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On a theorem of Alekseevskii concerning conformal transformations

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The purpose of this note is to give a direct proof to a theorem of Alekseevskii which asserts existence of a special kind of neighborhoods around a certain point of a Riemannian manifold.

Let M be a Riemannian manifold of dimension $m \geqq 3$. The following is known as Lichnerowicz's conjecture: *If the largest connected group $C_0(M)$ of conformal transformations of M is essential* (See §1 for the meaning of terminology), *then M is conformal either to a Euclidian sphere $\boldsymbol{S}^m$ or to a Euclidian space $\boldsymbol{E}^m$.* This conjecture is affirmatively answered by Lelong-Ferrand [2] and by Obata [3] in the case when M is compact, and also by others under some additional conditions (cf. [4]). Recently Alekseevskii [1] tried to assure the conjecture for the most general case. The theorem we shall establish in this paper is stated in a slightly weaker form in the paper [1] with a proof which seems incomplete. We shall show, in a way slightly different from Alekseevskii's, how our theorem is applied to a proof of Lichnerowicz's conjecture for the general case under the assumption that M admits an essential one-parameter subgroup of $C_0(M)$.

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§1. Statement of Theorem.

Let M be a Riemannian manifold. Throughout this paper, manifolds, functions etc. are assumed to be of class C^∞. We shall denote by $C(M)$ the group of all conformal transformations of M endowed with the compact-open topology and $C_0(M)$ its connected component of the identity. A subgroup G (resp. an element ϕ) of $C(M)$ is said to be *essential*, if G (resp. ϕ) is not contained in the group of all isometric transformations of the manifold M endowed with any Riemannian metric conformal to the original one.

Let Ψ be a family of diffeomorphisms of a Riemannian manifold M which leave a point $p \in M$ fixed. A neighborhood U of p is said to be *Ψ-admissible* if for any $\phi\in\Psi$ different from the identity transformation, and for any $x\in U$ one of the sequences $\{\phi^{(i)}(x)|i=1, 2, \cdots\}$ and $\{\phi^{(-i)}(x)|i=1, 2, \cdots\}$ lies entirely in U and converges to p, where $\phi^{(i)}$ and $\phi^{(-i)}$ are i-th iterated products of ϕ and ϕ^{-1} respectively. When Ψ consists of only one element ϕ, U is said to be *ϕ-admissible.*

We can now state:

THEOREM. *Let M be a Riemannian manifold of dimension $m\geqq 3$, and ϕ be an essential conformal transformation of (M, g) which leaves a point $p\in M$ fixed. Then there exists, in an arbitrary small neighborhood of p, a ϕ-admissible neighborhood U of the point p. Moreover, when the differential ϕ_{*p} of ϕ at p is an orthogonal transformation of the tangent space $T_p(M)$ to M at p, there exists a point $q\in U$ $(q\neq p)$ such that both sequences $\{\phi^{(i)}(q)|i=1, 2, \cdots\}$ and $\{\phi^{(-i)}(q)|i=1, 2, \cdots\}$ lie entirely in U and converge to p.*

REMARK. The condition that ϕ_{*p} is an orthogonal transformation of $T_p(M)$ is preserved under any conformal change of the metric g.

We make a few observations about the proof of this theorem in the Alekseevskii's paper [1]. Let $\phi^i(x)$ be the i-th coordinate of $\phi(x)$ with respect to geodesic coordinate x^i around p, ϕ being the transformation in the theorem, and let

$$\phi^i(x)=A^i_j x^j+\frac{1}{2!}A^i_{jk}x^j x^k+\frac{1}{3!}A^i_{jkl}x^j x^k x^l+\cdots$$

be the Taylor expansion of the function $\phi^i(x)$ around the point p. Here and in the following, $\cdots$ means a residue term of higher order, indices run over $1, \cdots, m$, and we use the Einstein's convention. Alekseevskii proved the theorem ([**1**], § 2, Proposition 2) assuming that all the coefficients of the third order A^i_{jkl} are zero. It is not clear whether this is true, and even if this fact is admitted, it seems that his proof is still not complete. In this paper, we shall prove the theorem by taking into account all the coefficients up to the third order of the above Taylor expansion. Our proof may clarify local behavior of the essential transformation ϕ leaving a point fixed.

We get the following:

COROLLARY 1. *Let M be a Riemannian manifold of dimension $m\geqq 3$, and $\Psi=\{\phi_t\}$ be an essential one-parameter subgroup which leaves a point $p\in M$ fixed. Then there exists, in an arbitrary small neighborhood of p, a Ψ-admissible neighborhood.*

COROLLARY 2. *The assumptions and notation being as in Corollary* 1, *suppose that the differential $(\phi_r)_{*p}$ at p of a transformation $\phi_r\in\Psi$ $(r\neq 0)$ is an orthogonal transformation of $T_p(M)$. Let U be a Ψ-admissible neighborhood of p, and put $\Psi U=\{\phi x|\phi\in\Psi,\ x\in U\}$. Then $\Psi U=M$.*

PROPOSITION 1. *The assumptions and notation being as in Corollary* 1, *suppose that the differential* $(\phi_r)_{*p}$ *at* p *of a transformation* $\phi_r \in \Psi$ $(r \neq 0)$ *is an orthogonal transformation of* $T_p(M)$. *Then* M *is conformal to* $\boldsymbol{S}^m$.

REMARK. An essential one-parameter subgroup of $C(M)$ always has a fixed point (see, for example, [**5**]).

The following has been proved.

PROPOSITION 2 (Avez [**6**], Obata [**5**]). *Suppose that a Riemannian manifold* M *of dimension* $m \geqq 3$ *admits a one-parameter subgroup of* $C(M)$ *such that, at each of its fixed points, the divergence of the corresponding vector field does not vanish. Then* M *is conformal to* $\boldsymbol{S}^m$ *or* $\boldsymbol{E}^m$.

By Proposition 1 and 2, we get the following theorem.

THEOREM A. *If a Riemannian manifold* M *of dimension* $m \geqq 3$ *admitis an essential one-parameter subgroup of* $C(M)$, *then* M *is conformal to* $\boldsymbol{S}^m$ *or* $\boldsymbol{E}^m$.

§ 2. Proof of Theorem.

We shall denote the k-dimensional standard real vector space by $\boldsymbol{R}^k$, and its dual space by $(\boldsymbol{R}^k)^*$. An element of $\boldsymbol{R}^k$ will be considered as a column vector, and an element of $(\boldsymbol{R}^k)^*$ as a row vector with respect to the canonical basis of $\boldsymbol{R}^k$. For $y \in \boldsymbol{R}^k$ (resp. $(\boldsymbol{R}^k)^*$) ${}^t y$ will denote the transpose of y. We denote by $(\ ,\ )$ the standard inner product on $\boldsymbol{R}^k$, and put $|y| = (y, y)^{1/2}$ for $y \in \boldsymbol{R}^k$. A linear transformation of $\boldsymbol{R}^k$ will be represented by a $k \times k$ matrix with respect to the canonical basis of $\boldsymbol{R}^k$. The orthogonal group of degree k is denoted by $O(k)$.

Let $(x^1, \cdots, x^m)$ be a system of geodesic coordinates around $p \in M$ with respect to the metric g $(m = \dim M)$. We shall identify the tangent space $T_p(M)$ (resp. the space $T_p^*(M)$ of covectors) of M at p with $\boldsymbol{R}^m$ (resp. $(\boldsymbol{R}^m)^*$), a tangent vector $\sum_{i=1}^m a^i(\partial/\partial x^i)_p$ (resp. a covector $\sum_{i=1}^m b^i(dx^i)_p$) being identified with ${}^t(a^1, \cdots, a^m) \in \boldsymbol{R}^m$ (resp. $(b_1, \cdots, b_m) \in (\boldsymbol{R}^m)^*$). A coordinate neighborhood around p will be identified with an open subset of $\boldsymbol{R}^m$ through the mapping $q \to {}^t(x^1(q), \cdots, x^m(q)) \in \boldsymbol{R}^m$.

Let A be the linear transformation of $\boldsymbol{R}^m$ corresponding to the differential ϕ_{*p} of the essential transformation ϕ at p. Let ν be the function on M defined by $\phi^* g = e^\nu g$ and ξ be the element in $\boldsymbol{R}^m$ such that ${}^t\xi$ corresponds to the differential $d\nu_p$ of ν at p. We may assume ${}^t\xi A = {}^t\xi$ ([**1**], § 2, Lemma 1). This implies that ${}^t A \xi = \xi$, ${}^t A$ being the transpose of A.

Now if $\nu(p) \neq 0$, then the theorem is easily proved. So we assume $\nu(p) = 0$. Then $A \in O(m)$ and we are under the assumption of the second half of the theorem. Since ϕ is essential, ξ is not 0 ([**1**], § 2, Proposition 1). ϕ can be expanded as follows in the geodesic coordinates around p with respect to the metric g:

$$\phi^i(x)=A^i_j x^j+\frac{1}{2!}A^i_{jk}x^j x^k+\frac{1}{3!}A^i_{jkl}x^j x^k x^l+\cdots. \tag{1}$$

REMARK. The assumption $\dim M\geqq 3$ in Theorem is necessary for the proof of Proposition 1 in the paper [1].

LEMMA 1. *We have*

$$\begin{aligned}|\phi(x)|^2=|x|^2&+\frac{1}{2}(\xi, x)|x|^2+\frac{1}{6}|x|^2((x, \xi)^2+{}^t x\phi x)\\&+\frac{1}{6}x^a x^b(g_{ab,st}-A^i_a A^j_b g_{ij,kl}A^k_s A^l_t)x^s x^t\\&-\frac{1}{48}|x|^4|\xi|^2+o(|x|^4),\end{aligned} \tag{2}$$

where

$$g_{ab,st}=\frac{\partial^2(g(\partial/\partial x^a, \partial/\partial x^b))}{\partial x^s\partial x^t}(o)$$

and ϕ *is the* $m\times m$ *matrix whose* (i, j) *components are*

$$\partial^2\nu/\partial x^i\partial x^j(o) \qquad (1\leqq i, j\leqq m).$$

PROOF. Since $\{x^i\}$ are geodesic coordinates, the function $g_{ij}=g(\partial/\partial x^i, \partial/\partial x^j)$ can be expanded as

$$g_{ij}(x)=\delta_{ij}+\frac{1}{2}g_{ij,kl}x^k x^l+\cdots. \tag{3}$$

From (1) and (3) follows

$$g_{ij}(\phi(x))=\delta_{ij}+\frac{1}{2}g_{ij,kl}A^k A^l x^s x^t+\cdots. \tag{4}$$

And by (1)

$$\partial\phi^i(x)/\partial x^a=A^i_a+A^i_{ak}x^k+\frac{1}{2}A^i_{akl}x^k x^l+\cdots, \tag{5}$$

$$\partial\phi^j(x)/\partial x^b=A^j_b+A^j_{bk}x^k+\frac{1}{2}A^j_{bkl}x^k x^l+\cdots. \tag{6}$$

The function e^ν is expanded as

$$e^{\nu(x)}=1+\xi_k x^k+\frac{1}{2}\xi_k\xi_l x^k x^l+\frac{1}{2}\Phi_{kl}x^k x^l+\cdots, \tag{7}$$

where $\xi_k=\partial\nu/\partial x^k(o)$ and $\Phi_{kl}=\partial^2\nu/\partial x^k\partial x^l(o)$. The local expression of $\phi^*g=e^\nu g$ is

$$g_{ij}(\phi(x))\frac{\partial\phi^i(x)}{\partial x^a}\frac{\partial\phi^j(x)}{\partial x^b}=e^{\nu(x)}g_{ab}(x). \tag{8}$$

Substituting the formulas (3), (4), (5), (6) and (7) in (8) and comparing the coefficients of the second order terms, we obtain

(9)
$$\sum_{j=1}^{m}\left(\frac{1}{2}A_b^jA_{akl}^jx^kx^l+A_{ak}A_{bl}^jx^kx^l+\frac{1}{2}A_a^jA_{bkl}^jx^kx^l\right)$$
$$+\frac{1}{2}A_a^iA_b^jg_{ij,kl}A_s^kA_t^lx^sx^t$$
$$=\frac{1}{2}g_{ab,kl}x^kx^l+\frac{1}{2}\delta_{ab}\xi_k\xi_lx^kx^l+\frac{1}{2}\delta_{ab}\Phi_{kl}x^kx^l .$$

Comparing the coefficients of the first order terms, we obtain
$$\sum_{j=1}^{m}(A_a^jA_{bk}^j+A_b^jA_{ak}^j)=\xi_k\delta_{ab} .$$
From this follows
$$A_xy=\frac{1}{2}A\{(\xi, x)y+(\xi, y)x-(x, y)\xi\},$$
where x and y are m-vectors and A_x is the $m\times m$ matrix whose (i, j) components are $A_{jk}^ix^k$ ([**1**], § 2, p. 292). Put $y=x$ in the above formula. Then we obtain

(10)
$$A_xx=(\xi, x)Ax-\frac{1}{2}|x|^2A\xi .$$
Therefore

(11)
$$|A_xx|^2=\frac{1}{4}|x|^4|\xi|^2 .$$
Now from (9) we get

(12)
$$\sum_{j=1}^{m}A_b^jA_{akl}^jx^ax^bx^kx^l+|A_xx|^2+\frac{1}{2}x^ax^bA_a^iA_b^jg_{ij,kl}A_s^kA_t^lx^sx^t$$
$$=\frac{1}{2}x^ax^bg_{ab,kl}x^kx^l+\frac{1}{2}|x|^2((x, \xi)^2+{}^tx\Phi x) .$$
From (1) we get

(13)
$$|\phi(x)|^2=|Ax|^2+(Ax, A_xx)+\frac{1}{3}\sum_{i=1}^{m}A_j^iA_{klt}^ix^jx^kx^lx^t$$
$$+\frac{1}{4}|A_xx|^2+o(|x|^4) .$$

Substituting the formulas (10), (11) and (12) in (13), it follows:
$$|\phi(x)|^2=|x|^2+\frac{1}{2}(\xi, x)|x|^2+\frac{1}{6}|x|^2((x, \xi)^2+{}^tx\Phi x)$$
$$+\frac{1}{6}x^ax^b(g_{ab,st}-A_a^iA_b^jg_{ij,kl}A^kA^l)x^sx^t$$
$$-\frac{1}{48}|x|^4|\xi|^2+o(|x|^4) .$$

This proves the formula (2) in Lemma 1.

LEMMA 2. *Let α be any positive constant less than $|\xi|/4$. Then there exists a positive constant $\varepsilon_1=\varepsilon_1(\alpha)$ with the following property: If $|x|<\varepsilon_1$ and $(\xi, x)<-\alpha|x|$, then $|\phi(x)|<|x|$ and $(\xi, \phi(x))<-\alpha|\phi(x)|$.*

PROOF. We note first

$$|\phi(x)|^2=|x|^2+\frac{1}{2}(\xi, x)|x|^2+o(|x|^3).$$

If $|x|$ $(x\neq o)$ is sufficiently small, then

$$|o(|x|^3)|/|x|^2<\alpha|x|/2.$$

If $|x|$ $(x\neq o)$ is sufficiently small and $(\xi, x)<-\alpha|x|$, then

$$\begin{aligned}|\phi(x)|^2&=|x|^2+|x|^2\Big(\frac{1}{2}(\xi, x)+\frac{o(|x|^3)}{|x|^2}\Big)\\&<|x|^2+|x|^2\Big(-\frac{1}{2}\alpha|x|+\frac{1}{2}\alpha|x|\Big)=|x|^2,\end{aligned}$$

and so $|\phi(x)|<|x|$.

If $-2\alpha|x|<(\xi, x)<-\alpha|x|$, then from (1) and (10) we see

$$\begin{aligned}(\xi, \phi(x))&=(\xi, x)+\frac{1}{2}(\xi, x)^2-\frac{1}{4}|x|^2|\xi|^2+o(|x|^2)\\&=(\xi, x)+\frac{1}{2}(\xi, x)^2-\frac{1}{8}|x|^2|\xi|^2-\frac{1}{8}|x|^2|\xi|^2+o(|x|^2)\\&<(\xi, x)+\frac{1}{2}|x|^2\Big(4\alpha^2-\frac{1}{4}|\xi|^2\Big)-\frac{1}{8}|x|^2|\xi|^2+o(|x|^2).\end{aligned}$$

Thus if $|x|$ $(x\neq o)$ is sufficiently small, then

$$(\xi, \phi(x))<(\xi, x)<-\alpha|x|<-\alpha|\phi(x)|.$$

Since $(\xi, \phi(x))-(\xi, x)=o(|x|)$ holds, we have

$$|(\xi, \phi(x))-(\xi, x)|<\alpha|x|$$

for any x $(x\neq o)$ whose $|x|$ is sufficiently small. Therefore if $(\xi, x)\leqq-2\alpha|x|$, then

$$(\xi, \phi(x))<-2\alpha|x|+\alpha|x|=-\alpha|x|<-\alpha|\phi(x)|.$$

This proves Lemma 2.

LEMMA 3. *Let α be a positive constant less than $|\xi|/\sqrt{24}$ and n be a sufficiently large positive integer. Then there exists a positive constant $\varepsilon_2=\varepsilon_2(n)$ with the following property: If $0<|x|<\varepsilon_2$ and $-\alpha|x|\leqq(\xi, x)\leqq 0$, then $|\phi^{(n)}(x)|<|x|$.*

PROOF. We see

$$(\phi^{(r)})^*g=e^{(\nu\phi(r-1)+\nu\phi(r-2)+\cdots+\nu\phi+\nu)}g\,,$$

where r is a positive integer. If we put $\nu(r)=\nu\phi^{(r-1)}+\nu\phi^{(r-2)}+\cdots+\nu\phi+\nu$, then

$$\frac{\partial\nu(r)}{\partial x^i}=\frac{\partial\nu}{\partial x^k}\frac{\partial x^k\phi^{(r-1)}}{\partial x^i}+\frac{\partial\nu}{\partial x^k}\frac{\partial x^k\phi^{(r-2)}}{\partial x^i}+\cdots+\frac{\partial\nu}{\partial x^k}\frac{\partial x^k\phi}{\partial x^i}+\frac{\partial\nu}{\partial x^i}$$

and therefore $d\nu(r)_p=r\xi$. Furthermore,

$$\frac{\partial^2\nu(r)}{\partial x^i\partial x^j}=\sum_{q=1}^{r}\Big(\frac{\partial^2\nu}{\partial x^k\partial x^s}\frac{\partial x^k\phi^{(q-1)}}{\partial x^i}\frac{\partial x^s\phi^{(q-1)}}{\partial x^j}+\frac{\partial\nu}{\partial x^k}\frac{\partial^2x^k\phi^{(q-1)}}{\partial x^i\partial x^j}\Big). \tag{14}$$

Since $d\nu(r)_p=r\xi$ and $(\phi^{(r)})_{*p}=A^r$, the formula (10) applied to $\phi^{(r)}$ is

$$(A_r)_x x=r\Big\{(\xi,\,x)A^r x-\frac{1}{2}|x|^2A^r\xi\Big\}\,, \tag{15}$$

where $(A_r)_x$ is the $m\times m$ matrix whose $(i,\,j)$ components are

$$\frac{\partial^2x^i\phi^{(r)}}{\partial x^j\partial x^k}(o)x^k\,.$$

Let $\Phi^{(n)}$ be the $m\times m$ matrix whose $(i,\,j)$ components are

$$\partial^2\nu(n)/\partial x^i\partial x^j(o)\,.$$

From (14) for $r=n$ we get

$${}^tx\Phi^{(n)}x=\sum_{q=1}^{n}{}^tx^t(A^{q-1})\Phi A^{q-1}x+\sum_{q=1}^{n}(\xi,\,(A_{q-1})_x x)\,.$$

From this equality and (15) for $r=q-1$, we get

$${}^tx\Phi^{(n)}x=\sum_{q=1}^{n}{}^tx^t(A^{q-1})\Phi A^{q-1}x+\sum_{q=1}^{n}(q-1)\Big\{(\xi,\,x)^2-\frac{1}{2}|x|^2|\xi|^2\Big\}\,.$$

Since $A\in O(m)$, there exist positive constants M_1 and M_2 such that

$${}^tx\Phi^{(n)}x\leqq nM_1|x|^2+\frac{n(n-1)}{2}\Big\{(\xi,\,x)^2-\frac{1}{2}|x|^2|\xi|^2\Big\}$$

and

$$x^ax^b(g_{ab,st}-(A^n)_a^i(A^n)_b^jg_{ij,kl}(A^n)_s^k(A^n)_t^l)x^sx^t\leqq M_2|x|^4\,.$$

Note that M_1 and M_2 are independent of the choice of n.

Applying (2) to $\phi^{(n)}$, it follows that

$$\begin{aligned}|\phi^{(n)}(x)|^2=&|x|^2+\frac{1}{2}n(\xi,\,x)|x|^2+\frac{1}{6}|x|^2(n^2(x,\,\xi)^2+{}^tx\Phi^{(n)}x)\\&+\frac{1}{6}x^ax^b(g_{ab,st}-(A^n)_a^i(A^n)_b^jg_{ij,kl}(A^n)_s^k(A^n)_t^l)x^sX^t\\&-\frac{1}{48}n^2|x|^4|\xi|^2+o(|x|^4)\,.\end{aligned}$$

From this equality and the above two inequalities, we get

$$|\phi^{(n)}(x)|^2 \leqq |x|^2 + \frac{1}{2} n(\xi, x)|x|^2$$

$$+\frac{1}{6}|x|^2 n^2\Big((x, \xi)^2 - \frac{1}{3}\cdot\frac{1}{8}|x|^2|\xi|^2\Big)$$

$$+\frac{1}{6}|x|^4\Big(nM_1 + M_2 - \frac{1}{3}\cdot\frac{1}{8}n^2|\xi|^2\Big)$$

$$+\frac{n(n-1)}{2}\,\frac{1}{6}|x|^2\Big((\xi, x)^2 - \frac{1}{2}|x|^2|\xi|^2\Big)$$

$$-\frac{1}{6}\cdot\frac{1}{3}\cdot\frac{1}{8}|x|^4 n^2|\xi|^2 + o(|x|^4).$$

If $0<\alpha<|\xi|/\sqrt{24}$, $-\alpha|x| \leqq (\xi, x) \leqq 0$ and $x \neq o$, then

$$(x, \xi)^2 - \frac{1}{3}\cdot\frac{1}{8}|x|^2|\xi|^2 < 0$$

and

$$(\xi, x)^2 - \frac{1}{2}|x|^2|\xi|^2 < 0.$$

If n is sufficiently large, then

$$nM_1 + M_2 - \frac{1}{3}\cdot\frac{1}{8}n^2|\xi|^2 < 0.$$

If $|x|$ $(x \neq o)$ is sufficiently small, then

$$-\frac{1}{6}\cdot\frac{1}{3}\cdot\frac{1}{8}|x|^4 n^2|\xi|^2 + o(|x|^4) < 0.$$

Thus we get Lemma 3.

Since $\phi^{(n)}$ is expanded as

$$\phi^{(n)}(x) = A^n x + \frac{1}{2}(A_n)_x x + \cdots,$$

we get from (15) for $r=n$

$$(\xi, \phi^{(n)}(x)) = (\xi, x) + \frac{1}{2}n(\xi, x)^2 - \frac{1}{4}n|x|^2|\xi|^2 + o(|x|^2)$$

$$= (\xi, x) + \frac{1}{2}n((\xi, x)^2 - \frac{1}{4}|x|^2|\xi|^2)$$

$$-\frac{1}{8}n|x|^2|\xi|^2 + o(|x|^2).$$

From this equality we get the following lemma.

LEMMA 4. *Let α be a positive constant less than $|\xi|/2$ and n be any positive integer. Then there exists a positive constant $\varepsilon_3 = \varepsilon_3(n)$ with the following* *property: If* $0<|x|<\varepsilon_3$ *and* $-\alpha|x|\leqq(\xi, x)\leqq 0$, *then* $(\xi, \phi^{(n)}(x))<(\xi, x)\leqq 0$.

LEMMA 5. *Let n be a sufficiently large integer. Then there exists a positive constant $\tau=\tau(n)$ with the following property: If $(\xi, x)\leqq 0$ and $0<|x|<\tau$, then $|\phi^{(n)}(x)|<|x|$ and $(\xi, \phi^{(n)}(x))\leqq 0$. And if $(\xi, x)\geqq 0$ and $0<|x|<\tau$, then $|\phi^{(-n)}(x)|<|x|$ and $(\xi, \phi^{(-n)}(x))\geqq 0$.*

PROOF. Let α be a sufficiently small positive constant, and ε be a positive constant less than $\varepsilon_1(\alpha)$, $\varepsilon_2(n)$ and $\varepsilon_3(n)$ in Lemmas 2, 3 and 4. Using Lemma 2 n times, we see that if $0<|x|<\varepsilon$ and $(\xi, x)<-\alpha|x|$, then $|\phi^{(n)}(x)|<|x|$ and $(\xi, \phi^{(n)}(x))<-\alpha|\phi^{(n)}(x)|$. From Lemmas 3 and 4 we see that if $0<|x|<\varepsilon$ and $-\alpha|x|\leqq(\xi, x)\leqq 0$, then $|\phi^{(n)}(x)|<|x|$ and $(\xi, \phi^{(n)}(x))<(\xi, x)\leqq 0$. Therefore if $0<|x|<\varepsilon$ and $(\xi, x)\leqq 0$, then

$$|\phi^{(n)}(x)|<|x| \quad \text{and} \quad (\xi, \phi^{(n)}(x))\leqq 0\,. \tag{16}$$

Now let $\tilde{\nu}$ be the function on M defined by $(\phi^{-1})^*g=e^{\tilde{\nu}}g$. Since the differential $(\phi^{-1})_{*p}$ of ϕ^{-1} at p is A^{-1} and the differential $d\tilde{\nu}_p$ of $\tilde{\nu}$ at p is $-{}^t\xi$, we can prove in the same way as above existence of a positive constant $\tilde{\varepsilon}$ with the following property: If $0<|x|<\varepsilon$ and $(-\xi, x)\leqq 0$, then $|\phi^{(-n)}(x)|<|x|$ and $(-\xi, \phi^{(-n)}(x))\leqq 0$. This implies that if $0<|x|<\tilde{\varepsilon}$ and $(\xi, x)\geqq 0$, then

$$|\phi^{(-n)}(x)<|x|, \quad \text{and} \quad (\xi, \phi^{(-n)}(x))\geqq 0\,. \tag{17}$$

Let τ be a positive constant less than ε and $\tilde{\varepsilon}$. Then τ satisfies the conditions in Lemma 5, as follows from (16) and (17).

LEMMA 6. *Let W be a subset of $\boldsymbol{R}^k$ containing the origin o of $\boldsymbol{R}^k$, and ϕ be a continuous map of the closure $\overline{W}$ of W into itself such that $\phi(o)=o$ and $|\phi(x)|<|x|$ for any point $x\in\overline{W}$ $(x\neq o)$. Then for any point $y\in W$ the sequence $\{\phi^{(i)}(y)|i=1, 2, \cdots\}$ converges to o.*

PROOF. Take any point $y\in W$. To prove the statement, we may assume $y\neq o$. Since the sequence $\{\phi^{(i)}(y)|i=1, 2, \cdots\}$ is bounded, there exists a subsequence $\{\phi^{(t_i)}(y)|i=1, 2, \cdots\}$ which converges to a point $y_0\in\overline{W}$. Then the sequence of positive numbers $\{|\phi^{(t_i)}(y)|\,|i=1, 2, \cdots\}$ converges to $|y_0|$. On the other hand, since the sequence $\{|\phi^{(i)}(y)|\,|i=1, 2, \cdots\}$ is monotone decreasing, it must converge to a certain number $\gamma\geqq 0$. We get $\gamma=|y_0|$, because $\{|\phi^{(t_i)}(y)|\,|i=1, 2, \cdots\}$ is a subsequence of $\{|\phi^{(i)}(y)|\,|i=1, 2, \cdots\}$. Since ϕ is a continuous map, the sequence $\{\phi^{(t_i+1)}(y)|i=1, 2, \cdots\}$ converges to $\phi(y_0)$. It follows that $\gamma=|\phi(y_0)|$ by the same reason as above. Therefore $|y_0|=|\phi(y_0)|$. Then we see $y_0=o$ by the assumption that $|x|<|\phi(x)|$ for any point $x\in\overline{W}$ $(x\neq o)$. So $\gamma=0$, which means that $\{\phi^{(i)}(y)|i=1, 2, \cdots\}$ converges to o. This proves Lemma 6.

PROOF OF THEOREM. Let n be a sufficiently large integer and $\tau=\tau(n)$ be the positive constant in Lemma 5. In $\boldsymbol{R}^m$ we consider two subsets W_1 and W_2

defined by

$$W_1 = \{x \mid |x| < \tau/2,\ (\xi, x) \leqq 0\},$$

$$W_2 = \{x \mid |x| < \tau/2,\ (\xi, x) \geqq 0\}.$$

Applying Lemma 6 to W_1 and $\phi^{(n)}$, we see that for any point $y \in W_1$ the sequence $\{\phi^{(kn)}(y) \mid k=1, 2, \cdots\}$ converges to o. Put $U_1 = \bigcup_{i=0}^{n-1} \phi^{(i)}(W_1)$, $\phi^{(0)}$ being the identity transformation of $\boldsymbol{R}^m$. Since W_1 is invariant by $\phi^{(n)}$, U_1 is invariant by ϕ. Let x be any point in U_1 and put $x = \phi^{(s)}(y)$, where $y \in W_1$ and s is a nonnegative integer less than n. Then the sequence $\{\phi^{(kn-s)}(x) \mid k=1, 2, \cdots\}$ converges to o. So the sequences $\{\phi^{(kn-s+t)}(x) \mid k=1, 2, \cdots\}$ converge to o $(t=0, 1, 2, \cdots, n-1)$. This implies that the sequence $\{\phi^{(i)}(x) \mid i=1, 2, \cdots\}$ converges to o.

Put $U_2 = \bigcup_{i=1}^{n-1} \phi^{(i)}(W_2)$. Then by the same reason as above it follows that U_2 is invariant by ϕ^{-1} and for any $x \in U_2$ the sequence $\{\phi^{(-i)}(x) \mid i=1, 2, \cdots\}$ converges to o. Put $W = W_1 \cup W_2$. Then the open set $U = \bigcup_{i=0}^{n-1} \phi^{(i)}(W)$ is clearly a ϕ-admissible neighborhood of the point o. Let V be an arbitrary neighborhood of o. Then we can choose in V the admissible neighborhood U, because we can choose an arbitrary small positive constant τ for the definition of the sets W_1 and W_2. Let q be any point such that $|q| < \tau/2$ and $(\xi, q) = 0$, then the both sequences $\{\phi^{(i)}(q) \mid i=1, 2, \cdots\}$ and $\{\phi^{(-i)}(q) \mid i=1, 2, \cdots\}$ converge to o and lie entirely in U. This completes our proof of Theorem.

§ 3. Proof of Corollaries.

To prove Corollary 1 we need the following:

LEMMA 7. *Let $\Psi = \{\phi_t\}$ be a one-parameter group of diffeomorphisms of a Riemannian manifold (M, g) which leaves a point $p \in M$ fixed, and V be a ϕ_r-admissible neighborhood of p $(r \neq 0)$. Then $U = \bigcup_{0 \leqq t < r} \phi_t(V)$ is a Ψ-admissible neighborhood.*

PROOF. Let x be any point in U and put $x = \phi_{s_0}(y)$, where $y \in V$ and $0 \leqq s_0 < r$. We put $\phi = \phi_r$ for brevity. Since V is a ϕ-admissible neighborhood, one of the sequences $\{\phi^{(i)}(y) \mid i=1, 2, \cdots\}$ and $\{\phi^{(-i)}(y) \mid i=1, 2, \cdots\}$ lies entirely in V and converges to p. Assume that the sequence $\{\phi^{(i)}(y) \mid i=1, 2, \cdots\}$ converges to p. Now

$$(\phi_t)^{(i)}(x) = \phi_{s_i}(\phi^{(k_i)}(y)) \quad (\text{resp. } (\phi_{-t})^{(-i)}(x) = \phi_{s_i}(\phi^{(k_i)}(y))) \tag{18}$$

for $\phi_t \in \Psi$, $t > 0$ (resp. $t < 0$), where $t_i + s_0 = s_i + k_i r$, $0 \leqq s_i < r$ and i is a positive integer. When i goes to infinity, k_i also goes to infinity, and so $\phi^{(k_i)}(y)$ converges to p. Let W be a relatively compact neighborhood of p, and d denote the distance between two points with respect to the metric g. Since the function $f(s, q)=d(\phi_s(q), p)$ on $[0, r]\times \overline{W}$ is uniformly continuous and $f(s, p)=0$ for any number s, the sequence $\{(\phi_t)^{(i)}(x)\mid i=1, 2, \cdots\}$ (resp. $\{(\phi_{-t})^{(-i)}(x)\mid i=1, 2, \cdots\}$) converges to p. We see easily from (18) that the sequence $\{(\phi_t)^{(i)}(x)\mid i=1, 2, \cdots\}$ (resp. $\{(\phi_{-t})^{(-i)}(x)\mid i=1, 2, \cdots\}$) lies entirely in U. In the case when the sequence $\{\phi^{(-i)}(y)\mid i=1, 2, \cdots\}$ converges to p, we can show in the same way as above that for any $t>0$ the sequences $\{(\phi_t)^{(-i)}(x)\mid i=1, 2, \cdots\}$ and $\{(\phi_{-t})^{(i)}(x)\mid i=1, 2, \cdots\}$ converge to p and lie entirely in U. Thus we get Lemma 7.

PROOF OF COROLLARY 1. First one can see that any conformal transformation $\phi_t\in\Psi$ $(t\neq 0)$ is essential ([1], § 2, Proposition 1). Let V be an arbitrary neighborhood of p. Then there exist a positive constant r and a neighborhood V' of p such that $\phi_t(x)\in V$ for any t $(0\leqq t<r)$ and $x\in V'$. By Theorem we can find, in an arbitrary small neighborhood of p, an admissible neighborhood of p. Let W be a ϕ_r-admissible neighborhood of p contained in V', and put $U=\bigcup_{0\leqq t<r}\phi_t(W)$. It is clear that U is contained in V. We see by Lemma 7 that U is a Ψ-admissible neighborhood, which proves Corollary 1.

PROOF OF COROLLARY 2. Suppose that $M-\Psi U$ is not empty and let x be any element in $M-\Psi U$. Then x is a fixed point under Ψ ([1], § 4, Lemma 6, Corollary). By Corollary 1, there exists a Ψ-admissible neighborhood V of x. Then $\Psi U-\{p\}=\Psi V-\{x\}$ ([1], § 4, Lemma 6, Corollary). By Theorem, there exists a point $q\in\Psi U-\{p\}$ such that both sequences $\{(\phi_r)^{(i)}(q)\mid i=1, 2, \cdots\}$ and $\{(\phi_r)^{(-i)}(q)\mid i=1, 2, \cdots\}$ converge to p. On the other hand, since $q\in\Psi V-\{x\}$, one of them converges to x, which is a contradiction. Thus we get Corollary 2.

§ 4. Proof of Proposition 1 and Theorem A.

PROOF OF PROPOSITION 1. Let U be a Ψ-admissible neighborhood of p. Put $W=\{q\in M\mid \lim_{t\to-\infty}\phi_t(q)=p\}$, $W^*=\{q\in M\mid \lim_{t\to\infty}\phi_t(q)=p\}$. By Corollary 2, $M=\Psi U$. Since $\Psi U\subset W\cup W^*$, M is conformally flat ([5], § 1, Lemma 1.2). Therefore M is conformal to an $\tilde{\Psi}$-invariant open submanifold $\tilde{V}$ of S^m, where $\tilde{\Psi}=\{\tilde{\phi}_t\}$ is an essential one-parameter subgroup of $C(S^m)$ with a fixed point $\tilde{p}\in V$, and $(\tilde{\phi}_r)_{*\tilde{p}}$ is an orthogonal transformation of $T_{\tilde{p}}(S^m)$ ([1], § 4, the proof of Lemma 5). Let $\tilde{U}(\subset\tilde{V})$ be a Ψ-admissible neighborhood of $\tilde{p}$. Then $\tilde{V}=\Psi\tilde{V}\supset\Psi\tilde{U}$. Therefore $\tilde{V}=S^m$ by Corollary 2, which proves Proposition 1.

LEMMA 8. *Let X be a vector field on a Riemannian manifold (M, g), and denote by* div *X the divergence of X and by L_x the Lie derivative with respect to X. Then*

$$\operatorname{trace}(L_X g)_q=2(\operatorname{div} X)_q .$$

PROOF. Recall that $(\operatorname{div} X)_q$ is the trace of the endomorphism $V\to\nabla_X X$ of

$T_q(M)$, where ∇ denotes the covariant derivation. Then

$$\begin{aligned}\operatorname{trace}(L_X g)_q &= g^{ij}(\nabla_i X_j + \nabla_j X_i) \\ &= 2g^{ij}\nabla_i X_j = 2(\operatorname{div} X)_q .\end{aligned}$$

LEMMA 9. *Let $\Psi=\{\phi_t\}$ be a one-parameter subgroup of $C(M)$ with a fixed point p, and X be the corresponding vector field. Then $(\operatorname{div} X)_q=0$ if and only if the differential $(\phi_1)_{*p}$ of ϕ_1 at p is an orthogonal transformation of $T_p(M)$.*

REMARK. If $(\phi_1)_{*p}$ is an orthogonal transformation of $T_p(M)$, then $(\phi_t)_{*p}$ is also an orthogonal transformation of $T_p(M)$ for any t.

PROOF. Let ν_t be the function on M defined by $\phi_t{}^*g=e^{\nu_t}g$. Then $(\phi_t{}^*g)_p=e^{t\nu_1(p)}g_p$. Therefore $(L_X g)_p=\nu_1(p)g_p$. By Lemma 8, if $\nu_1(p)=0$, then $(\operatorname{div} X)_p=0$ and if $\nu_1(p)\neq 0$, then $\operatorname{div} X\neq 0$, which proves Lemma 9.

PROOF OF THEOREM A. Let Ψ be an essential one-parameter subgroup of $C(M)$ and X be the corresponding vector filed. If Ψ has a fixed point p such that $(\operatorname{div} X)_p=0$, then by Corollary 2 and Lemma 9, M is conformal to $\boldsymbol{S}^m$. Otherwise, by Proposition 2, M is conformal to $\boldsymbol{S}^m$ or to $\boldsymbol{E}^m$.

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