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Some functors on Grothendieck exact sequences of type I

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Abstract. It has been shown that the exactness of Grothendieck exact sequences of type I, of abelian groups and their homomorphisms is preserved by the free functor F, the polynomial functors P_n, the n-th symmetric product functors Sp^n, and the direct limit functor $\varinjlim$. A necessary and sufficient condition has also been sought under which the polynomial functors Q_n preserve the exactness of Grothendieck exact sequences of type I.

1.1. Introduction.

Following M. N. Roby [6] we recall a sequence,

$$A \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} B \xrightarrow{\quad\beta\quad} C \tag{1.2}$$

of sets A, B, C and functions α_1, α_2 and β on them a Grothendieck exact sequence of type I, if;

(a) β is onto,

(b) For any two elements b_1, b_2 of B the following two conditions are equivalent,

(i) $\beta(b_1)=\beta(b_2)$,

(ii) There exists an $a\in A$ such that: $\alpha_1(a)=b_1$, $\alpha_2(a)=b_2$.

In the language of M. Barr [1] a Grothendieck exact sequence of type I, may be called a right exact sequence. In the proof of the theorem (2.1) of next section, we shall use the well known definitions and facts given in [3].

For groups we modify the definitions of Grothendieck exact sequences so as to that functions are replaced by group homomorphisms and consequently onto functions correspond to epimorphisms.

In this paper we have studied the behavior of such sequences with abelian groups under the free functor F, the polynomial functors P_n, Q_n, the n-th symmetric product functors Sp^n [2, 5] and the direct limit functor $\varinjlim$. It has been shown that except Q_n, all the functors under consideration preserve the exactness of Grothendieck exact sequences of type I. A necessary and sufficient condition has been sought under which Q_n, also preserve the exactness of type I.

For the sake of brevity we shall write GESI for the Grothendieck Exact Sequences of type I.

2.1. THEOREM. *Let*

$$A \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} B \xrightarrow{\beta} C \tag{2.2}$$

be a sequence of sets and functions on them and let F be the free functor. Then

$$F(A) \underset{F(\alpha_2)}{\overset{F(\alpha_1)}{\rightrightarrows}} F(B) \xrightarrow{F(\beta)} F(C) \tag{2.3}$$

is a GESI if and only if so is (2.2), *where $F(f)$ denotes the induced homomorphism corresponding to the function f.*

PROOF. In the terminology of M. Barr [1] a GESI means a right exact sequence. Let Ab and S denote the categories of abelian groups and sets respectively. If U is the forgetful functor, then

$$Ab \underset{F}{\overset{U}{\rightleftarrows}} S$$

is an adjunction. It is well known that the forgetful functor U is a limit preserving epifunctor [3], and hence preserves the kernel pair also. It is also clear that β coequalizes α_1 and α_2.

Conversely, since F preserves colimits, $F(\beta)$ will coequalize $F(\alpha_1)$ and $F(\alpha_2)$. Now it remains to prove that the image of $F(A)$ in $F(B)\times F(B)$ is kernel pair, and for this it suffices to show that it is an equivalence. Further we know that in any class of equationally defined algebraic systems, reflexivity implies the symmetry and transitivity. Therefore in the category of groups reflexivity implies the equivalence. Since free functor F is an epifunctor, the reflexivity will obviously be preserved. This completes the proof of (2.1).

Let A be an abelian group and let A_n denotes the free abelian group generated by all ordered sets $(a_1, a_2, \cdots, a_n)$, where $a_1, a_2, \cdots, a_n$ are arbitrary elements of A. Then it can be easily seen that the correspondence $F_n : A \longrightarrow A_n$ determines a functor, that we also denote by F_n, from the category of abelian groups into itself. We call F_n as n-free functor. The following corollary is an immediate consequence of the preceding theorem.

2.4. COROLLARY. *If F_n is the n-free functor and* (2.2) *is a GESI of abelian* *groups and their homomorphisms. Then*

$$F_n(A) \overset{F_n(\alpha_1)}{\underset{F_n(\alpha_2)}{\rightrightarrows}} F_n(B) \xrightarrow{F_n(\beta)} F_n(C)$$

is also a GESI.

Now we prove a lemma that will be used in sequel frequently.

3.1. LEMMA. *Let*

$$A \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} B \xrightarrow{\beta} C \tag{3.2}$$

be a GESI of abelian groups. Let L, M and N be subgroups of A, B and C respectively, satisfying $\alpha_i(L) \leqq M$, $i=1, 2$, and $N=\beta(M)$. Then

$$A/L \overset{\alpha_1^*}{\underset{\alpha_2^*}{\rightrightarrows}} B/M \xrightarrow{\beta^*} C/N$$

is also a GESI.

PROOF. Since $\alpha_i(L) \leqq M$ for $i=1, 2$ and $N=\beta(M)$, we certainly get the induced homomorphisms α_1^*, α_2^* and β^*. The onto property of β^* follows from that of β. Now suppose that b_1M, $b_2M \in B/M$ are such that $\beta^*(b_1M)=\beta^*(b_2M)$. Then $\beta(b_1b_2^{-1}) \in N$; therefore we get $\beta(b_1)=\beta(mb_2)$ for some $m \in M$. Since (3.2) is a GESI and b_1 and mb_2 are two elements of B with equal images under β, therefore there exists an $a \in A$ such that $\alpha_1(a)=b_1$, $\alpha_2(a)=mb_2$. Evidently $aL \in A/L$ is a desired element for which $\alpha_1^*(aL)=\alpha_1(a)M=b_1M$ and $\alpha_2^*(aL)=\alpha_2(a)M=mb_2M=b_2m^{b_2}M=b_2M$, as M is normal in B.

On the otherhand suppose that b_1M, $b_2M \in B/M$, and there exists an element $aL \in A/L$ such that $\alpha_i^*(aL)=b_iM$ for $i=1, 2$. Since $\alpha_i^*(aL)=\alpha_i(a)M$ and $\beta\alpha_1(a)=\beta\alpha_2(a)$ for all $a \in A$ as (3.2) is a GESI, therefore, $\beta^*(b_1M)=\beta(b_1)N=\beta\alpha_1(a)N=\beta\alpha_2(a)N=\beta(b_2)N=\beta^*(b_2M)$.

3.3. PROPOSITION. *Let T and U be two functors that preserve the exactness of GESI* (3.3). *Then*

$$T(A)\otimes U(A) \overset{T(\alpha_1)\otimes U(\alpha_1)}{\underset{T(\alpha_2)\otimes U(\alpha_2)}{\rightrightarrows}} T(B)\otimes U(B) \xrightarrow{T(\beta)\otimes U(\beta)} T(C)\otimes U(C)$$

is also a GESI, where α denotes the tensor product.

PROOF. In fact this proposition is equivalent to the following:

If

$$A \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} B \xrightarrow{\beta} C \quad \text{and} \quad L \overset{f_1}{\underset{f_2}{\rightrightarrows}} M \xrightarrow{g} N$$

are two GESI of abelian groups, then so is

$$A\otimes L \underset{\alpha_2\otimes f_2}{\overset{\alpha_1\otimes f_1}{\rightrightarrows}} B\otimes M \xrightarrow{\beta\otimes g} C\otimes N. \tag{3.4}$$

Now, if $Z(X, Y)$ denote the free abelian group generated by the set of symbols (x, y), $x\in X$ and $y\in Y$ then as in corollary (2.4) one can prove that

$$Z(A, L) \underset{(\alpha_2, f_2)}{\overset{(\alpha_1, f_1)}{\rightrightarrows}} Z(B, M) \xrightarrow{(\beta, g)} Z(C,N)$$

is also a GESI. Further if $W(X, Y)$ is the subgroup of $Z(X, Y)$ generated by symbols of the form:

(i) $(x_1+x_2, y)-(x_1, y)-(x_2, y)$,

(ii) $(x, y_1+y_2)-(x, y_1)-(x, y_2)$

and

(iii) $(xk, y)-(x, ky)$

where $x, x_1, x_2\in X, y, y_1, y_2\in Y$ and $k\in Z$, the set of integers, then it is quite immediate that:

$$(\alpha_i, f_i)[W(A, L)]\leqq W(B, M) \text{ for } i=1, 2 \text{ and } (\beta, g)[W(B, M)]=W(C, N),$$

and therefore in view of lemma (3.1) we conclude that (3.4) is a GESI.

4.1. Direct limit.

We recall first a few definitions. A set D with a partial order relation $\leqq$ is called a directed set if for each pair α, β of D there exists a γ in D such that $\alpha\leqq\gamma$, $\beta\leqq\gamma$. A directed system of sets over a directed set D is a function which attaches to each $\alpha\in D$, a set A^α and to each pair α, β such that $\alpha\leqq\beta$ in D a map $f_\alpha^\beta\colon A^\alpha\longrightarrow A^\beta$ such that, for each $\alpha\in D$, $f_\alpha^\alpha=$ identity on A^α and for $\alpha\leqq\beta\leqq\gamma$ in D, $f_\beta^\gamma f_\alpha^\beta=f_\alpha^\gamma$ and is denoted by $[A^\alpha, f_\alpha^\beta]_D$.

Let $[G^\alpha, f_\alpha^\beta]_D$ be a directed system of abelian groups and group homomorphisms over a directed set D. We denote the direct sum of G^α, $\alpha\in D$, by $\oplus\sum G^\alpha$. Let H be the subgroup of $\oplus\sum G^\alpha$, generated by all the elements of the form: $f_\alpha^\beta(x)-x$, $x\in G^\alpha$, $\alpha, \beta\in D$ and $\alpha\leqq\beta$. The quotient group $[\oplus\sum G^\alpha]/H$ is called the direct limit of the directed system $[G^\alpha, f_\alpha^\beta]_D$ and is generally denoted by $\varinjlim[G^\alpha, f_\alpha^\beta]_D$. We, for the sake of brevity, shall denote it by G^∞.

4.2. PROPOSITION. *Let $[A^\alpha, f_\alpha^\beta]_D$, $[B^\alpha, g_\alpha^\beta]_D$, $[C^\alpha, h_\alpha^\beta]_D$ be three directed systems of abelian groups and group homomorphisms over the same directed set D. If*

$$A^\alpha \underset{\delta_2^\alpha}{\overset{\delta_1^\alpha}{\rightrightarrows}} B^\alpha \xrightarrow{\mu^\alpha} C^\alpha \tag{4.3}$$

is a GESI for each $\alpha \in D$. Then so is

$$A^{\infty} \overset{\delta_1^{\infty}}{\underset{\delta_2^{\infty}}{\rightrightarrows}} B^{\infty} \xrightarrow{\mu^{\infty}} C^{\infty}. \tag{4.4}$$

PROOF. If $\delta_1^{\oplus}$, $\delta_2^{\oplus}$ and $\mu^{\oplus}$ denote the natural linear extensions of δ_1^{α}, δ_2^{α}, and μ^{α}, $\alpha \in D$ then first we prove that

$$\oplus \sum A^{\alpha} \overset{\delta_1^{\oplus}}{\underset{\delta_2^{\oplus}}{\rightrightarrows}} \oplus \sum B^{\alpha} \xrightarrow{\mu^{\oplus}} \oplus \sum C^{\alpha} \tag{4.5}$$

is a GESI. Since μ^{α} is an epimorphism for each $\alpha \in D$, $\mu^{\oplus}$ will obviously be an epimorphism. If b and b' be two elements of $\oplus \sum B^{\alpha}$ which are images of the same element a of $\oplus \sum A^{\alpha}$ under $\delta_1^{\oplus}$, $\delta_2^{\oplus}$ respectively, then we can express $b = \delta_1^{\oplus}[\sum a^{\alpha}] = \sum \delta_1^{\alpha}(a^{\alpha})$ and $b' = \delta_2^{\oplus}[\sum a^{\alpha}] = \sum \delta_2^{\alpha}(a^{\alpha})$. Since by hypothesis $\mu^{\alpha}\delta_1^{\alpha} = \mu^{\alpha}\delta_2^{\alpha}$ for each $\alpha \in D$, therefore

$$\mu^{\oplus}(b) = \sum [\mu^{\alpha}\delta_1^{\alpha}](a^{\alpha}) = \sum [\mu^{\alpha}\delta_2^{\alpha}(a^{\alpha})] = \mu^{\oplus}(b').$$

On the other hand if $\mu^{\oplus}(b) = \mu^{\oplus}(b')$, then we have a situation such that:

$$\sum \mu^{\alpha_i}(b_i) = \sum \mu^{\alpha_i}(b_i') \tag{4.6}$$

where b_i, $b_i' \in B^{\alpha_i}$ and $\alpha_i \in D$. Now uniqueness of the expression of an element in a direct sum gives $\mu^{\alpha_i}(b_i) = \mu^{\alpha_i}(b_i')$, for each i. Since (4.3) is a GESI, there exists $a_i \in A^{\alpha_i}$ for each i such that: $b_i = \delta_1^{\alpha_i}(a_i)$ and $b_i' = \delta_2^{\alpha_i}(a_i)$. And therefore, if we choose $a = \sum a_i$, then $b = \delta_1^{\oplus}(a)$ and $b' = \delta_2^{\oplus}(a)$. Thus we conclude that (4.5) is a GESI.

Finally if H_G denote the subgroup of $\oplus \sum G^{\alpha}$, generated by elements of the form $f_{\alpha}^{\beta}(x) - x$, $x \in G^{\alpha}$, $\alpha, \beta \in D$ and $\alpha \leqq \beta$; then obviously $\delta_i^{\oplus}(H_A) \leqq H_B$, for $i = 1, 2$ and $\mu^{\oplus}(H_B) = H_C$. Now in view of lemma (3.1) we conclude that (4.4) is a GESI.

5.1. Symmetric products.

Let S_n, the symmetry group of degree n, operate in a module M [i.e. each $g \in S_n$ defines a module endomorphism g^* of M such that $(g_1 g_2)^* = g_1^* g_2^*$, $e^* = Id.$, where $g_1, g_2 \in S_n$, e is the identity of S_n and $Id.$ is the identity endomorphism of the module M]. We denote by M/S_n the quotient of M by the submodule of M which is generated by the elements of the form $g^*(m) - m$, $m \in M$, $g \in S_n$. For any module M let $\otimes_n M = M \otimes \cdots \otimes M$, [$n$-factors], be its n-fold tensor product. The action of S_n extends to $\otimes_n M$, in a natural way, by permuting the factors. We define the n-th symmetric product of M to be the quotient $[\otimes_n M]/[S_n]$ and denote it by $Sp^n(M)$.

If $\alpha: A \longrightarrow B$, is a module homomorphism then its n-th tensor product $\otimes_n \alpha : \otimes_n A \longrightarrow \otimes_n B$ is a module homomorphism which commute with the operation of S_n. Therefore we have a unique homomorphism $Sp^n(\alpha): Sp^n(A) \longrightarrow Sp^n(B)$, the S_n-product of α, such that the diagram:

$$\begin{array}{ccc} \otimes_n A & \xrightarrow{\otimes_n \alpha} & \otimes_n B \\ \pi_A \downarrow & & \downarrow \pi_B \\ Sp^n(A) & \xrightarrow{Sp^n(\alpha)} & Sp^n(B) \end{array}$$

is commutative, where π_A, π_B are natural projections. For composite homomorphism $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$, we have $\otimes_n(\beta\alpha) = \otimes_n(\beta)\otimes_n(\alpha)$, and hence $Sp^n(\beta\alpha) = Sp^n(\beta)Sp^n(\alpha)$. Finally, since $Sp^n(Id.) = Id.$, we conclude that Sp^n is a functor from the category of modules into itself. And since every abelian group is trivially a Z-module, the n-th symmetric product for an abelian group is a particular case of the above.

5.2. THEOREM. *The n-th symmetric product functors Sp^n preserve the exactness of GESI of abelian groups.*

PROOF. Let G be a group and S a subset of G. Then denote the subgroup of G, generated by S by $\langle s \mid s \in S \rangle$. Consider a GESI,

$$A \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} B \xrightarrow{\beta} C$$

of abelian groups A, B, C. If L, M, N are subgroups of A, B, C respectively given by $L = \langle a - g^*(a) \mid a \in A,\ g \in S_n \rangle$, $M = \langle b - g^*(b) \mid b \in B,\ g \in S_n \rangle$ and $N = \langle c - g^*(c) \mid c \in C,\ g \in S_n \rangle$, then as action of S_n commutes with homomorphism, it is quite obvious that $\alpha_i(L) \leqq M$ and $\beta(M) = N$, $i = 1, 2$. And consequently by lemma (3.1) we get the GESI:

$$A/S_n \underset{\bar{\alpha}_2}{\overset{\bar{\alpha}_1}{\rightrightarrows}} B/S_n \xrightarrow{\bar{\beta}} C/S_n \tag{5.3}$$

Now using the proposition (3.3) inductively on (5.3) we conclude that

$$Sp^n(A) \underset{Sp^n(\alpha_2)}{\overset{Sp^n(\alpha_1)}{\rightrightarrows}} Sp^n(B) \xrightarrow{Sp^n(\beta)} Sp^n(C)$$

is a GESI.

6.1. Let X be a group and $Z(X)$ its integral groupring. If A_X is the corresponding augmentation ideal, then one can form the abelian groups: $P_n(X) = Z(X)/A_X^{n+1}$, $\bar{P}_n(X) = A_X/A_X^{n+1}$ and $Q_n(X) = A_X^n/A_X^{n-1}$, where A_X^i denotes the i-th power of A_X. These abelian groups have been studied by Passi [4], and called polynomial groups. In [5], he has proved their functorial nature.

6.2. THEOREM. *The polynomial functors P_n and $\bar{P}_n$ preserve the exactness of* GESI *of abelian groups.*

PROOF. Let $f: X \longrightarrow Y$ be a homomorphism from abelian group X into abelian group Y. Then f can be linearly extended to the ring homomorphism $f^*: Z(X) \longrightarrow Z(Y)$, by setting $f^*[\sum rx] = \sum rf(x)$, $r \in Z$, the set of integers and $x \in X$. Obviously f^* maps A_X^n into A_Y^n for all $n \geqq 1$; and if f is surjective so is f^*. Thus $\alpha_i^*(A_A) \leqq A_B$ for $i=1, 2$, and $\beta^*(A_B) \leqq A_C$. Also in view of (2.1)

$$Z(A) \underset{\alpha_2^*}{\overset{\alpha_1^*}{\rightrightarrows}} Z(B) \xrightarrow{\beta^*} Z(C)$$

is a GESI. Therefore lemma (3.1) implies that,

$$P_n(A) \underset{P_n(\alpha_2)}{\overset{P_n(\alpha_1)}{\rightrightarrows}} P_n(B) \xrightarrow{P_n(\beta)} P_n(C)$$

is a GESI.

For the second part it will suffice to show that

$$A_A \underset{\alpha_2^*}{\overset{\alpha_1^*}{\rightrightarrows}} A_B \xrightarrow{\beta^*} A_C \tag{6.3}$$

is a GESI. The nontrivial part to prove this is that if images of two elements $v_1, v_2 \in A_B$ under β^* are equal then there exists an element $u \in A_A$ which satisfies the condition $\alpha_1^*(u) = v_1$ and $\alpha_2^*(u) = v_2$.

Since A_X is Z-free with basis $[x - e_X \mid x \in X,\ x \neq e_X$ and e_X the identity of $X]$, it will suffice to make the verification on generators. Suppose that $y_1 - e_B$ and $y_2 - e_B$ are two generators of A_B such that $\beta^*(y_1 - e_B) = \beta^*(y_2 - e_B)$ i.e. $\beta(y_1) - e_C = \beta(y_2) - e_C$. Clearly, $\beta(y_1) = \beta(y_2)$ and therefore by hypothesis there exists an $x \in A$, such that $\alpha_1(x) = y_1$, and $\alpha_2(x) = y_2$. Evidently $\alpha_1^*(x - e_A) = y_1 - e_B$ and $\alpha_2^*(x - e_A) = y_2 - e_B$. Thus (6.3) is a GESI.

Passi [5] has shown that if G is an abelian group, then there is a natural epimorphism $\pi_n: Sp_n(G) \longrightarrow Q_n(G)$ for all $n \geqq 1$, and π_n is an isomorphism for $n=1, 2$. He has also proved that π_n is an isomorphism for all $n \geqq 1$, provided G is free abelian. Further Vermani [7] has added that π_n is an isomorphism as and when G is $\underline{|n}$-torsion free abelian.

In view of Passi's results [5], we infer that Q_1 and Q_2 will preserve the exactness of GESI of abelian groups as they can be identified with Sp^1 and Sp^2 respectively; and for the same reason Q_n will preserve the exactness of GESI of free abelian groups for all $n \geqq 1$. From Vermani's result mentioned in the above paragraph this fact extends to GESI of $\underline{|n}$-torsion free abelian groups. Next we investigate the case in general.

First of all we remark that the canonical epimorphism $\pi_n(G): Sp^n(G) \longrightarrow Q_n(G)$ is not injective for all abelian groups. Let Z_p denotes the cyclic group of order p, p a prime. Consider abelian group G given by,

$$G = Z_p \oplus \cdots \oplus Z_p, \ [m \text{ copies and } m \geqq 2].$$

Passi in ([4], Theorem 4.7) has proved that the polynomial groups $Q_n(G)$ have a constant structure for $n \geqq (m-1)(p-1)+1$, while this is not the case for $Sp^n(G)$. Dold in ([2], Theorem 8.4) has shown that:

$$Sp^n(A \oplus B) = Sp^n(B) \oplus (\oplus \sum_{r=1}^{n-1} Sp^r(A) \otimes_Z Sp^{n-r}(B)) \oplus Sp^n(A).$$

The iterative use of this formula will yield that $Sp^n(G)$ never attain a constant structure.

6.4. Let G be an abelian group with non-isomorphic $\pi_n(G)$, Let

$$R \overset{\alpha}{\rightarrowtail} F \overset{\beta}{\twoheadrightarrow} G$$

be a free presentation of G. Consider the direct sum $R \oplus F$ of R and F, with binary operation on it defined by:

$$(r_1, f_1) + (r_2, f_2) = (r_1 + r_2, f_1 + f_2).$$

If $\alpha_1, \alpha_2: R \oplus F \rightrightarrows F$ are given by $\alpha_1(r, f) = r + f$ and $\alpha_2(r, f) = f$, then it can be easily verified that α_1, α_2 are group homomorphisms, and

$$R \oplus F \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} F \xrightarrow{\beta} G$$

is a GESI. The nontrivial part for proving this can be established easily. For, if $\beta(f_1) = \beta(f_2)$, $f_1, f_2 \in F$, then $\beta(f_1 - f_2) = 0$, therefore $f_1 - f_2 \in R$. Obviously $(f_1 - f_2, f_2) \in R \oplus F$ is a suitable element which satisfies the required condition.

We now claim that:

$$Q_n(R \oplus F) \underset{Q_n(\alpha_2)}{\overset{Q_n(\alpha_1)}{\rightrightarrows}} Q_n(F) \xrightarrow{Q_n(\beta)} Q_n(G) \tag{6.5}$$

is not a GESI.

Since subgroups of free abelian groups are free, we notice that R is free and hence $R \oplus F$ is free and accordingly $\pi_n(R \oplus F): Sp^n(R \oplus F) \longrightarrow Q_n(R \oplus F)$ is an isomorphism. Now starting with a nonzero element x lying in the kernel of $\pi_n(G)$ and chasing the following commutative diagram one can easily conclude the non-GESI-ness of (6.5).

$$\begin{array}{ccccc} Sp^n(R\oplus F) & \overset{Sp^n(\alpha_1)}{\underset{Sp^n(\alpha_2)}{\rightrightarrows}} & Sp^n(F) & \xrightarrow{Sp^n(\beta)} & Sp^n(G) \\ \downarrow{\scriptstyle \pi_n(R\oplus F)} & & \downarrow{\scriptstyle \pi_n(F)} & & \downarrow{\scriptstyle \pi_n(G)} \\ Q_n(R\oplus F) & \overset{Q_n(\alpha_1)}{\underset{Q_n(\alpha_2)}{\rightrightarrows}} & Q_n(F) & \xrightarrow{Q_n(\beta)} & Q_n(G) \end{array} \tag{6.6}$$

6.7. THEOREM. *A necessary and sufficient condition that Q_n preserve the GESI-ness of a GESI of abelian groups is that the restriction of $Sp^n(\beta)$ to the kernel of $\pi_n(B)$ in the following commutative diagram is an epimorphism.*

$$\begin{array}{ccccc} Sp^n(A) & \overset{Sp^n(\alpha_1)}{\underset{Sp^n(\alpha_2)}{\rightrightarrows}} & Sp^n(B) & \xrightarrow{Sp^n(\beta)} & Sp^n(C) \\ \downarrow{\scriptstyle \pi_n(A)} & & \downarrow{\scriptstyle \pi_n(B)} & & \downarrow{\scriptstyle \pi_n(C)} \\ Q_n(A) & \overset{Q_n(\alpha_1)}{\underset{Q_n(\alpha_2)}{\rightrightarrows}} & Q_n(B) & \xrightarrow{Q_n(\beta)} & Q_n(C) \end{array} \tag{6.8}$$

PROOF. Suppose that the bottom sequence in the above diagram is a GESI. Consider a nonzero element x in the kernel of $\pi_n(C)$. Since $Sp^n(\beta)$ is an epimorphism, there exists a nonzero element $y \in Sp^n(B)$ such that $Sp^n(\beta)[y]=x$. Therefore, $\pi_n(C)Sp^n(\beta)[y]=0$. Using the commutativity of right square in (6.8), we get $Q_n(\beta)\pi_n(B)[y]=0$. Also $Q_n(\beta)[0]=0$. Since $\pi_n(A)$ is an epimorphism and the bottom row in (6.8) is a GESI, there exists an element $t \in Sp^n(A)$ such that:

$$Q_n(\alpha_1)\pi_n(A)[t]=\pi_n(B)[y] \quad \text{and} \quad Q_n(\alpha_2)\pi_n(A)[t]=0\,,$$

and from the commutativity of left square in (6.8), we have

$$\pi_n(B)Sp^n(\alpha_1)[t]=\pi_n(B)[y] \quad \text{and} \quad \pi_n(B)Sp^n(\alpha_2)[t]=0\,,$$

wherefrom we conclude that $y-Sp^n(\alpha_1)[t]$ and $Sp^n(\alpha_2)[t]$ are in the kernel of $\pi_n(B)$. Certainly $y+[Sp^n(\alpha_2)-Sp^n(\alpha_1)][t]$ is in the kernel of $\pi_n(B)$. We claim that this is a preimage of x, as, required, under the restriction of $Sp^n(\beta)$ to the kernel of $\pi_n(B)$, because,

$$\begin{aligned} Sp^n(\beta)[y+(Sp^n(\alpha_2)-Sp^n(\alpha_1))(t)] &= Sp^n(\beta)[y]+Sp^n(\beta)[Sp^n(\alpha_2)-Sp^n(\alpha_1)](t) \\ &= x+0=x\,, \end{aligned}$$

as the upper sequence in (6.8) is a GESI.

Conversely, if $Sp^n(\beta)$ restricted to the kernel of $\pi_n(B)$ is an epimorphism then $Sp^n(\beta)[\operatorname{Ker}\pi_n(B)]=\operatorname{Ker}\pi_n(C)$ and also $Sp^n(\alpha_i)[\operatorname{Ker}\pi_n(A)] \leqq \operatorname{Ker}\pi_n(B)$, for $i=1, 2$. Now conditions of lemma (3.1) hold, consequently we get the commutative diagram:

$$\begin{array}{ccccc} Sp^n(A)/\mathrm{Ker}\,\pi_n(A) & \overset{g_1^*}{\underset{g_2^*}{\rightrightarrows}} & Sp^n(B)/\mathrm{Ker}\,\pi_n(B) & \xrightarrow{h^*} & Sp^n(C)/\mathrm{Ker}\,\pi_n(C) \\ \downarrow{\scriptstyle \rho_n(A)} & & \downarrow{\scriptstyle \rho_n(B)} & & \downarrow{\scriptstyle \rho_n(C)} \\ Q_n(A) & \overset{Q_n(\alpha_1)}{\underset{Q_n(\alpha_2)}{\rightrightarrows}} & Q_n(B) & \xrightarrow{Q_n(\beta)} & Q_n(C) \end{array}$$

with $\rho_n(A)$, $\rho_n(B)$ and $\rho_n(C)$ as isomorphisms and g_1^*, g_2^* and h^* the homomorphisms induced by $Sp^n(\alpha_1)$, $Sp^n(\alpha_2)$ and $Sp^n(\beta)$ respectively and with upper row as a GESI. In this situation it is quite obvious that the lower sequence is also a GESI.

6.9. COROLLARY. *If $\pi_n(B)$ is an isomorphism in theorem* (6.7), *then so is* $\pi_n(C)$.

PROOF. Obviously, in this situation, the kernel of $\pi_n(C)$ is the identity.

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