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On the abstract linear evolution equations in Banach spaces

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§ 0. Introduction.

The objective of the present paper is to construct the evolution operator associated with an evolution equation

$$du/dt+A(t)u=f(t) \tag{E}$$

in a Banach space X. Here $u=u(t)$ and $f(t)$ are functions on $[0, T]$ to X and $A(t)$ is a function on $[0, T]$ to the set of linear operators acting in X. We assume (E) is of parabolic type, that is, $-A(t)$ are all infinitesimal generators of analytic semi-groups of bounded linear operators on X.

This problem has been considered already in many papers, for instance, [1], [2] and [3]. The main assumption of [1] is that the inequality

$$\|A(t)^{\rho}dA(t)^{-1}/dt\|\leqq N \tag{0.1}$$

is valid with some constant $\rho\in(0,1]$; and those of [2] are that the inequality of the form

$$\begin{aligned}&\|A(t)(\lambda-A(t))^{-1}(dA(t)^{-1}/dt)A(t)(\lambda-A(t))^{-1}\|\\&=\|\partial/\partial t(\lambda-A(t))^{-1}\|\leqq N/|\lambda|^{\rho}\end{aligned} \tag{0.2}$$

is valid with some constant $\rho\in(0,1]$ and $dA(t)^{-1}/dt$ is Hölder continuous. In [3] the following conditions are assumed: the domain of $A(t)^{\rho}$ is independent of t for some $\rho=1/m$ where m is a positive integer, and $A(t)^{\rho}A(0)^{-\rho}$ is Hölder continuous in t.

In this paper we assume the inequality

$$\|A(t)(\lambda-A(t))^{-1}dA(t)^{-1}/dt\|\leqq N/|\lambda|^{\rho} \tag{0.3}$$

with a constant $\rho\in(0,1]$. This inequality (0.3) is slightly weaker than the inequality (0.1), for (0.1) implies (0.3) by the equation

$$A(t)(\lambda-A(t))^{-1}dA(t)^{-1}/dt=A(t)^{1-\rho}(\lambda-A(t))^{-1}A(t)^{\rho}dA(t)^{-1}/dt$$

and the estimation

$$\|A(t)^{1-\rho}(\lambda-A(t))^{-1}\|\leqq M/|\lambda|^{\rho}.$$

On the other hand (0.3) is rather stronger than the inequality (0.2), for (0.3) is a direct consequence of (0.2) on account of the estimation

$$\|A(t)(\lambda-A(t))^{-1}\| \leqq M .$$

However we should note that in order to construct the evolution operator in [2] (0.2) alone is not sufficient and the Hölder continuity of $dA(t)^{-1}/dt$ must be assumed in addition.

In [1] and [2] the evolution operator is constructed directly by means of E. E. Levi's method. In the present case, however, this method does not work well to prove our theorem. Instead we use Yosida's approximation of $A(t)$ and some integral equations as was done in [3].

As a corollary of the theorem we will construct the evolution operator under the assumptions that for some constant $\rho\in(0, 1]$ the domain of $A(t)^{\rho}$ is independent of t, and $A(t)^{\rho}A(0)^{-\rho}$ is strongly continuously differentiable in t. Thus we can eliminate the condition assumed in [3] that ρ must be $1/m$ with some positive integer m. In our corrollary, however, we must require the strong differentiability as for the smoothness condition of $A(t)^{\rho}A(0)^{-\rho}$, though only its Hölder continuity was sufficient in [3].

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§ 1. Main theorem.

As our main result we claim the following.

THEOREM 1. *For each $t\in[0, T]$, let $A(t)$ be a densely defined, closed linear operator acting in the Banach space X. We assume the following conditions:*

(I) *For each $t\in[0, T]$ the resolvent set of $A(t)$ contains a fixed closed angular domain*

$$\Sigma=\{\lambda\in C;\ \arg\lambda\notin(-\theta_0, \theta_0)\},$$

where θ_0 belongs to $(0, \pi/2)$. For any $t\in[0, T]$ and $\lambda\in\Sigma$ the resolvent satisfies the inequality

$$\|(\lambda-A(t))^{-1}\| \leqq N_0/(1+|\lambda|)$$

with some positive constant N_0 independent of t and λ;

(II) *$A(t)^{-1}$ is strongly continuously differentiable in t, and the derivative $dA(t)^{-1}/dt$ satisfies*

$$\|A(t)(\lambda-A(t))^{-1}dA(t)^{-1}/dt\| \leqq N_1/|\lambda|^{\rho}$$

for any $t\in[0, T]$ and $\lambda\in\Sigma$, where N_1 and $\rho\in(0, 1]$ are independent of t and λ.

Then there exists a family $\{U(t, s);\ 0\leqq s\leqq t\leqq T\}$ of bounded operators on X having the following properties:

1) *$U(t, s)$ is strongly continuous in (t, s);*

2) *$U(t, r)U(r, s)=U(t, s)$, $0\leqq s\leqq r\leqq t\leqq T$, $U(s, s)=I$;*

3) *For $s<t$, $R(U(t, s))$ (the range of $U(t, s)$) is contained in $D(A(t))$ (the domain of $A(t)$), and the estimates*

$$\|A(t)U(t, s)A(s)^{-1}\|\leqq C_0$$

and

$$\|A(t)U(t, s)\|\leqq C_1/(t-s)$$

hold, where the constants C_0 and C_1 are determined by θ_0, ρ, T, N_0 and N_1 alone. Moreover $A(t)U(t, s)A(s)^{-1}$ is strongly continuous in $0\leqq s\leqq t\leqq T$, and $A(t)U(t, s)$ is strongly continuous in $0\leqq s<t\leqq T$;

4) *If f is a continuous function with values in X, then any strict solution u of* (E) *on $[s, T]$ with its initial value $u(s)\in X$ can be expressed in the form*

$$u(t)=U(t, s)u(s)+\int_s^t U(t, r)f(r)dr; \tag{1.1}$$

5) *Conversely if f is Hölder continuous, then any u defined by* (1.1) *with any $u(s)\in X$ gives the strict solution of* (E) *on $[s, T]$.*

By a strict solution of (E) on $[s, T]$ we mean a continuous function defined on $[s, T]$ which is continuously differentiable on $(s, T]$ and satisfies (E) on $(s, T]$.

PROOF OF THE THEOREM. For any integer $n\geqq 1$ and $t\in[0, T]$ let us define

$$A_n(t)=A(t)(I+n^{-1}A(t))^{-1}.$$

It is well known that the resolvent set of $A_n(t)$ contains Σ and the estimate

$$\|(\lambda-A_n(t))^{-1}\|\leqq M_0/(1+|\lambda|) \tag{1.2}$$

still holds for any $t\in[0, T]$, $\lambda\in\Sigma$ and n with some constant M_0 depending only on N_0 and θ_0 (e.g. see [3]). We know also that $(\lambda-A_n(t))^{-1}$ converges strongly to $(\lambda-A(t))^{-1}$ as $n\to\infty$ for each $t\in[0, T]$ and $\lambda\in\Sigma$. Since $A_n(t)^{-1}=A(t)^{-1}+n^{-1}$, $A_n(t)^{-1}$ is strongly continuously differentiable in t, and the estimate

$$\|A_n(t)(\lambda-A_n(t))^{-1}dA_n(t)^{-1}/dt\|\leqq M_1/|\lambda|^{\rho} \tag{1.3}$$

is valid for any $t\in[0, T]$, $\lambda\in\Sigma$ and n with some constant M_1 dependent only on θ_0, ρ and N_1.

Throughout this section $M_2, M_3, \cdots, M_{14}$ denote constants determined by θ_0, ρ, T, N_0 and N_1 alone.

From (1.2), $-A_n(t)$ generates an analytic semi-group $\exp(-rA_n(r))$ which is represented by

$$\exp(-rA_n(t))=\frac{-1}{2\pi i}\int_\Gamma e^{-\lambda r}(\lambda-A_n(t))^{-1}d\lambda \tag{1.4}$$

where Γ is a smooth path running in Σ from $\infty e^{-\theta_0 i}$ to $\infty e^{\theta_0 i}$.

By the differentiability of $(\lambda-A_n(t))^{-1}$ and (1.4), $\exp(-(t-s)A_n(s))$ is strongly continuously differentiable in $s\in[0, t)$ and the derivative is expressed by the following integral

$$(\partial/\partial s)\exp(-(t-s)A_n(s))$$
$$=\frac{-1}{2\pi i}\int_{\Gamma}e^{-\lambda(t-s)}\left\{\frac{\partial}{\partial s}(\lambda-A_n(s))^{-1}+\lambda(\lambda-A_n(s))^{-1}\right\}d\lambda. \tag{1.5}$$

Let $U_n(t, s)$ $(0\leqq s\leqq t\leqq T)$ be the evolution operator corresponding to the equation

$$du/dt+A_n(t)u=0,$$

and let

$$V_n(t, s)=A_n(t)U_n(t, s)A_n(s)^{-1},$$
$$W_n(t, s)=A_n(t)U_n(t, s)-A_n(s)\exp(-(t-s)A_n(s)).$$

Then we can construct three integral equations satisfied by U_n, V_n and W_n respectively. For U_n,

$$U_n(t, s)-\exp(-(t-s)A_n(s))=\int_s^t\frac{\partial}{\partial r}\{\exp(-(t-r)A_n(r))U_n(r, s)\}\,dr$$
$$=\int_s^t P_n(t, r)U_n(r, s)dr, \tag{1.6}$$

where

$$P_n(t, r)=(\partial/\partial t+\partial/\partial r)\exp(-(t-r)A_n(r)). \tag{1.7}$$

For V_n,

$$V_n(t, s)-\exp(-(t-s)A_n(s))$$
$$=\int_s^t\frac{\partial}{\partial r}\{\exp(-(t-r)A_n(r))V_n(r, s)\}\,dr$$
$$=\int_s^t\{P_n(t, r)+\exp(-(t-r)A_n(r))(dA_n(r)/dr)A_n(r)^{-1}\}\,V_n(r, s)dr$$
$$=\int_s^t Q_n(t, r)V_n(r, s)dr, \tag{1.8}$$

since $(dA_n(r)/dr)A_n(r)^{-1}=-A_n(r)dA_n(r)^{-1}/dr$, we see

$$Q_n(t, r)=P_n(t, r)-A_n(r)\exp(-(t-r)A_n(r))dA_n(r)^{-1}/dr. \tag{1.9}$$

Operating $A_n(s)$ to (1.8) from the right, we obtain the equation for W_n

$$W_n(t, s)=R_n(t, s)+\int_s^t Q_n(t, r)W_n(r, s)dr, \tag{1.10}$$

where

$$R_n(t, s)=\int_s^t Q_n(t, r)A_n(s)\exp(-(r-s)A_n(s))dr. \tag{1.11}$$

These equations will play an important role in the proof of the theorem.

We next introduce a notation for classes of operator-valued functions which was used in [3]. By $H(\nu, M)$ we denote the set of all operator-valued functions $K(t, s)$, defined and strongly continuous in $0\leqq s<t\leqq T$, such that

$$\|K(t, s)\|\leqq M(t-s)^{\nu-1}.$$

We see from (1.2) that

$$\exp(-(t-s)A_n(t)),\ \exp(-(t-s)A_n(s))\in H(1, M_2), \tag{1.12}$$

$$A_n(t)\exp(-(t-s)A_n(t)),\ A_n(s)\exp(-(t-s)A_n(s))\in H(0, M_3). \tag{1.13}$$

LEMMA 1. *The integral kernels P_n, Q_n and R_n belong to $H(\rho, M_4)$.*

To see this let us begin with P_n. By (1.5) and (1.7) we have

$$P_n(t, s)=\frac{-1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}(\partial/\partial s)(\lambda-A_n(s))^{-1}d\lambda$$

$$=\frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}A_n(s)(\lambda-A_n(s))^{-1}(dA_n(s)^{-1}/ds)A_n(s)(\lambda-A_n(s))^{-1}d\lambda. \tag{1.14}$$

Here we have used the formula

$$(\partial/\partial s)(\lambda-A_n(s))^{-1}$$

$$=-A_n(s)(\lambda-A_n(s))^{-1}(dA_n(s)^{-1}/ds)A_n(s)(\lambda-A_n(s))^{-1}. \tag{1.15}$$

From (1.3) and (1.14) we can see easily that P_n is strongly continuous in $s<t$ and satisfies the estimate

$$\|P_n(t, s)\|\leqq\frac{(M_0+1)M_1}{2\pi}\int_\Gamma e^{-\mathrm{Re}\lambda(t-s)}|\lambda|^{-\rho}|d\lambda|$$

$$\leqq M_5(t-s)^{\rho-1}.$$

By (1.4), (1.9) and (1.14) we have the representation

$$Q_n(t, s)=\frac{1}{2\pi i}\int_\Gamma \lambda e^{-\lambda(t-s)}A_n(s)(\lambda-A_n(s))^{-1}\frac{dA_n(s)^{-1}}{ds}(\lambda-A_n(s))^{-1}d\lambda. \tag{1.16}$$

From (1.16) we obtain $Q_n\in H(\rho, M_6)$ by the same argument as above for P_n.

To prove $R_n\in H(\rho, M_4)$, we need some preparations. By (1.3) and (1.15)

$$\|(\lambda-A_n(t))^{-1}-(\lambda-A_n(s))^{-1}\|\leqq(M_0+1)M_1(t-s)|\lambda|^{-\rho},$$

and therefore

$$\|A_n(t)\exp(-(t-s)A_n(t))-A_n(s)\exp(-(t-s)A_n(s))\|$$
$$\leqq M_7(t-s)^{\rho-1}. \tag{1.17}$$

Using the expression (1.16), we obtain

$$\|Q_n(t, s)A_n(s)\| \leqq M_8(t-s)^{\rho-2}. \tag{1.18}$$

Now, (1.11) can be rewritten in the following form

$$\begin{aligned} R_n(t, s) &= \int_s^t Q_n(t, r)A_n(r)\exp(-(r-s)A_n(r))dr \\ &\quad + \int_s^t Q_n(t, r)\{A_n(s)\exp(-(r-s)A_n(s)) \\ &\quad - A_n(r)\exp(-(r-s)A_n(r))\}\, dr \\ &= R_n^1(t, s) + R_n^2(t, s). \end{aligned} \tag{1.19}$$

$R_n^1(t, s)$ is strongly continuous for $s<t$ by (1.12), (1.13), (1.18) and the fact $Q_n \in H(\rho, M_6)$. $\|R_n^1(t, s)\|$ is estimated as follows

$$\begin{aligned} \|R_n^1(t, s)\| &\leqq \int_{(t+s)/2}^t \|Q_n(t, r)\| \|A_n(r)\exp(-(r-s)A_n(r))\| dr \\ &\quad + \int_s^{(t+s)/2} \|Q_n(t, r)A_n(r)\| \|\exp(-(r-s)A_n(r))\| dr \\ &\leqq M_9(t-s)^{\rho-1}. \end{aligned}$$

(1.17) and the fact $Q_n \in H(\rho, M_6)$ give $R_n^2 \in H(2\rho, M_{10})$. Thus we have $R_n \in H(\rho, M_{11})$.

Finally we have only to put $M_4 = \mathrm{Max}\,\{M_5, M_6, M_{11}\}$.

Let us define $P(t, s)$, $Q(t, s)$ and $R(t, s)$ by the equalities (1.7), (1.9) and (1.11) with $A_n(t)$ replaced by $A(t)$, respectively. Then the whole proof of Lemma 1 holds for P, Q and R; namely P, Q and R are elements of $H(\rho, M_4)$. This fact enables us to define U, V and W as the solutions of the following integral equations;

$$U(t, s) = \exp(-(t-s)A(s)) + \int_s^t P(t, r)U(r, s)dr, \tag{1.20}$$

$$V(t, s) = \exp(-(t-s)A(s)) + \int_s^t Q(t, r)V(r, s)dr, \tag{1.21}$$

$$W(t, s) = R(t, s) + \int_s^t Q(t, r)W(r, s)dr. \tag{1.22}$$

LEMMA 2. *Let* $\varphi \in H(\nu, L)$ *and* $\psi \in H(\mu, M)$ *with* $\nu>0$ *and* $\mu>0$, *and let* Y *be the solution of the equation*

$$Y(t, s)=\varphi(t, s)+\int_s^t \psi(t, r)Y(r, s)dr,$$

then $Y\in H(\nu, N)$ *with some constant* N *determined by* ν, μ, L, M *and* T *alone.*

Moreover, let $\varphi_n\in H(\nu, L)$, $\psi_n\in H(\mu, M)$ *and* Y_n *be the solution of the equation*

$$Y_n(t, s)=\varphi_n(t, s)+\int_s^t \psi_n(t, r)Y_n(r, s)dr.$$

If $\varphi_n(t, s)$ *and* $\psi_n(t, s)$ *converge to* $\varphi(t, s)$ *and* $\psi(t, s)$ *strongly as* $n\to\infty$ *for each* $0\leqq s<t\leqq T$, *then* $Y_n(t, s)$ *converges to* $Y(t, s)$ *strongly.*

The proof is a simple calculation when we use the theorem of dominated convergence (see [3]).

We want to apply Lemma 2 to the equations (1.6), (1.20), (1.8), (1.21) and (1.10), (1.22). To this end it is sufficient to check the convergence of P_n, Q_n and R_n, for we have already showed Lemma 1 and stated the remark to it.

In view of (1.14) and (1.16), we observe easily that $P_n(t, s)$ and $Q_n(t, s)$ are strongly convergent to $P(t, s)$ and $Q(t, s)$. Rewriting $R_n^1(t, s)$ defined by (1.19) in the form

$$R_n^1(t, s)=\Big(\int_{(t+s)/2}^t+\int_s^{(t+s)/2}\Big)Q_n(t, r)A_n(r)\exp(-(r-s)A_n(r))dr,$$

we verify the strong convergence of $R_n^1(t, s)$ to $R^1(t, s)$. The strong convergence of $R_n^2(t, s)$ to $R^2(t, s)$ is rather obvious. Here R^1 and R^2 are defined correspondingly. In this way we have seen that $R_n(t, s)$ converges to $R(t, s)$ strongly for each $0\leqq s<t\leqq T$.

Therefore, noting (1.12) and Lemma 1, and applying Lemma 2, we deduce that U_n, U, V_n and V are all elements of $H(1, M_{12})$; and that

$$W_n,\ W\in H(\rho, M_{13}); \tag{1.23}$$

and that $U_n(t, s)$, $V_n(t, s)$ and $W_n(t, s)$ converge strongly to $U(t, s)$, $V(t, s)$ and $W(t, s)$, respectively, for each $0\leqq s<t\leqq T$.

Now we will show that $U(t, s)$ has the properties 1)$\sim$5) mentioned in our theorem. First of all, letting $n\to\infty$ in the equation

$$A_n(t)^{-1}V_n(t, s)=U_n(t, s)A_n(s)^{-1}$$

we have

$$A(t)^{-1}V(t, s)=U(t, s)A(s)^{-1},$$

which implies $R(U(t, s)A(s)^{-1})\subset D(A(t))$ and

$$V(t, s)=A(t)U(t, s)A(s)^{-1}.$$

The term $\exp(-(t-s)A(s))$ in the equations (1.20) and (1.21) is strongly continuous in $0 \leqq s \leqq t \leqq T$, and this property is inherited by those of the solutions $U(t, s)$ and $V(t, s)$. The property 2) of U_n implies that of U. Thus we have obtained 1), 2) and a part of 3).

Next, let us prove 5), dividing the proof into two steps. For the first step we assume $f=0$. Letting $n \to \infty$ in the following

$$U_n(t, s) = A_n(t)^{-1}\{W_n(t, s) + A_n(s) \exp(-(t-s)A_n(s))\},$$

we obtain that $R(U(t, s)) \subset D(A(t))$ for $s<t$ and

$$A(t)U(t, s) = W(t, s) + A(s) \exp(-(t-s)A(s)).$$

This shows that $A(t)U(t, s)$ is strongly continuous in $0 \leqq s < t \leqq T$ and $A_n(t)U_n(t, s)$ converges to $A(t)U(t, s)$ strongly. On the other hand by (1.13) and (1.23)

$$\|A_n(t)U_n(t, s)\| \leqq M_{14}(t-s)^{-1}, \tag{1.24}$$

and hence

$$\|A(t)U(t, s)\| \leqq M_{14}(t-s)^{-1}.$$

Therefore, for any $\varepsilon > 0$ and $t \geqq s+\varepsilon$ it follows from

$$U_n(t, s)u(s) - U_n(s+\varepsilon, s)u(s) = -\int_{s+\varepsilon}^{t} A_n(r)U_n(r, s)u(s)dr$$

that

$$u(t) = U(s+\varepsilon, s)u(s) - \int_{s+\varepsilon}^{t} A(r)U(r, s)u(s)dr,$$

which shows that u is a strict solution on $[s, T]$. At the same time we have deduced the remaining part of 3).

For the second step we assume $u(s)=0$. Put for integer $n \geqq 1$

$$u_n(t) = \int_s^t U_n(t, r)f(r)dr, \tag{1.25}$$

then by the definition of U_n we can write

$$u_n(t) = \int_s^t \{f(r) - A_n(r)u_n(r)\}\, dr. \tag{1.26}$$

Multiplying (1.25) by $A_n(t)$, we have

$$\begin{aligned} A_n(t)u_n(t) = & \int_s^t A_n(t)U_n(t, r)[f(r)-f(t)]dr \\ & + \Big[\int_s^t \{A_n(r) \exp(-(t-r)A_n(r)) - A_n(t) \exp(-(t-r)A_n(t))\}\, dr \\ & + \int_s^t W_n(t, r)dr + \int_s^t A_n(t) \exp(-(t-r)A_n(t))dr\Big] f(t). \end{aligned} \tag{1.27}$$

From (1.27), noting

$$\int_s^t A_n(t)\exp(-(t-r)A_n(t))dr = I - \exp(-(t-s)A_n(t))$$

and using (1.12), (1.17), (1.23), (1.24) and the Hölder continuity of f; we conclude that

$$\|A_n(t)u_n(t)\| \leqq M_{15}, \tag{1.28}$$

where M_{15} depends on f besides θ_0, ρ, T, N_0 and N_1, and that $A_n(t)u_n(t)$ converges to a function $v(t)$ pointwise. v is then defined on $[s, T]$ by the right hand side of (1.27) with A_n, U_n and W_n replaced by A, U and W, respectively. v is continuous on $[s, T]$. In a similar way to V_n we can see $u(t)\in D(A(t))$ and $A(t)u(t)=v(t)$ for any $t\in[s, T]$. Therefore if n tends to infinity in (1.26), the pointwise convergence of $A_n(r)u_n(r)$ and (1.28) give us

$$u(t)=\int_s^t \{f(r)-A(r)u(r)\}\,dr,$$

which means that u is a strict solution on $[s, T]$. Now the proof of 5) is complete with the above two steps.

Finally we will prove 4). The strong continuity of $V(t, s)$ at $t=s$ implies the strong right differentiability of $U(t, s)A(s)^{-1}$ in t at $t=s$. And then 2), 3) and this property give us

$$\lim_{\Delta s\to+0}\frac{U(t, s+\Delta s)u_0-U(t, s)u_0}{\Delta s}=U(t, s)A(s)u_0$$

for any $u_0\in D(A(s))$ and $t>s$. Hence for any strict solution u on $[s, T]$, $U(t, r)u(r)$ is continuously differentiable from the right in $r\in(s, t)$ and

$$(\partial^+/\partial r)U(t, r)u(r)=U(t, r)f(r). \tag{1.29}$$

Integrating (1.29) on (s, t), we conclude

$$u(t)=U(t, s)u(s)+\int_s^t U(t, r)f(r)dr.$$

§2. A consequence of the main theorem.

As a consequence of Theorem 1 we have

THEOREM 2. *Under the same situation as the theorem we assume the following conditions:*

(I′) *The same condition as* (I) *stated in the theorem;*

(II′) *There exists a constant* $\rho\in(0, 1]$ *such that the domain of* $A(t)^\rho$ *is independent of* t *and* $A(t)^\rho A(0)^{-\rho}$ *is strongly continuously differentiable in* t.

Then there exists a family $\{U(t, s);\ 0\leqq s\leqq t\leqq T\}$ *of bounded operators having the properties* 1)$\sim$5) *stated in the theorem. In this case the constants*

C_0 and C_1 which appear in the statement of 3) *are determined by* θ_0, ρ, T, N_0, $N_2=\sup_{t\in[0,T]}\|dA(t)^{\rho}A(0)^{-\rho}/dt\|$ *and* $N_3=\sup_{t\in[0,T]}\|A(0)^{\rho}A(t)^{-\rho}\|$ *alone, the existence of finite* N_2 *and* N_3 *being ensured by* (II$'$). *Moreover under these assumptions U have the following property:*

6) *For any* $u_0\in D(A(0)^{\rho})$, $U(t, s)u_0$ *is continuously differentiable in* $s\in[0, t)$ *and*

$$\partial/\partial s U(t, s)u_0=\overline{U(t, s)A(s)^{1-\rho}}A(s)^{\rho}u_0 .$$

PROOF. By the assumption (II$'$) we shall show

$$\|A(t)^{\rho}dA(t)^{-1}/dt\|\leqq M_{16} . \tag{2.1}$$

Unless there is any specification, $M_{16}\cdots M_{21}$ denote the constants depending only on $\theta_0, \rho, T, N_0, N_2$ and N_3. Actually if $\rho=1$, $A(t)^{-1}$ is strongly continuously differentiable with its derivative

$$dA(t)^{-1}/dt=-A(t)^{-1}(dA(t)A(0)^{-1}/dt)A(0)A(t)^{-1} ,$$

which implies (2.1). If $0<\rho<1$, then we can write

$$A(t)^{-1}=\frac{-1}{2\pi i}\int_{\Gamma}\lambda^{-\frac{1}{\rho}}(\lambda-A(t)^{\rho})^{-1}d\lambda .$$

Since $(\lambda-A(t)^{\rho})^{-1}$ is strongly continuously differentiable in t, so is $A(t)^{-1}$, whose derivative is expressed by

$$dA(t)^{-1}/dt=\frac{-1}{2\pi i}\int_{\Gamma}\lambda^{-\frac{1}{\rho}}(\lambda-A(t)^{\rho})^{-1}\frac{dA(t)^{\rho}A(0)^{-\rho}}{dt}A(0)^{\rho}(\lambda-A(t)^{\rho})^{-1}d\lambda .$$

From this expression we obtain (2.1). Thus we have proved that the condition (II$'$) implies (2.1), and hence (II).

Now we have only to show 6). To this end we need a lemma concerned with the fractional powers of $A_n(t)$.

LEMMA 3. *Let* $\delta\in(0, \rho)$, *then* $A_n(t)^{\rho-\delta}A(t)^{-\rho}$ *converges to* $A(t)^{-\delta}$ *strongly as* $n\to\infty$.

Since

$$A_n(t)^{\rho-\delta}=\frac{\sin(\rho-\delta)\pi}{\pi}\int_0^{\infty}\mu^{\rho-\delta-1}(\mu+A_n(t))^{-1}d\mu A_n(t) , \tag{2.2}$$

we see that $A_n(t)^{\rho-\delta}A(t)^{-\rho}x$ converges to $A(t)^{-\delta}x$ for any $x\in D(A(t)^{1-\rho})$. Thus it suffices to show

$$\|A_n(t)^{\rho-\delta}A(t)^{-\rho}\|\leqq M_{17} . \tag{2.3}$$

If $\rho=1$, this is clear from (2.2). If $0<\rho<1$, it is obvious from the equation

$$A_n(t)^{\rho-\delta}A(t)^{-\rho}=\frac{\sin\rho\pi}{\pi}\int_0^{\infty}\mu^{-\rho}A_n(t)^{\rho-\delta}(\mu+A(t))^{-1}d\mu$$

and the well-known inequality

$$\|A_n(t)^{\rho-\delta}(\mu+A(t))^{-1}\| \leqq M_{18}\|(\mu+A(t))^{-1}\|^{(1-\rho+\delta)}\|A_n(t)(\mu+A(t))^{-1}\|^{(\rho-\delta)}.$$

Thus we have proved (2.3). M_{17} and M_{18} are dependent also on δ.

Let $\eta\in[0, 1)$. Operating $A_n(s)^\eta$ to (1.6) from the right, we have

$$\begin{aligned} U_n(t, s)A_n(s)^\eta = A_n(s)^\eta \exp(-(t-s)A_n(s)) \\ +\int_s^t P_n(t, r)U_n(r, s)A_n(s)^\eta dr. \end{aligned} \tag{2.4}$$

In (2.4), the term $A_n(s)^\eta \exp(-(t-s)A_n(s))$ belongs to $H(1-\eta, M_{19})$ and converges to $A(s)^\eta \exp(-(t-s)A(s))$ strongly, therefore Lemma 2 is applicable to (2.4) again. Hence it follows that

$$U_n(t, s)A_n(s)^\eta \in H(1-\eta, M_{20}) \tag{2.5}$$

and $U_n(t, s)A_n(s)^\eta$ is strongly convergent to the bounded extension

$$\overline{U(t, s)A(s)^\eta} \in H(1-\eta, M_{20}). \tag{2.6}$$

Here M_{19} and M_{20} depend on η.

Now let $\delta\in(0, \rho)$ be fixed and write $\eta=1+\delta-\rho$. By the definition of U_n

$$u_0-U_n(t, s)u_0=\int_s^t U_n(t, r)A_n(r)^\eta A_n(r)^{\rho-\delta}A(r)^{-\rho}A(r)^\rho u_0 dr.$$

Letting $n\to\infty$ in this equation with the aid of Lemma 3, (2.3) and (2.5); we obtain

$$u_0-U(t, s)u_0=\int_s^t \overline{U(t, r)A(r)^\eta}A(r)^{-\delta}A(r)^\rho u_0 dr.$$

Taking $\eta=1-\rho$ in (2.6), we have

$$\overline{U(t, r)A(r)^\eta}A(r)^{-\delta}=\overline{U(t, r)A(r)^{1-\rho}} \in H(\rho, M_{21})$$

and the desired relation.

§ 3. Remark.

Professor M. Watanabe informed the author that our main theorem can be applied to the initial-boundary value problems for parabolic equations.

Let Ω be a bounded region in R^n with a sufficiently smooth boundary,

$$A(x, t, D)=\sum_{|\alpha|\leqq 2m} a_\alpha(x, t)D^\alpha, \qquad x\in\bar{\Omega},\ t\in[0, T]$$

be uniformly strongly elliptic differential operators with smooth coefficients, and

$$B_j(x, t, D)=\sum_{|\beta|\leqq m_j} b_{j\beta}(x, t)D^\beta\,, \qquad j=1, \cdots, m$$

be normal boundary differential operators having smooth coefficients. We consider the following mixed problem

$$\begin{cases} \partial u(x, t)/\partial t+A(x, t, D)u(x, t)=f(x, t)\,, & x\in\Omega,\ t\in(0, T] \\ u(x, 0)=u_0(x)\,, \quad x\in\Omega & \\ B_j(x, t, D)u(x, t)=0\,, \quad x\in\partial\Omega,\ t\in(0, T] & \end{cases}$$

in $L^p(\Omega)$, $1<p<\infty$. For each $t\in[0, T]$, we define the linear operator $A(t)$ acting in $L^p(\Omega)$ as follows,

$$D(A(t))=\{u\in W_p^{2m}(\Omega)\,;\ B_j(x, t, D)u(x)=0,\ x\in\partial\Omega,\ j=1, \cdots, m\}$$

and for $u\in D(A(t))$

$$(A(t)u)(x)=A(x, t, D)u(x)\,.$$

We denote by $W_p^l(\Omega)$ the set of all complex-valued functions defined in Ω whose distribution derivatives $D^\alpha u$ belong to $L^p(\Omega)$ for any α with $0\leqq|\alpha|\leqq l$.

Then, it can be shown that $A(t)$ satisfies the conditions (I) and (II) of the theorem provided that we add a sufficiently large positive number to $A(t)$ if necessary.

In truth it is possible to show with the aid of the interpolation theory that such $A(t)$ satisfies the stronger condition (0.1) with a sufficiently small $\rho>0$; however, the proof is not elementary. We give below a simple proof that (II) is satisfied with

$$\rho=\begin{cases} 1\,, & \text{if } m=1 \text{ and } m_1=0 \\ \mathrm{Min}\,\{m_j/2m\,;\ m_j\geqq 1,\ j=1, \cdots, m\}\,, & \text{otherwise}\,. \end{cases}$$

For any $f\in L^p(\Omega)$, we have by the definition

$$A(x, t, D)A(t)^{-1}f(x)=f(x)\,, \qquad x\in\Omega \tag{3.1}$$

and

$$B_j(x, t, D)A(t)^{-1}f(x)=0\,, \qquad x\in\partial\Omega\,. \tag{3.2}$$

Since $A(t)^{-1}$, the bounded operator from $L^p(\Omega)$ into $W_p^{2m}(\Omega)$, is strongly continuously differentiable, (3.1) and (3.2) give

$$A(x, t, D)(dA(t)^{-1}/dt)f(x)=-\dot{A}(x, t, D)A(t)^{-1}f(x)\,, \qquad x\in\Omega\,, \tag{3.3}$$

and

$$B_j(x, t, D)(dA(t)^{-1}/dt)f(x)=-\dot{B}_j(x, t, D)A(t)^{-1}f(x)\,, \qquad x\in\partial\Omega\,, \tag{3.4}$$

where $\dot{A}$ and $\dot{B}_j$ are defined by

$$\dot{A}(x, t, D)=\sum_{|\alpha|\leqq 2m}(\partial a_\alpha(x, t)/\partial t)D^\alpha$$

and

$$\dot{B}_j(x, t, D)=\sum_{|\beta|\leqq m_j}(\partial b_{j\beta}(x, t)/\partial t)D^\beta .$$

Next, for any $\lambda\in\Sigma$ and $u\in W_p^{2m}(\Omega)$

$$\begin{aligned}(\lambda-A(x, t, D))A(t)(\lambda-A(t))^{-1}u&\\ &=(\lambda-A(x, t, D))\{\lambda(\lambda-A(t))^{-1}-1\}u\\ &=A(x, t, D)u ,\end{aligned}$$

therefore using (3.3), we have

$$\begin{aligned}(\lambda-A(x, t, D))A(t)(\lambda-A(t))^{-1}(dA(t)^{-1}/dt)f(x)&\\ =-\dot{A}(x, t, D)A(t)^{-1}f(x), \qquad x\in\Omega .&\end{aligned} \tag{3.5}$$

Similarly for $\lambda\in\Sigma$ and $u\in W_p^{2m}(\Omega)$

$$B_j(x, t, D)A(t)(\lambda-A(t))^{-1}u=-B_j(x, t, D)u$$

on $\partial\Omega$, and hence in view of (3.4) we get

$$\begin{aligned}B_j(x, t, D)A(t)(\lambda-A(t))^{-1}(dA(t)^{-1}/dt)f(x)&\\ =\dot{B}_j(x, t, D)A(t)^{-1}f(x), \qquad x\in\partial\Omega .&\end{aligned} \tag{3.6}$$

It is known that for any $\lambda\in\Sigma$ and $t\in[0, T]$ the estimate

$$\begin{aligned}|\lambda|\,\|u\|_p+\|u\|_{2m,p}\leqq M_{22}\{&\|(\lambda-A(x, t, D))u\|_p\\ &+\sum_{j=1}^m|\lambda|^{(1-m_j/2m)}\|g_j\|_p+\sum_{j=1}^m\|g_j\|_{2m-m_j,p}\}\end{aligned}$$

holds for $u\in W_p^{2m}(\Omega)$ with some constant M_{22} independent of λ and t. Here g_j is an arbitrary function which belongs to $W_p^{2m-m_j}(\Omega)$ and coincides with $B_j(x, t, D)u(x)$ on $\partial\Omega$. Substituting $A(t)(\lambda-A(t))^{-1}(dA(t)^{-1}/dt)f$ for u in this estimate, we obtain from (3.5) and (3.6)

$$\begin{aligned}|\lambda|\,\|A(t)(\lambda-A(t))^{-1}(dA(t)^{-1}/dt)f\|_p&\\ \leqq M_{23}\{\|A(t)^{-1}f\|_{2m,p}+\sum_{j=1}^m|\lambda|^{(1-m_j/2m)}\|h_j\|_p\} ,&\end{aligned}$$

where $h_j\in W_p^{2m-m_j}(\Omega)$ and

$$h_j(x)=\dot{B}_j(x, t, D)A(t)^{-1}f(x), \qquad x\in\partial\Omega .$$

If $m_j=0$, then $B_j(x, t, D)=1$, and therefore we can take $h_j=0$. Thus we conclude

$$\|A(t)(\lambda-A(t))^{-1}dA(t)^{-1}/dt\| \leqq M_{24}/|\lambda|^{\rho}$$

with

$$\rho=\begin{cases} 1, & \text{if} \quad m=1 \text{ and } m_1=0 \\ \mathrm{Min}\{m_j/2m\,;\ m_j\geqq 1,\ j=1, \cdots, m\}, & \text{otherwise}. \end{cases}$$

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