi}\gamma'}\\ \pi^*(M^b\otimes P_{\beta'+\gamma'})\cong L^b\otimes P_{\beta+\hat{\pi}\gamma'}\end{cases}$$

or

$$\begin{cases}\pi_*(L^a\otimes P_{\alpha-\hat{\pi}\gamma'})\cong \sum_{\lambda'\in\hat{H}} M^a\otimes P_{\alpha'-\gamma'+\lambda'}\\ \pi_*(L^b\otimes P_{\beta+\hat{\pi}\gamma'})\cong \sum_{\lambda'\in\hat{H}} M^b\otimes P_{\beta'+\gamma'+\lambda'}\,,\end{cases}$$

i. e.,

$$\begin{cases}\Gamma(L^a\otimes P_{\alpha-\hat{\pi}\gamma'})\cong \sum_{\lambda'\in\hat{H}} \Gamma(M^a\otimes P_{\alpha'-\gamma'+\lambda'})\\ \Gamma(L^b\otimes P_{\beta+\hat{\pi}\gamma'})\cong \sum_{\lambda'\in\hat{H}} \Gamma(M^b\otimes P_{\beta'+\gamma'+\lambda'})\,.\end{cases}$$

Hence we obtain a commutative diagram:

$$\begin{array}{ccc}
\sum_{\gamma\in\hat{H}(d)}\Gamma(L^a\otimes P_{\alpha-\gamma})\otimes\Gamma(L^b\otimes P_{\beta+\gamma}) & \longrightarrow & \Gamma(L^{a+b}\otimes P_{\alpha+\beta}) \\
\wr\big| & & \\
\sum_{\gamma'\in K}((\sum_{\lambda'\in\hat{H}}\Gamma(M^a\otimes P_{\alpha'-\gamma'+\lambda'}))\otimes(\sum_{\mu'\in\hat{H}}\Gamma(M^b\otimes P_{\beta'+\gamma'+\mu'})) & & \wr\big| \\
\wr\big| & & \\
\sum_{\nu'\in\hat{H}}(\sum_{\lambda'+\mu'=\nu'}(\sum_{\gamma'\in K}\Gamma(M^a\otimes P_{\alpha'+\lambda'-\gamma'})\otimes\Gamma(M^b\otimes P_{\beta'+\mu'+\gamma'}))) & \longrightarrow & \sum_{\nu'\in\hat{H}}\Gamma(M^{a+b}\otimes P_{\alpha'+\beta'+\nu'})\,.
\end{array}$$

Therefore we have been able to reduce our theorem to some principal cases. Q. E. D.

Lastly, we assume that $p\neq 2, 3, 5$. Then we have

THEOREM 2.3. *Let L be any ample invertible sheaf of separable type on X; let α, β be two closed points on $\hat{X}$. Then*

$$\Gamma(L^a\otimes P_\alpha)\otimes\Gamma(L^b\otimes P_\beta)\longrightarrow\Gamma(L^{a+b}\otimes P_{\alpha+\beta})$$

is surjective for all integers a, b such that $a\geqq 2$, $b\geqq 3$.

PROOF. Applying Theorem 2.2 in the case of $a=2$, $b=3$ and $d=1$, we obtain the surjectivity of the map $\Gamma(L^2\otimes P_\alpha)\otimes\Gamma(L^3\otimes P_\beta)\rightarrow\Gamma(L^5\otimes P_{\alpha+\beta})$. For general a, b, the assertion can be inductively reduced to the case by Mumford's lemma and his method in [4], pp. 68-70. Q. E. D.

References

[1] S. Koizumi, Theta relations and projective normality of abelian varieties, to appear in Amer. J. Math.

[2] D. Mumford, On the equations defining abelian varieties, I, Inv. Math., 1 (1966), 287-354.

[3] D. Mumford, Abelian Varieties, Tata Inst. Studies in Math., Oxford Univ. Press, London and New York, 1970.

[4] D. Mumford, Varieties defined by quadratic equations, Questioni sulle varieta algebraiche, Corsi dal C. L. M. E., Edizioni Cremonese, Roma, 1969.

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