J. Math. Soc. Japan
Vol. 28, No. 2, 1976

Representation of pseudo-holomorphic functions of several complex variables

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(Received Jan. 30, 1975)
(Revised Aug. 7, 1975)

§ 0. Introduction.

Hitotumatu [6] was the first to introduce K-quasi-conformal functions of several complex variables for a positive number $K>1$ and derived, in a function-theoretic approach, some properties similar to holomorphic functions (the maximum modulus principle and that the set of zeros of a K-quasi-conformal function in a neighborhood of its ordinary point is an $(n-1)$-dimensional complex manifold if not empty etc.) Moreover he obtained several properties of non singular mappings determined by them.

A function $g(t, \bar{t})$ of class C^1 on an open set Δ in the complex plane $\boldsymbol{C}$ is *K-quasi-conformal* on Δ if and only if it satisfies the Beltrami equation $\partial_{\bar{t}} g=\mu\partial_t g$ with $|\mu|\leqq(K-1)/(K+1)$ at each point of Δ. We use the notation $\bar{z}=(\bar{z}_1, \cdots, \bar{z}_n)\in\boldsymbol{C}^n$. A function $f(z, \bar{z})$ of class C^1 on a domain D in $\boldsymbol{C}^n$ is said to be *K-quasi-holomorphic* on D if, for any $(c, z_0)\in(\boldsymbol{C}^n-\{0\})\times\boldsymbol{C}^n$, the function $f(ct+z_0, \bar{c}\bar{t}+\bar{z}_0)$ is K-quasi-conformal on the open set $\{t\in\boldsymbol{C};\ ct+z_0\in D\}$ in the complex plane $\boldsymbol{C}$ of the complex variable t.

He proved also that a K-quasi-holomorphic function f on D satisfies the system of differential equation $\partial\bar{f}=\kappa(z, \bar{z})\partial f$ everywhere on D, where $|\kappa|\leqq(K-1)/(K+1)$ and that, conversely, given any κ continuous on D such that $|\kappa|\leqq k_0<1$, solutions of class C^1 on D of the above equations are $(1+k_0)/(1-k_0)$-quasi-holomorphic functions where ∂ denotes the operator $\partial=dz_1\partial_{z_1}+\cdots+dz_n\partial_{z_n}$ and the bar is the complex conjugate. The κ is called the *characteristic* (function) of f.

In [7] we dealt with those generalized analytic functions of several complex variables which satisfy the system of differential equations

$$(0.1)\qquad \bar{\partial}f=(a_1f+b_1\bar{f})d\bar{z}_1+\cdots+(a_nf+b_n\bar{f})d\bar{z}_n\,,\quad \text{where } \bar{\partial}=d\bar{z}_1\partial_{\bar{z}_1}+\cdots+d\bar{z}_n\partial_{\bar{z}_n}\,.$$

Under suitable assumptions on the coefficients of the equation (0.1), we showed by using a special nonsingular *K-quasi-holomorphic mapping*, and by a homeomorphism satisfying a Beltrami equation, that (0.1) can be transformed into the equation

$$(0.2) \qquad \partial_{\bar{t}} w = aw + b\bar{w}\,.$$

In this paper the method of obtaining a representation of the quasi-holomorphic functions (hereafter "pseudo" is used for "quasi") will be developed from the point of view of differential equations and, through such a representation, some of the properties similar to holomorphic functions will be obtained.

If the characteristic function κ satisfies the condition in Theorem 19, then each pseudo-holomorphic function associated with such κ is written by a pseudo-analytic function of the second kind and a holomorphic function ϕ of several complex variable (ϕ depends only on κ), or in other words, by a generalized analytic function (pseudo-analytic one of the first kind which is a solution of the equation of the type (0.2)) and a ϕ (§3.1). When the coefficients a, b of (0.2) are analytic in $\mathrm{Re}(t)$ and $\mathrm{Im}(t)$, the new representation of solutions of (0.2) was obtained by Vekua [**10**]. Hence in case that κ is analytic in $\mathrm{Re}(z_j)$ and $\mathrm{Im}(z_j)$, we can have a precise representation of our functions. A recent paper related to our topic is Bauer and Ruscheweyh [**2**], where the explicit representation is investigated in detail in a very special case.

On the contrary if κ is anti-holomorphic, in other words, if it satisfies the Frobenius-Nirenberg condition [**8**], our functions are expressible as a simple form (§4).

We note that the arguments in this paper can be extended over complex manifolds.

The author expresses his sincere thanks to the referees for their useful advices.

§1. Properties of $S(\kappa\,;D)$.

1.1. Throughout this paper, all the functions under consideration are defined on a subset of $\boldsymbol{C}^n$ and of class C^∞ on the set considered.

If a function f is defined on a set M, $f|N$ means the restriction of f to a subset N of M.

$U_z^n(a\,;r)$ or $U^n(a\,;r)$ denotes the polydisc with center a and polyradius $r=(r_1,\cdots,r_n)$: $\{z\in\boldsymbol{C}^n\,;\,|z_j-a_j|<r_j,\ j=1,\cdots,n\}$. In the case $n=1$ we shall denote it by $U_z(a\,;r)$ or $U(a\,;r)$. When no confusion is likely, we shall use U^n for $U^n(a\,;r)$.

If W is an open set, the set $\{z\in\boldsymbol{C}^n\,;\,|z_j-a_j|<r_j,\ j=1,\cdots,n,\ j\neq k\,;\ z_k\in W\}$ is denoted by $U_z^{k-1}\times W\times U_z^{n-k}$.

$f(z)$ and $f(\bar{z})$ mean that they are holomorphic and anti-holomorphic in z, respectively.

A function f defined on an open set G is said to be nondegenerate or degenerate on G according as $df\wedge d\bar{f}\neq\boldsymbol{0}$ or $=\boldsymbol{0}$ on G, where $d=\partial+\bar{\partial}$.

Let us consider a function $\kappa(z, \bar{z})$ defined on an open set D_0 such that on D_0

$$(H_0) \qquad \partial_{z_n}\kappa \neq 0\,,$$

and introduce the differential operators

$$X_j = (\partial_{z_j}\kappa)\partial_{z_n} - (\partial_{z_n}\kappa)\partial_{z_j}\,,$$

$$\bar{X}_j = \overline{(\partial_{z_j}\kappa)}\partial_{\bar{z}_n} - \overline{(\partial_{z_n}\kappa)}\partial_{\bar{z}_j}\,,$$

where j runs from 1 to $n-1$. We require the κ to satisfy the condition

$$(H_1) \qquad (\partial_{z_j}\kappa)\bar{X}_k\partial_{z_n}\kappa - (\partial_{z_n}\kappa)\bar{X}_k\partial_{z_j}\kappa = 0$$

on D_0, where j and k run from 1 to $n-1$.

1.2. We shall consider the system of differential equations

$$(1.2.1) \qquad \begin{cases} \partial\kappa \wedge \partial w = \mathbf{0}\,, \\ \partial\kappa \wedge \partial\bar{w} = \mathbf{0}\,. \end{cases}$$

Let D be a subdomain of D_0. Introducing the notation

$$S(\kappa\,; D) = \{w\,;\ \partial\kappa \wedge \partial w = \mathbf{0},\ \partial\kappa \wedge \partial\bar{w} = \mathbf{0} \text{ on } D\}\,,$$

we describe the fundamental properties of $S(\kappa\,; D)$.

PROPOSITION 1. (i) *$S = S(\kappa\,; D)$ is a vector space over $\mathbf{C}$, the field of complex numbers.*

Let w with or without sub-script belong to S in the following.

(ii) *$\bar{w}$, w_1w_2 and w_1/w_2 ($w_2 \neq 0$) also belong to S.*

(iii) *For a function F of a variable defined on $w(D)$, the composite function $F \circ w$ is also in S.*

(iv) *Let w_0 be nondegenerate on D. For any point a of D there exist a small neighborhood V of a, $V \subset D$ and a function F defined on $w_0(V)$ such that $w = F \circ (w_0 | V)$.*

(v) *For w nondegenerate on D, the inverse image of a point under the map w is an $(n-1)$-dimensional complex manifold if not empty.*

(vi) *If w_0 is nondegenerate on D, then either $\partial_{z_n}w_0 \neq 0$ or $\partial_{z_n}\bar{w}_0 \neq 0$ on D. If $\partial_{z_n}w_0 \neq 0$ and $\partial_{z_j}w_0 = 0$ at a point z^0 of D for $j \neq n$, then for any w we have $\partial_{z_j}w = 0$ at z^0. $\bar{w}_0$ also has the same result.*

(vii) *For w such that it is degenerate on D and $\partial w \neq \mathbf{0}$, let a function μ be defined by the relation $\partial\bar{w} = \mu(z, \bar{z})\partial w$. Then μ also is in S.*

(iv) and (v) were proved in [6, 7]. The others are obvious. The above Proposition will be frequently used in the sequel.

PROPOSITION 2. *Let μ be such that $\partial\mu \neq \mathbf{0}$ and $\partial\kappa \wedge \partial\mu = \mathbf{0}$ on D. Then*

$$S(\mu\,;D)=S(\kappa\,;D)\,.$$

Observe that, since we may assume that $\partial_{z_n}\mu\neq 0$ on D, the above μ also satisfies the assumption (H_1) on D. The proof is obvious.

Let a be any point of D and σ a nondegenerate element of $S(\kappa\,;D)$ such that $\partial_{z_n}\sigma\neq 0$ and $\sigma(a,\bar{a})=0$. Putting $t'=(t_1,\cdots,t_{n-1})$ and $z'=(z_1,\cdots,z_{n-1})$, let us consider the change of variables T on $U^n(a\,;r)$

$$\begin{cases} t'=z'\,, \\ t_n=\sigma(z,\bar{z}) \end{cases}$$

such that $U^n(a\,;r)$ is homeomorphic to the neighborhood of the origin $T(U^n)$. From (1.2.1) it is seen that there is a function ρ defined on D such that $\partial\kappa=\rho\partial\sigma$. In this way we have

$$X_j=\rho\{(\partial_{z_j}\sigma)\partial_{z_n}-(\partial_{z_n}\sigma)\partial_{z_j}\}\,,\qquad j=1,\cdots,n-1\,. \tag{1.2.2}$$

The operators X_j are transformed into

$$Y_j=\widehat{(\rho\partial_{z_n}\sigma)}\partial_{t_j}\,, \tag{1.2.3}$$

where the hat denotes the function transformed by T.

Let f be a function on D such that

$$\partial f\wedge\partial\sigma=\mathbf{0}\,. \tag{1.2.4}$$

If we take the change of variables T, then (1.2.4) becomes, by (1.2.3),

$$\partial_{t_j}\hat{f}=0\,,\qquad j=1,\cdots,n-1\,.$$

Thus we see that $\hat{f}$ is anti-holomorphic in t' such that $t\in T(U^n)=\hat{U}_1\times\cdots\times\hat{U}_{n-1}\times\sigma(U^n)$, where $\hat{U}_j=U_{t_j}(0\,;r)$, $j=1,\cdots,n-1$. Therefore, f can be written in the form

$$f=F(\bar{z}',\sigma(z,\bar{z}),\overline{\sigma(z,\bar{z})})\qquad\text{on}\quad U^n\,. \tag{1.2.5}$$

Conversely, considering f defined on D and given by (1.2.5) near any point of D, we can see easily that it satisfies (1.2.4) on D.

Thus we reached the following

PROPOSITION 3. *Let σ be a nondegenerate element of $S(\kappa\,;D)$ and f a function on D satisfying* (1.2.4). *For any point of D there are a polydisc $U^n(a\,;r)$ and a function $F(\bar{t}',t_n,\bar{t}_n)$ anti-holomorphic in t' for t of $T(U^n(a\,;r))$ such that f is written in the form* (1.2.5) *on $U^n(a\,;r)$. Conversely, if f has the form* (1.2.5) *near each point of D, then f satisfies* (1.2.4) *on D.*

COROLLARY 4 (see Proposition 1, (iv)). *Let σ be a nondegenerate element* *of $S(\kappa;D)$. Then w is in $S(\kappa;D)$ if and only if, near each point of D, w is of the form*

$$w=W\circ\sigma. \tag{1.2.6}$$

PROOF. Let w be in $S(\kappa;D)$ and a any point of D. By Proposition 3, we have $w=F(\bar{z}', \sigma, \bar{\sigma})$. From this, if we set $\mu=(\partial_{z_j}\bar{\sigma})/(\partial_{z_j}\sigma)$, we have

$$\partial\bar{w}=\partial'\bar{F}+(\partial_{t_n}\bar{F}+\mu\partial_{\bar{t}_n}\bar{F})\partial\sigma,$$

where $\partial'\bar{F}=\partial_{t_1}\bar{F}dz_1+\cdots+\partial_{t_{n-1}}\bar{F}dz_{n-1}$. Since $\partial\bar{w}\wedge\partial\sigma=\mathbf{0}$ on $U^n(a;r)$, we obtain that $\partial'\bar{F}\wedge\partial\sigma=\mathbf{0}$ and hence $\partial'\bar{F}=\mathbf{0}$, because $\partial_{z_n}\sigma\neq 0$ on U^n. In this way (1.2.6) is obtained. The rest is clear.

If $a\in D$, Proposition 3 shows that we can associate κ with a function K and $U^n(a;r)$ such that $\kappa=K(\bar{z}', \sigma, \bar{\sigma})$ on U^n. In the arguments to follow, we shall say that κ is expressed in terms of K and σ on U^n, or briefly, has a *local expression* (K, σ, U^n, n). Whenever we handle a nondegenerate element σ of $S(\kappa;D)$, considering the complex conjugate $\bar{\sigma}$ if necessary, we may assume that σ always has the property $|\partial_{z_n}\sigma|^2-|\partial_{\bar{z}_n}\sigma|^2>0$.

In general, a local expression (K, σ, U^n, k) of κ means that on U^n

$$\kappa(z,\bar{z})=K(\bar{z}_1,\cdots,\bar{z}_{k-1},\sigma(z,\bar{z}),\overline{\sigma(z,\bar{z})},\bar{z}_{k+1},\cdots,\bar{z}_n),$$

where σ satisfies that on U^n, $|\partial_{z_k}\sigma|^2-|\partial_{\bar{z}_k}\sigma|^2>0$ and where $K(\bar{t}_1,\cdots,\bar{t}_{k-1},t_k,\bar{t}_k,\bar{t}_{k+1},\cdots,\bar{t}_n)$ defined on $T_k(U^n)$ is anti-holomorphic in each $t_j\in\hat{U}_j$, $j\neq k$, and T_k is the change of variables:

$$\begin{cases} t_j=z_j, & j\neq k, \\ t_k=\sigma(z,\bar{z}) & \end{cases} \tag{1.2.7}$$

such that $T_k(U^n)$ is homeomorphic to U^n.

For convenient reference we shall exhibit the

COROLLARY 5. *Let a be in D and $U^n(a;r)$ a polydisc such that the mapping* (1.2.7) *with ϕ, holomorphic on U^n, for σ is biholomorphic. The following statements are equivalent:*

(1) *$\phi(z)$ is in $S(\kappa;U^n)$.*

(2) *$\kappa(z,\bar{z})$ has a local expression (K,ϕ,U^n,k) for some number k, $1\leqq k\leqq n$.*

The following lemma is a basic one in this paper.

LEMMA 6 [7]. *Assume that a function κ, defined on D_0, satisfies (H_0) and (H_1). Then for any point a of D_0 and for any point $b\in\boldsymbol{C}$ there are a neighborhood $U^n(a;r)$ and a nondegenerate function σ, satisfying* (1.2.1) *on U^n and $\sigma(a,\bar{a})=b$.*

§ 2. Pseudo-holomorphic functions.

2.1. In the following we shall consider the system of differential equations

$$\partial\bar{w}=\kappa(z, \bar{z})\partial w\ , \tag{2.1.1}$$

whose coefficient κ is defined on C^n, has a compact support and the sup-norm $\|\kappa\|<1$.

From now on D denotes a domain. Let a function w satisfy the equations (2.1.1) on D. We say that w is a *pseudo-holomorphic* function of the second kind of several complex variables, or briefly, pseudo-holomorphic on D and, following Hitotumatu, call the κ the characteristic (function) of w. In case of $n=1$, w is a so-called pseudo-analytic function of the second kind which was introduced by Bers [**3**, **4**, **5**, **9**].

PROPOSITION 7 (Identity theorem). *Any pseudo-holomorphic function on D vanishing on a subdomain of D is identically zero on D.*

PROPOSITION 8 (Maximum modulus principle) [**6**]. *No nonconstant pseudo-holomorphic function on D has any absolute maximum point in D.*

Both propositions are proved by induction on the dimension n and by using the representation theorem for a complex variable [**4**, **5**, **9**]. Since the technique used in the proof is standard, we shall describe only the proof of the latter.

We may assume that the origin is in D and our function w has an absolute maximum $|c|\neq 0$ at the origin. Taking a polydisc $U^k(0\,;\varepsilon)$ in D, we can prove that $w\equiv c$ on $U^k(0\,;\varepsilon)$. To do this, putting $\varepsilon'=(\varepsilon_1, \cdots, \varepsilon_{k-1})$, $z'=(z_1, \cdots, z_{k-1})$, and $\hat{w}(z', \bar{z}')=w(z', 0, \bar{z}', 0)$, consider $\hat{w}(\not\equiv 0)$ on $U^{k-1}(0\,;\varepsilon')$. Then $\hat{w}(z', \bar{z}')\equiv c$ on $U^{k-1}(0\,;\varepsilon')$. Let $\xi=(\xi_1, \cdots, \xi_k)$ be any point of $U^k(0\,;\varepsilon)$ and fix it. Setting $\tilde{w}(z_k, \bar{z}_k)=w(\xi', z_k, \bar{\xi}', \bar{z}_k)\not\equiv 0$, consider it on $U_{z_k}(0\,;\varepsilon_k)$, then $\tilde{w}(0)=c$, which leads to $w(\xi, \bar{\xi})=c$. Hence we see from Proposition 7 that a contradiction is derived.

We see from Proposition 7 that if the set of nonordinary points of w, $N=\{z\in D\,;\,\partial w=\mathbf{0}\}$ has an inner point, then w is constant, so that N is nowhere dense in D unless w is constant (see Theorem 10 and Proposition 25).

2.2. In this section we shall discuss the properties of pseudo-holomorphic functions with the characteristic κ such that $\partial\kappa\neq\mathbf{0}$. Let D_0 denote the set $\{z\in \boldsymbol{C}^n\,;\,\partial\kappa\neq\mathbf{0}\}$ and $w(z, \bar{z})$ a nonconstant pseudo-holomorphic function on D contained in D_0. Then we can see at once that w needs to satisfy the system of equations (1.2.1) on D.

For a point a of D we may assume that $\partial_{z_n}\kappa\neq 0$ on a polydisc $U^n(a\,;r)$ in D. Then it is found that κ needs to satisfy the hypothesis (H_1) on U^n [**7**]. In view of the purpose of this paper we may assume without loss of generality that the κ satisfies the condition (H_0) on D_0. Since at a point where $\partial\kappa=\mathbf{0}$, of course, we have (H_1), from now on we may consider κ subject to (H_1) on the whole space $\boldsymbol{C}^n$.

Let z^0 be any point of D. By Lemma 6 we have a polydisc $U^n(z^0; r)$ and a function $\sigma(z, \bar{z})$ nondegenerate on U^n such that σ satisfies (1.2.1) on U^n and $\sigma(z^0, \bar{z}^0)=0$. Since w satisfies also (1.2.1), it is seen by virtue of Proposition 1, (iv) that, restricting U^n further if necessary, $w|U^n$ is written in the form $F\circ\sigma$, where F is defined on a neighborhood $\sigma(U^n)$ of the origin in $\boldsymbol{C}$.

It follows from Proposition 1, (vi) that there exists, perhaps after restricting U^n to a smaller polydisc, a function μ defined on U^n such that

$$\partial\bar{\sigma}=\mu(z, \bar{z})\partial\sigma\,, \qquad \|\mu\|<1\,. \tag{2.2.1}$$

By virtue of Proposition 3, κ has a local expression (K, σ, U^n, n). From (2.2.1) we have $\partial\mu\wedge\partial\sigma=\boldsymbol{0}$. Again, using Proposition 3, we see that μ has a local expression (L, σ, U^n, n).

We write, for simplicity, w and κ for $w|U^n$ and $\kappa|U^n$ respectively. Insert $w=F\circ\sigma$ into (2.1.1), then we obtain

$$\kappa(z, \bar{z})=\frac{\partial_t\bar{F}+\overline{\partial_t F}\mu(z, \bar{z})}{\partial_t F+\partial_{\bar{t}}F\mu(z, \bar{z})}\bigg|_{t=\sigma(z,\bar{z})}. \tag{2.2.2}$$

On eliminating $\partial_t\bar{F}$ from (2.2.2), we have

$$\partial_t F|_{t=\sigma(z,\bar{z})}=\{\alpha(z, \bar{z})\partial_t F+\beta(z, \bar{z})\overline{\partial_t F}\}\,|_{t=\sigma(z,\bar{z})}\,,$$

where

$$\alpha(z, \bar{z})=-\frac{1-|\kappa(z, \bar{z})|^2}{1-|\kappa(z, \bar{z})\mu(z, \bar{z})|^2}\,\overline{\mu(z, \bar{z})}\,,$$

$$\beta(z, \bar{z})=\frac{1-|\mu(z, \bar{z})|^2}{1-|\kappa(z, \bar{z})\mu(z, \bar{z})|^2}\,\overline{\kappa(z, \bar{z})}\,.$$

Making use of $\|\kappa\|<1$ and $\|\mu\|<1$, we see

$$\|\alpha\|+\|\beta\|<1\,.$$

In this way we have reached the following statement which is convenient for later reference.

LEMMA 9. *Let $w(z, \bar{z})$ be a pseudo-holomorphic function on D with κ satisfying (H_0) on D_0. For any z^0 of D there exist a polydisc $U^n(z^0; r)$ in D, functions σ nondegenerate on U^n and F of one complex variable defined on $\sigma(U^n)$ such that $w=F\circ\sigma$ on U^n, where σ and F satisfy the following conditions:*

(i) *σ is in $S(\kappa; U^n)$,*

(ii) *Define μ by the relation $\partial\bar{\sigma}=\mu\partial\sigma$. $\kappa|U^n$ and μ have the local expressions (K, σ, U^n, n) and (L, σ, U^n, n) respectively, and*

(iii) *F satisfies the differential equation*

(2.2.3) $$\partial_{\bar{t}} F(t, \bar{t})|_{t=\sigma} = \alpha(z', \bar{z}', t, \bar{t}) \partial_t F(t, \bar{t})|_{t=\sigma} + \beta(z', \bar{z}', t, \bar{t}) \overline{\partial_t F(t, \bar{t})}|_{t=\sigma},$$

where

$$\alpha(z', \bar{z}', t, \bar{t}) = -\frac{1-|K(\bar{z}', t, \bar{t})|^2}{1-|K(\bar{z}', t, \bar{t})L(\bar{z}', t, \bar{t})|^2} \overline{L(\bar{z}', t, \bar{t})},$$

$$\beta(z', \bar{z}', t, \bar{t}) = \frac{1-|L(\bar{z}', t, \bar{t})|^2}{1-|K(\bar{z}', t, \bar{t})L(\bar{z}', t, \bar{t})|^2} \overline{K(\bar{z}', t, \bar{t})},$$

and $\|\alpha\| + \|\beta\| < 1$ ($\|*\|$ *denotes the supremum of* $|*|$ *on* U^n).

REMARK 1. We can always choose σ whose characteristic μ (defined by (2.2.1)) satisfies $\partial\mu(z^0, \bar{z}^0) \neq 0$. In fact, if $\partial\mu(z^0, \bar{z}^0) = 0$, we consider on $U^n(z^0; r)$

$$\hat{\sigma} = \begin{cases} G \circ \sigma & \text{when} \quad \mu(z^0, \bar{z}^0) \neq 0, \\ G \circ G \circ \sigma & \text{when} \quad \mu(z^0, \bar{z}^0) = 0, \end{cases}$$

where $G(t, \bar{t}) = 2t + |t|^2 + \bar{t}$.
By a simple computation we obtain that if $\mu(z^0, \bar{z}^0) \neq 0$,

$$\partial\hat{\mu}(z^0, \bar{z}^0) = \frac{2\mu(z^0, \bar{z}^0)\{1-\hat{\mu}(z^0, \bar{z}^0)\}}{2+\mu(z^0, \bar{z}^0)} \partial\sigma(z^0, \bar{z}^0) \neq \mathbf{0},$$

where $\hat{\mu}(z, \bar{z}) = \{1+2\mu(z, \bar{z})\} / \{2+\mu(z, \bar{z})\}$, and that if $\mu(z^0, \bar{z}^0) = 0$,

$$\partial\hat{\mu}(z^0, \bar{z}^0) = (4/5)\{1-\hat{\mu}(z^0, \bar{z}^0)\} \partial\sigma(z^0, \bar{z}^0) \neq \mathbf{0},$$

where $\hat{\mu}(z^0, \bar{z}^0) = 4/5$. Thus the desired result is obtained.

In this way, by virtue of Proposition 2, it follows that

$$S(\kappa | U^n, U^n) = S(\sigma, U^n) = S(\mu, U^n).$$

Let κ be the coefficient of the system (1.2.1) and bounded on D. We may assume that $\|\kappa\| < 1$ on D. There does not always exist a pseudo-holomorphic function whose characteristic one is κ (Example 4, § 2.4). The above relation shows that, exchanging κ for μ if necessary, one can consider κ of the system (1.2.1) (in local) the characteristic of certain pseudo-holomorphic function. The remark ends.

As easily seen from the proof of the above lemma, whenever we think of a pseudo-holomorphic function with κ such that $\partial\kappa \neq \mathbf{0}$ on D, we can associate with each point of D a *triple* (F, σ, U^n), where U^n has the center at that point. It should be noted that U^n is so small that any element of $S(\kappa; U^n)$ is expressed by the composite function $G \circ \sigma$, where G is a function defined on $\sigma(U^n)$.

We say that a pseudo-holomorphic function w has a triple (F, σ, U^n) at each point of D.

THEOREM 10. *Let* w *be nonconstant pseudo-holomorphic on* D. *The set* N

of nonordinary points of w is an $(n-1)$-dimensional complex manifold unless N is empty.

PROOF. Let z^0 be any point of N. Lemma 9 shows that w has a triple (F, σ, U^n) at z^0. On account of $w=F\circ\sigma$, we find that

$$\begin{aligned}\partial w &= \partial_t F \partial\sigma + \partial_{\bar{t}} F \partial\bar{\sigma} \\ &= \{\partial_t F + \mu(z, \bar{z})\partial_{\bar{t}} F\}\partial\sigma .\end{aligned}$$

Since $\partial\sigma \neq \mathbf{0}$ at z^0 in N,

$$(\partial_t F)\circ\sigma + \mu(z, \bar{z})(\partial_{\bar{t}} F)\circ\sigma = 0 \quad \text{at} \quad z=z^0 . \tag{2.2.4}$$

On the other side, from the differential equation (2.2.3) it follows the inequality

$$|\partial_t F|^2 - |\partial_{\bar{t}} F|^2 \geqq 0 \quad \text{on} \quad \sigma(U^n) .$$

Hence from this and (2.2.4) it is verified that at $t=0$ (note that $\sigma(z^0, \bar{z}^0)=0$)

$$\partial_t F = 0 .$$

Let ξ be any point of $N \cap U^n$. A similar argument shows $\partial_t F(\eta, \bar{\eta})=0$, where $\eta=\sigma(\xi, \bar{\xi})$. It is obvious that $\sigma^{-1}(\eta)\cap U^n \subset N \cap U^n$.

Now we want to prove that the set $P=\{t\in\sigma(U^n) ; \partial_t F=0\}$ is isolated. If it had been shown, the connected component, which contains ξ, of $N\cap U^n$ would be mapped under σ to the zero point of $\partial_t F$. Therefore, on using Proposition 1, (v), we obtain that, by restricting to a smaller polydisc, $N\cap U^n$ is a connected $(n-1)$-dimensional complex manifold.

Using the change of variables in the proof of Proposition 3, the equation (2.2.3) leads to the relation

$$\partial_{\bar{t}_n} F(t_n, \bar{t}_n) = \hat{\alpha}(t, \bar{t})\partial_{t_n} F(t_n, \bar{t}_n) + \hat{\beta}(t, \bar{t})\overline{\partial_{t_n} F(t_n, \bar{t}_n)} . \tag{2.2.5}$$

Differentiate both sides of (2.2.5) with respect to t_n, it is seen that, with the notation $p=\partial_{t_n} F$ and $s=t_n$, we have

$$\partial_{\bar{s}} p = \hat{\alpha}\partial_s p + \hat{\beta}\partial_s \bar{p} + (\partial_s \hat{\alpha})p + (\partial_s \hat{\beta})\bar{p} . \tag{2.2.6}$$

On considering the complex conjugate of both sides of (2.2.6) and eliminating $\partial_s \bar{p}$ from these relations, we obtain

$$\begin{aligned}\partial_{\bar{s}} p = {} & A(t', \bar{t}', s, \bar{s})\partial_s p + B(t', \bar{t}', s, \bar{s})\overline{\partial_s p} \\ & + C(t', \bar{t}', s, \bar{s})p + D(t', \bar{t}', s, \bar{s})\bar{p} ,\end{aligned} \tag{2.2.7}$$

where

$$A = \hat{\alpha}(1-|\hat{\beta}|^2)^{-1} ,$$

$$B=\bar{\hat{\alpha}}\hat{\beta}(1-|\hat{\beta}|^2)^{-1},$$

$$C=(\partial_s\hat{\alpha}+\hat{\beta}\overline{\partial_s\hat{\beta}})(1-|\hat{\beta}|^2)^{-1},$$

and

$$D=(\partial_s\hat{\beta}+\hat{\beta}\overline{\partial_s\hat{\alpha}})(1-|\hat{\beta}|^2)^{-1}.$$

Note that $\|\hat{\alpha}\|+\|\hat{\beta}\|<1$ leads to $\|A\|+\|B\|<1$ ($\|*\|=\sup|*|$ on $T(U^n)$). It is important to remark that for arbitrary t' in $U_{t'}^{n-1}(0;r')$ the equation (2.2.7) is fulfilled. Let t' be any point of $U_{t'}^{n-1}$ and fixed. Then the well known representation theorem shows that the set of zeros of p is isolated and hence so is the set P. This is what we want.

EXAMPLE 1. Consider $w=3(z_1^2+z_2)^2+2(\bar{z}_1^2+\bar{z}_2)^3$ on $U=\{z\in \boldsymbol{C}^2;\ |z_1|^2+|z_2|<k<1,\ k:\text{a constant}\}$. Then w satisfies on U: $\partial\bar{w}=(z_1^2+z_2)\partial w$. Putting $\kappa=z_1^2+z_2$, we see $\partial_{z_2}\kappa=1$ and N is the 1-dimensional complex manifold.

The above theorem does not always apply to the case where $\partial\kappa$ has a zero point. An example for this situation is as follows.

EXAMPLE 2. Let w be defined on a small neighborhood U of the origin of $\boldsymbol{C}^2$ by the equation

$$(1/2)(\bar{w}+z_1+z_2)^2-w=z_1^2\cos z_2\,,\qquad w(0)=0\,. \tag{2.2.8}$$

Then we see that w satisfies, setting $\kappa=w+\bar{z}_1+\bar{z}_2$, $\partial\bar{w}=\kappa\partial w$ on U, where we consider such U that $\|\kappa\|<1$. It is seen that, with the notation $N_j=\{z\in U;\ \partial_{z_j}w=0\}$, $j=1, 2$,

$$N_1=\{z\in U;\ 2z_1^2\cos^2 z_2-z_1^2\cos z_2-w=0 \text{ and } (2.2.8)\}\,,$$

$$N_2=\{z\in U;\ z_1^4\sin^2 z_2-2z_1^2\cos z_2-2w=0 \text{ and } (2.2.8)\}\,,$$

and $N_1\cap N_2$ is the origin only. We see $\partial\kappa=\boldsymbol{0}$ at the origin.

On the contrary, for w defined on U by the equation

$$(1/2)(\bar{w}+z_1+z_2)^2-w=0\,,\qquad w(0)=0\,,$$

we have that $N=\{z\in U;\ z_1+z_2=0\}$ and $\partial\kappa=\boldsymbol{0}$ on N.

THEOREM 11 [6]. *For w nonconstant pseudo-holomorphic on D, the inverse image of a point under the map w is an $(n-1)$-dimensional complex manifold, if not empty.*

PROOF. Let M_a be the inverse image of a under w and not empty. Let z^0 be any point of M_a. Associate with z^0 a triple (F, σ, U^n). Since F satisfies the equation (2.2.3) on $\sigma(U^n)$, F is light. Let $t^0=\sigma(z^0, \bar{z}^0)$. If we restrict U^n to a smaller polydisc V, $(F|V)^{-1}(a)\cap(\sigma|V)(U^n)=\{t^0\}$. Thus we obtain

$$M_a\cap V=\{z\in V;\ w(z, \bar{z})=a\}$$
$$=\{z\in V;\ \sigma(z, \bar{z})=t^0\}\ .$$

By Proposition 1, (v) we see that $M_a \cap V$ is an $(n-1)$-dimensional complex manifold, which completes the proof.

REMARK 2. As was seen in the proofs of Theorems 10 and 11, if $\partial \kappa \neq \mathbf{0}$ on D, one recognizes that a triple being associated with each point of D plays an essential role. By using a triple the maximum modulus principle (Proposition 8) is obtained as follows: Let z^0 be the point of D at which a pseudo-holomorphic function w attains the absolute maximum. Let a triple (F, σ, U^n) be associated with z^0. Then, since σ is an open mapping and F has the maximum modulus principle, F vanishes on U^n, and hence w does. By Proposition 7 we have the result.

2.3. We proceed with the study of properties of pseudo-holomorphic functions with κ such that $\partial \kappa \neq \mathbf{0}$.

From Lemma 9 the following result is obtained at once.

LEMMA 12. *Let w and W be pseudo-holomorphic on D. Assume that they have the same characteristic κ such that $\partial_{z_n} \kappa \neq 0$ at a point z^0 of D. If W is nondegenerate at z^0, then there are a neighborhood $U^n(z^0; r)$ and a function F defined on $W(U^n)$ such that $w = F \circ W$ on U^n, where*

$$\partial_{\bar{t}} F(t, \bar{t}) = \frac{\overline{K(\bar{s}', t, \bar{t})}}{1 + |K(\bar{s}', t, \bar{t})|^2} (-\partial_t F + \overline{\partial_t F}), \qquad s' = (s_1, \cdots, s_{n-1}) \tag{2.3.1}$$

and $K(\bar{s}', t, \bar{t})$ defined on $U_{s'}^{n-1}(s^{0\prime}; r) \times W(U^n)$ is derived from Proposition 3 *($s^0 = T(z^0)$ and $s = (s', t) \in \boldsymbol{C}^n$).*

LEMMA 13. *Let the assumption of Lemma* 12 *be satisfied. Furthermore let the coefficient $K(\bar{s}', t, \bar{t})$ of the equation* (2.3.1) *be subject to the following: for a number j_0, $1 \leqq j_0 \leqq n-1$,*

$$\partial_{\bar{s}_{j_0}} K \neq 0 \tag{2.3.2}$$

on $U_{s'}^{n-1}(s^{0\prime}; r) \times W(U^n)$, except possibly a nowhere dense set. Then

$$w = aW + b \qquad on \quad U^n,$$

where a and b are constants and a is real.

PROOF. Assume that $\partial_t F \neq \overline{\partial_t F}$ at a point t^* of $W(U^n)$. Then there exists a neighborhood $\tilde{U}$ $(\subset W(U^n))$ of t^* on which $\partial_t F \neq \overline{\partial_t F}$. From (2.3.1) we see that, for every s_j, $\partial_{\bar{s}_j} K = 0$ on $U_{s'}^{n-1}(s^{0\prime}; r) \times \tilde{U}$, which contradicts the assumption (2.3.2). Hence it follows that on $W(U^n)$

$$\partial_t F = \overline{\partial_t F}. \tag{2.3.3}$$

Thus, from (2.3.1) we obtain that F is holomorphic and hence, from (2.3.3), that $F = at + b$. The rest is clear.

EXAMPLE 3. Let w be defined on a sufficiently small $U^2(0;\varepsilon)$ by the equation

$$(\bar{w}+z_1+z_2)^2-2w=2z_2, \qquad w(0)=0. \tag{2.3.4}$$

Then w satisfies on U^2

$$\partial\bar{w}=(w+\bar{z}_1+\bar{z}_2)\partial w. \tag{2.3.5}$$

On setting $\kappa=w(z,\bar{z})+\bar{z}_1+\bar{z}_2$, we have $\partial_{z_2}\kappa=\partial_{z_2}w\neq 0$ on U^2. From (2.3.4) it is obtained that

$$z_1+z_2=-\bar{w}+1-(1-2z_1+2w-2\bar{w})^{1/2},$$

where $(\ )^{1/2}$ denotes the branch such that $(1)^{1/2}=1$. We have

$$\kappa(z,\bar{z})=1-(1-2\bar{z}_1+2\bar{w}-2w)^{1/2}.$$

On changing the variables: $s=z_1$, $t=w(z,\bar{z})$, we have

$$K(\bar{s},t,\bar{t})=1-(1-\bar{s}+2\bar{t}-2t)^{1/2}.$$

Clearly we see that $\partial_{\bar{s}}K\neq 0$ on $U_s(0;\varepsilon)\times W(U^2)$.

THEOREM 14. *Let W be a nonconstant pseudo-holomorphic function on D with such κ as does not belong to $S(\kappa;D)$. If w is any pseudo-holomorphic function on D with the κ and if, for a_1 and a_2 such that $W(a_1,\bar{a}_1)\neq W(a_2,\bar{a}_2)$, $w(a_j,\bar{a}_j)=W(a_j,\bar{a}_j)$ $(j=1,2)$, then $w=W$ on D.*

PROOF. By virtue of Theorem 10, the set N of nonordinary points of W, if not empty, is an $(n-1)$-dimensional complex manifold. Then W is nondegenerate on $D-N$. It follows from the assumption on κ that for a point a of D there is a neighborhood $U^n(a;\varepsilon)$ in D such that

$$\partial\kappa\wedge\partial\bar{\kappa}\neq \mathbf{0} \qquad \text{on} \quad U^n(a;\varepsilon). \tag{2.3.6}$$

For a point z^0 in $U^n(a;\varepsilon)\cap(D-N)$, consider a neighborhood $U^n(z^0;r)\subset U^n(a;\varepsilon)\cap(D-N)$. Then, from Lemma 12, restricting $U^n(z^0;r)$ if necessary, we have (2.3.1) on $U^{n-1}_{s'}(s^{0\prime};r)\times W(U^n(z^0;r))$. On the other side, by Corollary 4 we find that (2.3.6) is equivalent to (2.3.2). By Lemma 13, $w=aW+b$ on $U^n(z^0;r)$, so that on D by Proposition 7. It is easy to see that $w=W$ on D, which completes the proof.

2.4. In this section we want to discuss the existence of solutions of the system of equations (2.1.1). The assumptions (H_0) and (H_1) do not always assure the existence of a nonconstant solution of this system. The following example illustrates this situation.

EXAMPLE 4. Let $\kappa=\bar{z}_1+z_2$ on $D: |z_j|<(1/2)$, $j=1,2$. Since $\partial\kappa=dz_2$, all solutions of (1.2.1) do not have the variables z_1 and $\bar{z}_1$. Therefore only the constant is the solution of (2.1.1). It should be noted that κ does not belong to $S(\kappa; D)$.

In the section 2.2 we have seen that, whenever one considers a pseudo-holomorphic function with κ such that $\partial\kappa\neq\mathbf{0}$ on D, with each point of D it is associated a triple (F, σ, U^n) and that the functions κ and μ are of the form

$$\kappa|U^n = K(\bar{z}', \sigma(z, \bar{z}), \overline{\sigma(z, \bar{z})}),$$

$$\mu = L(\bar{z}', \sigma(z, \bar{z}), \overline{\sigma(z, \bar{z})})$$

respectively, where μ is defined by the relation: $\partial\bar{\sigma}=\mu(z, \bar{z})\partial\sigma$, $(\partial\mu\neq\mathbf{0}$ on $U^n)$.

We now consider the following four cases: for brevity we shall use κ and S in place of $\kappa|U^n$ and $S(\kappa; U^n)$, respectively.

(I) $\kappa\notin S$ and $\mu\notin S$ (II) $\kappa\notin S$ and $\mu\in S$

(III) $\kappa\in S$ and $\mu\notin S$ (IV) $\kappa\in S$ and $\mu\in S$.

From Remark 1 it is seen that each case does not depend on a choice of σ. Noting that $\mu\in S$ is equivalent to $\mu=L\circ\sigma$, we see from (2.2.2) that $\kappa\in S$. The converse also is similar. Therefore, for nonconstant pseudo-holomorphic functions, cases (II) and (III) are excluded out of discussion. Thus we are now in a position to state the

LEMMA 15. *Let w be a nonconstant solution of the equation* (2.1.1) *on D and let a triple (F, σ, U^n) be associated with a point of D. Then*

(i) *$\kappa\notin S$ is equivalent to $\mu\notin S$,*

(ii) *$\kappa\in S$ is equivalent to $\mu\in S$.*

REMARK 3. Example 3 in the preceding section shows that case (I) actually occurs.

LEMMA 16. *Let functions κ and σ be defined on a polydisc $U^n(a; r)$, $a\in\mathbf{C}^n$. Assume they satisfy the following conditions on U^n:*

(i) *(H_0) is fulfilled.*

(ii) *σ is in $S(\kappa; U^n)$ and has the property $|\partial_{z_n}\sigma|^2-|\partial_{\bar{z}_n}\sigma|^2\geqq\varepsilon_0>0$, where ε_0 is a constant.*

(iii) *Any element of S is of the composite form $F\circ\sigma$.*

Define a function μ on U^n by the equation $\partial\bar{\sigma}=\mu\partial\sigma$. The following statements are equivalent:

(1) *μ belongs to S.*

(2) *There is a nondegenerate function $\phi(z)$ holomorphic on U^n and belonging to S.*

PROOF. (1)$\Rightarrow$(2). Choose a function F defined on $\sigma(U^n)$ such that $|\partial_t F|^2-|\partial_{\bar{t}} F|^2>0$ $(t=\sigma(z, \bar{z}))$. With the notation $f=F\circ\sigma$, we obtain

$$\bar{\partial}f=(\partial_{\bar{t}}F+\bar{\mu}\partial_t F)_{t=\sigma}\bar{\partial}\bar{\sigma}.$$

The assumptions show that there is a function L defined on $\sigma(U^n)$ such that $\mu(z, \bar{z})=(L\circ\sigma)(z, \bar{z})$.

In this way, if we choose a nondegenerate solution F such that

$$\partial_{\bar{t}} F(t, \bar{t})+\overline{L(t, \bar{t})}\partial_t F(t, \bar{t})=0$$

on $\sigma(U^n)$, we have (2). Such a function, however, exists by the well-known theorem in one variable [5, **9**].

(2) ⇒ (1). From assumption (iii) we have

$$\phi(z)=(\Phi\circ\sigma)(z). \tag{2.4.1}$$

Hence we have, by $\partial\sigma\neq\mathbf{0}$,

$$\partial_{\bar{t}}\Phi+\bar{\mu}\partial_t\Phi=0. \tag{2.4.2}$$

On the other side, we have that on U^n

$$d\phi=(\partial_t\Phi+\mu\partial_{\bar{t}}\Phi)\partial\sigma\neq\mathbf{0}, \tag{2.4.3}$$

because ϕ is nondegenerate on U^n. From (2.4.2) and (2.4.3) we obtain that $\partial_t\Phi\neq 0$ on U^n, so that from Proposition 1, (ii) the desired result.

From (2.4.3), noting $\partial_{z_n}\sigma\neq 0$ on U^n, we obtain the

COROLLARY 17. *Under the same assumption as in the preceding lemma, if* μ *is in* $S(\kappa; U^n)$, *the set* $N_j=\{z\in U^n; \partial_{z_j}\sigma=0\}$, $j=1, \cdots, n-1$, *is an analytic variety in* U^n *unless* N_j *is empty.*

COROLLARY 18. *Under the same assumption as in Corollary* 17, *the set* $M_j=\{z\in U^n; \partial_{z_j}\kappa=0\}$, $j=1, \cdots, n-1$, *is an analytic variety unless it is empty.*

PROOF. Let M_j, for a number j, be not empty. When U^n in the corollary is such that κ has a local expression on it, the result is immediately obtained from the above corollary, Propositions 1, (vi) and 3 (it is also obtained only from Corollary 5).

In general, for each point a in U^n, consider a neighborhood V of a, contained in U^n, on which κ has a local expression, so we see that $M_j\cap V=\{z\in V; \partial_{z_j}\sigma=0\}$ and hence M_j is a local variety. Since M_j is closed in U^n, we obtain the desired result.

THEOREM 19. *Assume that* w *is a nonconstant pseudo-holomorphic function on* D *with such* κ *as belongs to* $S(\kappa; D)$. *Then, for any point* a *of* D, *there exist a polydisc* $U^n(a; r)$, *functions* ϕ *holomorphic on* $U^n(a; r)$ *and* F *quasi-conformal on* $\phi(U^n)$ *such that* w *is written by the form* $F\circ\phi$. *Moreover the set* $N_j=\{z\in D; \partial_{z_j}w=0\}$ *is an analytic variety in* D *if not empty.*

PROOF. From Lemmas 15 and 16 it follows that at each point a of D w has a triple $(F, \phi, U^n(a; r))$ whose component ϕ is holomorphic on U^n. This shows $w=F\circ\phi$ on U^n. We have next

$$N_j \cap U^n = \{z \in U^n\,;\ \partial_t(F\circ\phi)(z, \bar{z})\partial_{z_j}\phi = 0\}\,.$$

On noting the proof of Theorem 10, we obtain the result.

THEOREM 20. *Let the coefficient* κ *of the system of equations* (2.1.1) *belong to* $S(\kappa\,; D)$. *Suppose that*

(2.4.4) *there is a function* ϕ *in* $S(\kappa\,; D)$, *nondegenerate and holomorphic on* D,

then the equation (2.1.1) *has a nonconstant solution on a neighborhood of each point of* D.

PROOF. Owing to Proposition 1, (iv), the assumption on κ and (2.4.4), for any point a of D we have a polydisc $U^n(a\,; r)\subset D$ and a function K defined on $\phi(U^n)$ such that κ is written by the form

$$\kappa = (K\circ\phi)(z, \bar{z})\,. \tag{2.4.5}$$

We want to seek a solution w in the form $F\circ\phi$. Using (2.2.3) and (2.4.5), we have the equation in a single variable t

$$\partial_{\bar{t}}F = \overline{K(t, \bar{t})\partial_t F}\,, \qquad \|K\| < 1\,. \tag{2.4.6}$$

It is well known that the equation (2.4.6) has a nonconstant solution on $\phi(U^n)$ [**5**, **9**].

A relation between Lemma 16, Theorems 19 and 20 is formulated as follows.

THEOREM 21. *Let* κ, σ *and* μ *be the same as in Lemma* 16. *Let there be the following three conditions:*

(1) *There exists a nonconstant pseudo-holomorphic function on* U^n *with* κ.

(2) $\kappa^* \in S(\kappa^*\,; U^n)$, $\kappa^* = \kappa\,|\,U^n$.

(3) $\mu \in S(\kappa^*\,; U^n)$.

If any two of the above conditions are satisfied, then the third is derived.

REMARK 4. (i) There does not always exist a function K defined on $\phi(D)$ such that κ can be written by the form $K\circ\phi$ on D (see Proposition 1, (iv)). However if κ is, for example, holomorphic on D, by taking κ as ϕ one can have a global solution (§ 3.2). In general we shall not be able to expect a global solution.

(ii) As seen in Proposition 1, (vii), if w is degenerate and satisfies $\partial w \neq 0$ on D, its characteristic function (*in a wide sense*) is also in $S(\kappa\,; D)$.

Assume that $S(\kappa\,; D)$ has a nondegenerate holomorphic element ϕ. If w is in $S(\kappa\,; D)$ and satisfies $\partial w \neq 0$ on D, then we see from Corollary 4 that any point a of D has a polydisc $U^n(a\,; r)$ and a function F defined on $\phi(U^n)$ such that $w = F\circ\phi$ on U^n. Using again Corollary 4, μ defined by $\partial\bar{w} = \mu\partial w$ is also in $S(\kappa\,; D)$, because the point a is arbitrary in D.

From the above, if w is a pseudo-holomorphic function in Theorem 19, the characteristic function of κ (in a wide sense) is also in $S(\kappa; D)$, which completes the remark.

§ 3. Connection with generalized analytic functions.

3.1. In the preceding section we have discussed the existence of local solutions of the equations (2.1.1). In this section we shall show such an existence in the second way which is found in [4, 5].

Let w be a pseudo-holomorphic function on D with κ. Then the function g, defined by

$$g+\bar{\kappa}\bar{g}=w\,, \tag{3.1.1}$$

satisfies the following differential equation

$$\bar{\partial}g=\frac{\bar{\kappa}g}{1-|\kappa|^2}\bar{\partial}\kappa-\frac{\bar{g}}{1-|\kappa|^2}\bar{\partial}\bar{\kappa}\,. \tag{3.1.2}$$

That is, g is a generalized analytic function in several complex variables mentioned in the introduction [7].

By virtue of Proposition 1, (ii), it follows at once from (3.1.1) that, under the assumption that κ is in $S(\kappa; D)$, $g\in S(\kappa; D)$ is equivalent to $w\in S(\kappa; D)$. The following is easily seen. If the function κ has the condition $\|\kappa\|<1$ on D, then the function g satisfying (3.1.2) on D, through (3.1.1), leads to the function w pseudo-holomorphic on D and having κ as the characteristic.

We have the local existence theorem for the equation (3.1.2).

THEOREM 22. *Assume that κ satisfies the same assumptions as in Theorem* 20. *Then there exists a nonconstant local solution of the generalized Cauchy-Riemann equation* (3.1.2).

PROOF. Using the same notations and techniques as in the proof of Theorem 20, we have the equation

$$\partial_{\bar{t}}G=\frac{\bar{K}\partial_{\bar{t}}K}{1-|K(t,\bar{t})|^2}G-\frac{\overline{\partial_t K}}{1-|K(t,\bar{t})|^2}\bar{G}\,, \tag{3.1.3}$$

which has a nonconstant solution on $\phi(U^n)$ [5, 9, 10], and $g=G\circ\phi$ is the desired function.

We can conclude that, in case κ satisfies the assumption in the above theorem, the existence of pseudo-holomorphic function with the κ is equivalent to that of generalized analytic function satisfying the equation (3.1.2), and each case may be reduced to the case of a complex variable.

Bauer and Ruscheweyh [2] have been obtained the explicit representation of a family of pseudo-analytic functions (of the first kind) on a simply connected domain (in C) in terms of the differential operator. However we have a question: Is there a nonconstant function K defined on $\phi(U^n)$ such that the equation (3.1.3) is reduced to that of Bauer and Ruscheweyh's type? In case K is holomorphic (§ 3.2), we can see easily that the answer is *no*.

3.2. In this section we shall consider the special case where κ is holomorphic in a simply connected domain $D \subset \mathrm{Int}\,(\mathrm{supp}\,\kappa)$, the set of inner points of the support of κ. If w is pseudo-holomorphic on D, we see from the equation (2.1.1)

$$\bar{\partial}\partial\bar{w} = \kappa\bar{\partial}\partial w\,. \tag{3.2.1}$$

From this, by using $\partial\bar{\partial}+\bar{\partial}\partial=0$ and $\|\kappa\|<1$,

$$\bar{\partial}\partial w = 0\,, \tag{3.2.2}$$

from which it follows that ∂w is a holomorphic form on D and that, noting (3.2.1), so is $\partial\bar{w}$.

On the other side, since w is in $S(\kappa\,;D)$, there exist the functions α and β holomorphic on D such that on D

$$\begin{cases} \partial w = \alpha(z)d\kappa\,, \\ \partial\bar{w} = \beta(z)d\kappa\,. \end{cases} \tag{3.2.3}$$

There must be the following compatibility conditions: on D

$$\begin{cases} d\kappa \wedge d\alpha = 0\,, \\ d\kappa \wedge d\beta = 0\,. \end{cases} \tag{3.2.4}$$

And from (2.1.1) and (3.2.3) it follows that on D

$$\beta = \kappa\alpha\,. \tag{3.2.5}$$

Conversely it is obvious that (3.2.3) with (3.2.5) leads to (2.1.1).

We consider the first equation of (3.2.3) with the first condition of (3.2.4). The solution on D of this equation is uniquely determined up to an additive anti-holomorphic function on D. However, since $\alpha d\kappa$ is a d-closed holomorphic form and D is simply connected, this equation has a solution ϕ holomorphic on D. Therefore the general solution of the first equation of (3.2.3) is of the form

$$w = \phi(z) + \overline{\psi(z)}\,, \tag{3.2.6}$$

where ψ is holomorphic on D. Take ψ such that $d\psi = \beta d\kappa$ on D. The second condition of (3.2.4) guarantees the existence of such a function. From (3.2.5) we obtain that $d\psi = \kappa d\phi$ on D.

Thus we are now in a position to state the following

PROPOSITION 23. *Let D be a simply connected domain and κ holomorphic on D such that $\|\kappa\|<1$ and $d\kappa\neq 0$. Then w is a pseudo-holomorphic function with the κ if and only if w is of the form* (3.2.6), *where ϕ and ψ have the relations:*

$$d\kappa\wedge d\phi=0, \tag{3.2.7}$$

$$d\psi=\kappa d\phi. \tag{3.2.8}$$

Note that the above proposition holds without the assumption $d\kappa\neq 0$.

By using Corollary 4, it is easily seen from (3.2.7) and (3.2.8) that ϕ and ψ locally are of the form, respectively: $\Phi\circ\kappa$ and $\overline{\kappa(\Phi\circ\kappa)-\hat{\Phi}\circ\kappa}$, where $\hat{\Phi}(t)$ is a primitive function of $\Phi(t)$. Putting $F(t,\bar{t})=\Phi(t)+\overline{t\Phi(t)-\hat{\Phi}(t)}$, we have a local representation of w: $w=F\circ\kappa$ (Theorem 19).

We shall note that the above Proposition is immediately obtained by Theorem 20 and does not depend on whether D is simply connected or not. As readily seen from Remark 4, (i), if there is a global function K, then we have a global solution. Since κ is nondegenerate and holomorphic on D, we can take κ as ϕ in Theorem 20, so that the equation (2.4.6) become $\partial_{\bar{t}}F=\bar{t}\overline{\partial_t F}$, $|t|\leqq\|\kappa\|$. It is convenient to treat more general equation than this

$$\partial_{\bar{t}}F=\overline{K(t)}\overline{\partial_t F}, \qquad \|K\|<1, \tag{3.2.9}$$

where K is holomorphic on the unit disc $\Delta\subset \boldsymbol{C}$. This equation, considering on Δ, has the general solution

$$F=\int_0^t H(\zeta)d\zeta+\overline{\int_0^t K(\zeta)H(\zeta)d\zeta}, \qquad t\in\Delta, \tag{3.2.10}$$

where H is any holomorphic function on Δ. From this we obtain (3.2.6), in which $\phi(z)=(F_1\circ\kappa)(z)$ and $\psi(z)=(F_2\circ\kappa)(z)$, where F_1 and F_2 are the first and the complex conjugate of the second terms of (3.2.10), respectively.

On the contrary, if we consider the equation (3.2.9) on $\kappa(D)$, then, in general, we have the local general solution only.

§ 4. Case where $\partial\kappa=0$.

4.1. If $\partial\kappa=0$ on $D\subset\operatorname{Int}(\operatorname{supp}\kappa)$, the situation is more simpler than in the case where $\partial\kappa\neq 0$. The equation (3.1.2) is of the following form

$$\bar{\partial}g=\frac{\bar{\kappa}g}{1-|\kappa|^2}\bar{\partial}\kappa. \tag{4.1.1}$$

Because $\bar{\kappa}(1-|\kappa|^2)^{-1}\bar{\partial}\kappa$ is $\bar{\partial}$-closed, there exists locally a nonzero solution. In fact, the general solution of (4.1.1) is given by the formula

$$g=\frac{h(z)}{1-|\kappa|^2}, \tag{4.1.2}$$

where h is any holomorphic function on D. Substituting (3.1.1) for (4.1.2), we can obtain the desired function w in the explicit form

$$w=\frac{h(z)+\overline{\kappa(\bar{z})h(z)}}{1-|\kappa|^2}. \tag{4.1.3}$$

We define $A_\kappa(D)$ to be the set of pseudo-holomorphic functions on D with κ given by the formula (4.1.3). In particular, $A_0(D)$ is the family of all the functions holomorphic on D. The family $A_\kappa(D)$ is a vector space over $\boldsymbol{R}$, the real number field. It is seen from (4.1.3) that there is an $\boldsymbol{R}$-isomorphism from $A_\kappa(D)$ onto $A_0(D)$.

Because the explicit form (4.1.3) is very simple, we can obtain easily some properties of $A_\kappa(D)$ which are weaker than in the case of $\partial\kappa\neq\mathbf{0}$.

Noting the relation

$$h-c+\bar{\kappa}\bar{c}=(w-c)-\bar{\kappa}(\bar{w}-\bar{c})$$

and that $h-c+\bar{\kappa}\bar{c}$ is holomorphic, where c is a constant, we have the following proposition.

PROPOSITION 24. *The inverse image of a point under w in $A_\kappa(D)$ is a complex analytic variety in D.*

The following proposition is also weaker than Theorem 10.

PROPOSITION 25. *Let w be in $A_\kappa(D)$. The set N of nonordinary points of w is a complex analytic variety in D.*

PROOF. Note that $N=\{z\in D;\ dh+\bar{w}d\bar{\kappa}=\mathbf{0}\}$. Consider a sufficiently small neighborhood $U^n(a;r)$ in D such that $N\cap U^n(a;r)$ is a real analytic irreducible variety, where a is on N. Owing to the definition of N, we see that, for any point of N, w is the constant $c=w(a,\bar{a})$ and hence that

$$\begin{aligned}N\cap U^n&=\{z\in U^n;\ w(z,\bar{z})=c\}\cap\{z\in U^n;\ dh(z)+\overline{w(z,\bar{z})}d\overline{\kappa(\bar{z})}=\mathbf{0}\}\\&=\{z\in U^n;\ w(z,\bar{z})=c\}\cap\{z\in U^n;\ dh(z)+\bar{c}d\overline{\kappa(\bar{z})}=\mathbf{0}\}\ .\end{aligned}$$

From Proposition 24 it follows that $N\cap U^n$ is complex analytic. Because of N being closed in D, we have the desired result.

It follows at once from (4.1.3) that a subfamily of $A_\kappa(D)$, uniformly bounded on any compact set in D, is a normal family (Montel type theorem), and that $A_\kappa(D)$ has the "Riemann extension theorem", that is, "Let V be a complex analytic variety such that $D-V$ is dense. Let w be in $A_\kappa(D-V)$ and locally bounded in D. Then there is a unique function $\tilde{w}$ in $A_\kappa(D)$ such that $\tilde{w}|D-V=w$ for $z\in D-V$".

4.2. Hitotumatu had dealt with a pseudo-holomorphic mapping of which characteristic functions are *all distinct*. In this case, however, it is necessary to take very careful note of "characteristic functions". Let κ_j be in $S(\sigma; U^n)$ and $\partial\kappa_j \neq \mathbf{0}$ for all j, where σ and U^n are the same as in Lemma 16. Let each w_j be a pseudo-holomorphic function on U^n with κ_j. The pseudo-holomorphic mapping $W=W(z, \bar{z})$, defined by them, from U^n into C^n is always singular by virtue of Proposition 2 and Corollary 4. On the contrary, in case that $\partial\kappa=\mathbf{0}$ on D, we have the following

THEOREM 26. *There exist n functions w_j in $A_\kappa(D)$ such that on D*

$$\partial w_1 \wedge \partial w_2 \wedge \cdots \wedge \partial w_n \neq \mathbf{0}.$$

We note that the Jacobian J of the mapping $W=(w_1, \cdots, w_n)$ whose components are in $A_\chi(D)$ is given by

$$J=(1-|\kappa|^2)^n \left| \frac{\partial(w_1, \cdots, w_n)}{\partial(z_1, \cdots, z_n)} \right|^2 \quad ([6]).$$

PROOF. Let $h_j(z)$ be $\exp(cz_j)$, $j=1, \cdots, n$, where c is a nonzero constant which is determined later. Assume that each w_j is given in terms of (4.1.3) with h_j in place of h.

By an elementary but lengthy computation we obtain

$$\begin{aligned} (4.2.1) \qquad & \partial w_1 \wedge \partial w_2 \wedge \cdots \wedge \partial w_n \\ &= \frac{c^n \exp c(z_1+\cdots+z_n)}{(1-|\kappa|^2)^n}\left[1+\frac{1}{c(1-|\kappa|^2)}\sum_{k=1}^{n}\{\kappa+\exp(\overline{cz_k}-cz_k)\partial_{z_k}\bar{\kappa}\}\right] \\ & \qquad dz_1 \wedge \cdots \wedge dz_n. \end{aligned}$$

From this we have

$$\begin{aligned} (4.2.2) \qquad & |\text{the term in } [\;] \text{ of the right side of } (4.2.1)| \\ & \geqq 1-\frac{1}{|c|(1-|\kappa|^2)}\sum_{k=1}^{n}(1+|\kappa|)|\partial_{z_k}\bar{\kappa}| \\ & \geqq 1-\frac{nK}{|c|(1-\|\kappa\|)}, \end{aligned}$$

where $K=\sup_{C^n}(|\partial_{z_1}\bar{\kappa}|, \cdots, |\partial_{z_n}\bar{\kappa}|)<\infty$. In this way we can take c such that the first term of (4.2.2) is bounded away from zero, which shows the desired result.

The following is a very special case of Frobenius-Nirenberg Theorem A′ [8] except that the latter is a local one. This special case can be easily proved directly. In fact, it is obtained from Proposition 1, (iv) and the above theorem.

THEOREM 27. *The differential equations* (2.1.1) *have n solutions on D such* *that the Jacobian of the transformation defined by them is not zero there if and only if the coefficient κ is anti-holomorphic on D.*

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