

A note on automorphisms of real semisimple Lie algebras

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Let \mathfrak{g} be a real semisimple Lie algebra, \mathfrak{h} a Cartan subalgebra of \mathfrak{g} , \mathfrak{g}_c and \mathfrak{h}_c the complexifications of \mathfrak{g} and \mathfrak{h} respectively, and σ the conjugation of \mathfrak{g}_c with respect to \mathfrak{g} . Denote by $W(\mathfrak{h}_c)$ the Weyl group of \mathfrak{g}_c acting on \mathfrak{h}_c , and let $W_\sigma(\mathfrak{h})$ be the subgroup of $W(\mathfrak{h}_c)$ consisting of elements leaving \mathfrak{h} stable or

$$W_\sigma(\mathfrak{h}) = \{w \in W(\mathfrak{h}_c); w \circ \sigma = \sigma \circ w \text{ on } \mathfrak{h}_c\}.$$

The purpose of this paper is to study the structure of the group $W_\sigma(\mathfrak{h})$, as was done in [7, Appendix] for such \mathfrak{h} which has the maximal vector part.

§1. Statement of Theorem.

For any linear form λ on \mathfrak{h}_c , we define $\sigma\lambda$ as $(\sigma\lambda)(X) = \overline{\lambda(\sigma X)}$ ($X \in \mathfrak{h}_c$), where \bar{a} denotes the conjugate number of $a \in \mathbb{C}$. If α is a root of $(\mathfrak{g}_c, \mathfrak{h}_c)$, then so is $\sigma\alpha$. Let r be the root system of $(\mathfrak{g}_c, \mathfrak{h}_c)$, then it is a σ -system of roots in the sense of [1] with the involutive automorphism σ . A root α is called real or imaginary if $\sigma\alpha = \alpha$ or $\sigma\alpha = -\alpha$ respectively. We see that α is real or imaginary if and only if it takes only real or purely imaginary values on \mathfrak{h} respectively. Denote by r_R and r_I the sets of all real or imaginary roots in r respectively. Let $W_R(\mathfrak{h})$ and $W_I(\mathfrak{h})$ be the groups generated by $\{s_\alpha; \alpha \in r_R\}$ and $\{s_\alpha; \alpha \in r_I\}$ respectively, where s_α denotes the reflexion corresponding to a root α . Then they are normal subgroups of $W_\sigma(\mathfrak{h})$, because $W_\sigma(\mathfrak{h})$ leaves r_R and r_I stable.

Let G be a connected Lie group with Lie algebra \mathfrak{g} . For a subset F of \mathfrak{g} , let $N_G(F)$ and $Z_G(F)$ be the normalizer and the centralizer of F in G respectively, and put $W_G(F) = N_G(F)/Z_G(F)$. Then $W_G(\mathfrak{h})$ is considered canonically as a subgroup of $W(\mathfrak{h}_c)$, and also of $W_\sigma(\mathfrak{h})$ since $\text{Ad}(g) \circ \sigma = \sigma \circ \text{Ad}(g)$ on \mathfrak{g}_c for any $g \in G$. We know that $W_R(\mathfrak{h})$ is a subgroup of $W_G(\mathfrak{h})$ (see for instance [2, p. 256]). An imaginary root α is called compact if $(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) \cap \mathfrak{g}$ is isomorphic to $\mathfrak{su}(2)$, where $\mathfrak{g}_{\pm\alpha}$ are the spaces of root vectors corresponding to $\pm\alpha$. If $\alpha \in r_I$ is compact, then $s_\alpha \in W_G(\mathfrak{h})$ [2, p. 256]. We define the vector part \mathfrak{h}^- and the toroidal part \mathfrak{h}^+ of \mathfrak{h} as follows:

$\mathfrak{h}^- = \{X \in \mathfrak{h}; \text{all eigenvalues of } \text{ad } X \text{ are real}\},$

$\mathfrak{h}^+ = \{X \in \mathfrak{h}; \text{all eigenvalues of } \text{ad } X \text{ are purely imaginary}\}.$

Every element in $W_\sigma(\mathfrak{h})$ leaves \mathfrak{h}^- and \mathfrak{h}^+ stable.

THEOREM. (a) *An element w in $W_\sigma(\mathfrak{h})$ induces on \mathfrak{h}^- or on \mathfrak{h}^+ the identity transformation if and only if w belongs to $W_I(\mathfrak{h})$ or $W_R(\mathfrak{h})$ respectively.*

(b) *The group $W_\sigma(\mathfrak{h})$ is a product of $W_G(\mathfrak{h})$ and $W_I(\mathfrak{h})$: $W_\sigma(\mathfrak{h}) = W_G(\mathfrak{h})W_I(\mathfrak{h})$.*

(c) *The image of $W_G(\mathfrak{h})$ under the homomorphism $w \rightarrow w|_{\mathfrak{h}^-}$ is equal to $W_G(\mathfrak{h}^-)$. That of $W_G(\mathfrak{h})$ under the homomorphism $w \rightarrow w|_{\mathfrak{h}^+}$ is equal to $W_G(\mathfrak{h}^+)$.*

As a consequence of this theorem, we see that the homomorphism $w \rightarrow w|_{\mathfrak{h}^-}$ of $W_\sigma(\mathfrak{h})$ has the kernel $W_I(\mathfrak{h})$ and the image $W_G(\mathfrak{h}^-)$, and that the homomorphism $w \rightarrow w|_{\mathfrak{h}^+}$ of $W_\sigma(\mathfrak{h})$ has the kernel $W_R(\mathfrak{h})$ and the image $W_G(\mathfrak{h}^+)W_I(\mathfrak{h})$, where $W_I(\mathfrak{h})$ is identified canonically with its image.

These results were applied in [6] to the theory of invariant eigendistributions on real semisimple Lie groups. We note that some parts of this theorem were known as is explained in § 2. In particular, in the case where the σ -system of roots \mathfrak{r} is normal or $\sigma\alpha - \alpha \in \mathfrak{r}$ for any $\alpha \in \mathfrak{r}$, the theorem is essentially known in [1, 7], and moreover we know that the set \mathfrak{r}^- of linear forms on \mathfrak{h}^- obtained by restricting the elements of $\{\alpha \in \mathfrak{r}; \sigma\alpha \neq -\alpha\}$ to \mathfrak{h}^- , is canonically a root system in the dual space of \mathfrak{h}^- , and $W_G(\mathfrak{h}^-)$ is equal to the Weyl group W^- of \mathfrak{r}^- . When the dimension of \mathfrak{h}^- is maximal, \mathfrak{r} is normal and all imaginary roots are compact and hence $W_I(\mathfrak{h}) \subset W_G(\mathfrak{h})$, so $W_\sigma(\mathfrak{h}) = W_G(\mathfrak{h})$ in (b). When $\mathfrak{h} = \mathfrak{h}^+$ or $\mathfrak{r} = \mathfrak{r}_I$, $W_\sigma(\mathfrak{h})$ is generated by $\{s_\alpha; \alpha \in \mathfrak{r}_I \text{ compact}\}$ (see [3, Corollary 2 of Lemma 6, p. 461]).

Let Δ be a fundamental system of \mathfrak{r} . It is called a σ -fundamental system if the linear order $>$ in \mathfrak{r} corresponding to it has the property: $\alpha > 0, \sigma\alpha \neq -\alpha \Rightarrow \sigma\alpha > 0$. Then the following assertion is known to hold when the σ -system of roots \mathfrak{r} is normal [1, 7], but we do not know until now whether this is true or false in general.

(d) *Let Δ and Δ^* be two σ -fundamental systems of \mathfrak{r} . Then there exists an element w in $W_\sigma(\mathfrak{h})$ such that $w\Delta = \Delta^*$.*

§ 2. Proofs of the assertions (a) and (c).

We prove here the assertions (a) and (c). The proof of (b) will be given in the next section.

PROOF OF (a). The proof is essentially due to I. Satake [7, p. 107]. Let $\{H_1, H_2, \dots, H_l\}$ be a basis of $\mathfrak{h}^- + \sqrt{-1}\mathfrak{h}^+$ such that $\{H_1, H_2, \dots, H_p\}$ forms a basis of \mathfrak{h}^- . Introduce in \mathfrak{r} the lexicographic order with respect to this basis.

and let $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ be the fundamental system corresponding to this order. We may assume that α_i for $1 \leq i \leq p$ are not imaginary but so are the rests. Then $\Delta_I = \{\alpha_i; p < i \leq l\}$ is a fundamental system of r_I . Take an element $w \in W_\sigma(\mathfrak{h})$ such that $w|\mathfrak{h}^- = 1$. Then $w\Delta_I$ is also a fundamental system of r_I , and therefore there exists a $w_0 \in W_I(\mathfrak{h})$ such that $w\Delta_I = w_0\Delta_I$. Put $w' = w_0^{-1}w$, then $w'\Delta_I = \Delta_I$, and moreover $w'\alpha_i > 0$ for $1 \leq i \leq p$ because $w'|\mathfrak{h}^- = 1$. Thus we get $w'\alpha_i > 0$ for any i and so $w' = 1$ and $w = w_0 \in W_I(\mathfrak{h})$. Thus the assertion (a) for \mathfrak{h}^- is proved. That for \mathfrak{h}^+ can be proved similarly taking a basis $\{H_1, H_2, \dots, H_l\}$ of $\mathfrak{h}^- + \sqrt{-1}\mathfrak{h}^+$ such that $\{H_1, \dots, H_{l-p}\}$ forms a basis for $\sqrt{-1}\mathfrak{h}^+$.

PROOF OF (c). The assertion (c) for \mathfrak{h}^- is known, and for instance follows from [4, Proposition 2.2, p. 245] and [8, Proposition 5 and Theorem 3].

The assertion for \mathfrak{h}^+ is proved as follows. Let g be an element in G such that $\text{Ad}(g)\mathfrak{h}^+ = \mathfrak{h}^+$. Let \mathfrak{g}_1 be the centralizer of \mathfrak{h}^+ in \mathfrak{g} , then it is reductive and is left stable under $\text{Ad}(g)$. The algebra \mathfrak{g}_1 and its center \mathfrak{c}_1 are given as follows :

$$\begin{aligned} \mathfrak{g}_1 &= (\mathfrak{h}_c + \sum_{\alpha \in r_R} \mathfrak{g}_\alpha) \cap \mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in r_R} \{X + \sigma X; X \in \mathfrak{g}_\alpha\}, \\ \mathfrak{c}_1 &= \{X \in \mathfrak{h}; \alpha(X) = 0 \quad \text{for all } \alpha \in r_R\}. \end{aligned}$$

Both \mathfrak{h} and $\mathfrak{h}_g = \text{Ad}(g)\mathfrak{h}$ are Cartan subalgebras of \mathfrak{g}_1 and so

$$\mathfrak{h} = \mathfrak{c}_1 + \mathfrak{h} \cap \mathfrak{g}'_1, \quad \mathfrak{h}_g = \mathfrak{c}_1 + \mathfrak{h}_g \cap \mathfrak{g}'_1,$$

where $\mathfrak{g}'_1 = [\mathfrak{g}_1, \mathfrak{g}_1]$. Since $\mathfrak{c}_1 \supset \mathfrak{h}^+$, the Cartan subalgebras $\mathfrak{h} \cap \mathfrak{g}'_1$ and $\mathfrak{h}_g \cap \mathfrak{g}'_1$ of \mathfrak{g}'_1 both have the trivial toroidal parts. Therefore, by [7, Proposition 2, p. 81], they are conjugate under the analytic subgroup G'_1 of G corresponding to \mathfrak{g}'_1 , that is, $\text{Ad}(g_1)(\mathfrak{h}_g \cap \mathfrak{g}'_1) = \mathfrak{h} \cap \mathfrak{g}'_1$ for some $g_1 \in G'_1$. Hence, putting $g_0 = g_1 g$, we get $\text{Ad}(g_0)\mathfrak{h} = \mathfrak{h}$ and $\text{Ad}(g_0)|\mathfrak{h}^+ = \text{Ad}(g)|\mathfrak{h}^+$, because $\text{Ad}(g_1)|\mathfrak{c}_1 = 1$. This proves our assertion.

§ 3. Proof of the assertion (b).

Put $W'(\mathfrak{h}) = W_\sigma(\mathfrak{h})W_I(\mathfrak{h})$. Then we wish to prove $W_\sigma(\mathfrak{h}) = W'(\mathfrak{h})$. To this end, we employ two lemmas, one of which is the following.

LEMMA 1. *Assume that there exists for any $w \in W_\sigma(\mathfrak{h})$ a $w' \in W'(\mathfrak{h})$ such that $w'w|\mathfrak{h}^- = 1$. Then $W_\sigma(\mathfrak{h}) = W'(\mathfrak{h})$.*

PROOF. This follows immediately from the assertion (a). Q. E. D.

Let us rewrite this lemma in more convenient form. Let \mathfrak{h}_0 be a Cartan subalgebra of \mathfrak{g} such that the dimension of its vector part $\mathfrak{a} = \mathfrak{h}_0^-$ is maximal. Let W^- be the Weyl group of the root system r^- of $(\mathfrak{g}, \mathfrak{a})$, the restricted root system. Then $W_\sigma(\mathfrak{a}) = W^-$ as is noted in § 1. Moreover we may assume $\mathfrak{h}^- \subset \mathfrak{a}$ up to conjugacy of \mathfrak{h} under $\text{Ad}(G)$ [8, Theorem 2, p. 383].

LEMMA 1'. Assume that there exists for any $w \in W_\sigma(\mathfrak{h})$ a $w'' \in W^-$ such that $w|\mathfrak{h}^- = w''|\mathfrak{h}^-$. Then $W_\sigma(\mathfrak{h}) = W'(\mathfrak{h})$.

PROOF. It follows from Theorem (c) and Lemma 2 below that

$$\{w'|\mathfrak{h}^-; w' \in W_\sigma(\mathfrak{h})\} = \{w''|\mathfrak{h}^-; w'' \in W^-, w''\mathfrak{h}^- = \mathfrak{h}^-\}.$$

This proves our assertion by Lemma 1.

Q. E. D.

LEMMA 2 (see [4, Proposition 2.2, p. 245]). Let α' be a subset of \mathfrak{a} . Assume that $\text{Ad}(g)\alpha' \subset \mathfrak{a}$ for some $g \in G$, then there exists $w'' \in W^-$ such that $w''X = \text{Ad}(g)X$ ($X \in \alpha'$).

PROOF. Let θ be a Cartan involution of \mathfrak{g} such that $\theta\mathfrak{h}_0 = \mathfrak{h}_0$, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the corresponding decomposition of \mathfrak{g} . Then $\mathfrak{a} \subset \mathfrak{p}$. We see from [3, p. 480] that there exists a $k \in K$, the analytic subgroup corresponding to \mathfrak{k} , such that $\text{Ad}(g)X = \text{Ad}(k)X$ ($X \in \alpha'$). Then the assertion follows from [4, Proposition 2.2, p. 245].

Q. E. D.

To give our second tool, Lemma 3, we need some preparation. Let α be a root of a Cartan subalgebra \mathfrak{h}' . We denote by H_α the element in \mathfrak{h}'_c corresponding to α under the Killing form of \mathfrak{g}_c . If α is real, we can choose root vectors $X_\alpha, X_{-\alpha}$ from \mathfrak{g} in such a way that $H_\alpha = [X_\alpha, X_{-\alpha}]$. Put $X'_{\pm\alpha} = \sqrt{2}|\alpha|^{-1}X_{\pm\alpha}$ and

$$\nu_\alpha = \exp \left\{ -\sqrt{-1} \frac{\pi}{2} \text{ad}(X'_\alpha + X'_{-\alpha}) \right\},$$

where $|\alpha|$ denotes the length of the root α . Then $\mathfrak{h}'' = \nu_\alpha(\mathfrak{h}'_c) \cap \mathfrak{g}$ is a Cartan subalgebra of \mathfrak{g} not conjugate to \mathfrak{h}' under G , and $\beta = \nu_\alpha\alpha$ is an imaginary, not compact root of \mathfrak{h}'' . Let σ_α be the hyperplane of \mathfrak{h}' defined by $\alpha = 0$, then \mathfrak{h}' and \mathfrak{h}'' are expressed as

$$\mathfrak{h}' = \sigma_\alpha + \mathbf{R}H_\alpha, \quad \mathfrak{h}'' = \sigma_\alpha + \mathbf{R}\sqrt{-1}H_\beta,$$

where $2|\beta|^{-2}H_\beta = \sqrt{-1}(X'_\alpha - X'_{-\alpha})$. Denote this relation between \mathfrak{h}' and \mathfrak{h}'' simply by $\mathfrak{h}' \rightarrow \mathfrak{h}''$. Then, as is remarked in [5, §3], we see from the classification of conjugate classes of Cartan subalgebras in [8] that any Cartan subalgebra \mathfrak{h} can be obtained up to conjugacy under G by a successive operation of " \rightarrow " on \mathfrak{h}_0 . Thus we may assume that there exists a set of real roots $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$ of \mathfrak{h} such that $\alpha_i \perp \alpha_j$ for $i \neq j$ and

$$\mathfrak{h} = \sigma + \mathbf{R}\sqrt{-1}H_{\nu\alpha_1} + \mathbf{R}\sqrt{-1}H_{\nu\alpha_2} + \dots + \mathbf{R}\sqrt{-1}H_{\nu\alpha_q},$$

$$\mathfrak{h}_0 = \sigma + \mathbf{R}H_{\alpha_1} + \mathbf{R}H_{\alpha_2} + \dots + \mathbf{R}H_{\alpha_q},$$

where $\nu = \nu_{\alpha_1} \nu_{\alpha_2} \dots \nu_{\alpha_q}$ and $\sigma = \sigma_{\alpha_1} \cap \sigma_{\alpha_2} \cap \dots \cap \sigma_{\alpha_q}$.

LEMMA 3. Let the notations be as above. Assume that for any $w \in W_\sigma(\mathfrak{h})$, there exists a $w' \in W'(\mathfrak{h})$ such that $w'w$ leaves $\mathbf{R}\sqrt{-1}H_{\nu\alpha_1} + \mathbf{R}\sqrt{-1}H_{\nu\alpha_2} + \dots +$

$\mathbf{R}\sqrt{-1}H_{\nu\alpha_q}$ stable. Then $W_\sigma(\mathfrak{h})=W'(\mathfrak{h})$.

PROOF. We prove that there exists $w'_1 \in W'(\mathfrak{h})$ such that $w'_1 w | \mathfrak{h}^- = 1$. Then the lemma follows directly from Lemma 1. Put $w_0 = w' w$ and $\nu X = \nu^{-1} w_0 \nu X$ ($X \in (\mathfrak{h}_0)_c = \nu^{-1} \mathfrak{h}_c$). Then $\nu \in W_\sigma(\mathfrak{h}_0)$ because $\nu X = X$ ($X \in \sigma$) and $\nu H_{\alpha_j} = H_{\nu\alpha_j}$ ($1 \leq j \leq q$). Moreover $w_0 | \mathfrak{h}^- = \nu | \mathfrak{h}^-$ because $\mathfrak{h}^- \subset \sigma$.

On the other hand, the assertion (b) for \mathfrak{h}_0 is valid, namely $W_\sigma(\mathfrak{h}_0) = W'(\mathfrak{h}_0) = W_G(\mathfrak{h}_0)W_I(\mathfrak{h}_0)$. Hence

$$\{u | \mathfrak{a}; u \in W_\sigma(\mathfrak{h}_0)\} = \{u | \mathfrak{a}; u \in W_G(\mathfrak{h}_0)\} = W_G(\mathfrak{a}),$$

where $\mathfrak{a} = \mathfrak{h}_0^-$ as before. Therefore we see as in the proof of Lemma 1' that there exists $w_1 \in W_\sigma(\mathfrak{h})$ such that $\nu | \mathfrak{h}^- = w_1 | \mathfrak{h}^-$. Put $w'_1 = w_1^{-1} w' \in W'(\mathfrak{h})$, then $w'_1 w | \mathfrak{h}^- = 1$.
Q. E. D.

Let \mathfrak{h}_I and \mathfrak{h}_R be the subspaces of \mathfrak{h}^+ and \mathfrak{h}^- spanned by $\sqrt{-1}H_\gamma$ ($\gamma \in \mathfrak{r}_I$) and H_γ ($\gamma \in \mathfrak{r}_R$) respectively. Then $W_\sigma(\mathfrak{h})$ leaves \mathfrak{h}_I and \mathfrak{h}_R invariant. Note that

$$\mathfrak{h}_I \supset \mathbf{R}\sqrt{-1}H_{\nu\alpha_1} + \mathbf{R}\sqrt{-1}H_{\nu\alpha_2} + \cdots + \mathbf{R}\sqrt{-1}H_{\nu\alpha_q}.$$

COROLLARY. If the equality holds in the above inclusion relation, then $W_\sigma(\mathfrak{h}) = W'(\mathfrak{h})$. In particular, if \mathfrak{g} is a normal real form of \mathfrak{g}_c or \mathfrak{g} has a Cartan subalgebra with trivial toroidal part, then $W_\sigma(\mathfrak{h}) = W'(\mathfrak{h})$ for any Cartan subalgebra \mathfrak{h} of \mathfrak{g} .

PROOF. In general, every element w in $W_\sigma(\mathfrak{h})$ leaves \mathfrak{h}_I stable. Hence the first assertion by Lemma 3. If \mathfrak{g} is normal, $\mathfrak{h}^+ = \mathfrak{h}_I = \mathbf{R}\sqrt{-1}H_{\nu\alpha_1} + \cdots + \mathbf{R}\sqrt{-1}H_{\nu\alpha_q}$ for any \mathfrak{h} . This proves the second assertion.
Q. E. D.

We prove the assertion (b) for all simple Lie algebras \mathfrak{g} according to their types. This has been already done for the cases where \mathfrak{g} is normal or has only one conjugate class of Cartan subalgebras. Hence, in the notation in [8], the types of \mathfrak{g} left to study are the following:

classical type: $A_n III_m$ ($1 \leq m \leq [(n+1)/2]$), $B_n I_m$ ($1 \leq m \leq n-1$), $C_n II_m$ ($1 \leq m \leq [n/2]$), $D_n I_m$ ($1 \leq m \leq n-1$), $D_n III$, where $n = \text{rank } \mathfrak{g}$ and m denotes the integer such that $\mathfrak{g} = \mathfrak{g}_m$;

exceptional type: EII , $EIII$ (real forms of E_6); EVI , $EVII$ (of E_7); EIX (of E_8); FII (of F_4).

(1°) The cases of $A_n III_m$, $C_n II_m$ and $D_n III$. In these cases, $\mathfrak{h}^- = \mathfrak{h}_R$ for any \mathfrak{h} and \mathfrak{r}_R is always a multiple of type A_1 , possibly \emptyset . Moreover the restricted root system \mathfrak{r}^- of \mathfrak{h}_0 is of type BC_m or C_m . Here a root system is called of type BC_m if it is isomorphic to $\{\pm e_i \pm e_j$ ($1 \leq i < j \leq m$), $\pm e_i$, $\pm 2e_i$ ($1 \leq i \leq m$)\}, where $\{e_i\}$ is an orthonormal system in a Euclidean space. Since $\mathfrak{h}^- = \mathfrak{h}_R$, the restriction of $w \in W_\sigma(\mathfrak{h})$ on \mathfrak{h}^- is completely determined by the automorphism of \mathfrak{r}_R induced by w . Therefore the assertion (b) follows from Lemma 1' if we admit the following lemma.

LEMMA 4. *In the above cases, any automorphism of \mathfrak{r}_R induced by an element $w \in W_\sigma(\mathfrak{h})$, can be realized by some element $u \in W^-$, the Weyl group of \mathfrak{r}^- .*

This lemma can be proved by applying the explicit forms of \mathfrak{h} given in [8].

(2°) The case of $B_n I_m$. In this case, $\mathfrak{h}^- = \mathfrak{h}_R$ for any \mathfrak{h} and $\mathfrak{r}^- = B_m$. Moreover \mathfrak{r}_R 's for various \mathfrak{h} 's are listed in [6, Table 9.1].

We see as in (1°) that the same conclusion as in Lemma 4 holds also in this case.

(3°) The case of $D_n I_m$. In this case, $\mathfrak{h}^- = \mathfrak{h}_R$ except when \mathfrak{h} is conjugate to the Cartan subalgebra corresponding to $F(l, k)$ [8, p. 403] with $2l + 2k = m - 1$ (in this case, the codimension of \mathfrak{h}_R in \mathfrak{h}^- is equal to 1). Moreover $\mathfrak{r}^- = B_m$ for $m < n$, and \mathfrak{r}_R 's are listed in [6, Table 9.1].

Only the cases where $\mathfrak{h}^- \neq \mathfrak{h}_R$, are different from (2°). But we can check without difficulty that the assumption in Lemma 1' is also satisfied in these cases.

(4°) The cases of exceptional type. In these cases, $\mathfrak{h}^- = \mathfrak{h}_R$ for any \mathfrak{h} , $\mathfrak{r}_R = A_1$, $A_1 \oplus A_1$ or $A_1 \oplus A_1 \oplus A_1$ for \mathfrak{h} not compact and not conjugate to \mathfrak{h}_0 . If $\mathfrak{r}_R = A_1$, the assumption in Lemma 1' is clearly satisfied. The case where $\mathfrak{r}_R = A_1 \oplus A_1$ or $A_1 \oplus A_1 \oplus A_1$ can occur only for EII , EVI , $EVII$ and EIX . The types of \mathfrak{r}^- are F_4 , F_4 , C_3 and F_4 respectively [1, p. 33]. Note that \mathfrak{r}_R consists of long roots in \mathfrak{r}^- and is a subsystem of \mathfrak{r}^- in the sense that $\alpha, \beta \in \mathfrak{r}_R$ and $\alpha + \beta \in \mathfrak{r}^-$, then $\alpha + \beta \in \mathfrak{r}_R$. Then we see that any automorphism of the subsystem \mathfrak{r}_R of \mathfrak{r}^- is induced by some element $u \in W^-$ in these cases. Thus the assumption of Lemma 1' is satisfied and hence the assertion (b).

The assertion (b) is now completely proved.

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