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Asymptotic properties of eigenvalues of a class of Markov chains induced by direct product branching processes

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§ 1. Introduction.

Karlin and McGregor introduced in [9] a class of finite state Markov chains induced by direct product branching processes (see also [8] and [10]). The class includes many Markov chains of interest in population genetics. In the case of absence of selection force (in genetics language), they made a deep investigation on eigenvalues and eigenvectors of the transition matrices. In simplest cases, their Markov chains are as follows. Let $\{Z(n)=(Z_1(n), Z_2(n))$; $n=0, 1, 2, \cdots\}$ be a two type direct product branching processes, $\{Z_1(n)\}$ and $\{Z_2(n)\}$ being independent Galton-Watson processes with a common offspring distribution. That is, $\{Z_1(n)\}$ and $\{Z_2(n)\}$ are independent Markov chains taking values in nonnegative integers satisfying, for $p=1, 2$,

$$P(Z_p(n+1)=k \mid Z_p(n)=j)=\text{coefficient of } s^k \text{ in } f(s)^j \tag{1.1}$$

where

$$f(s)=\sum_{k=0}^{\infty} c_k s^k , \tag{1.2}$$

$$c_k=P(Z_p(n+1)=k \mid Z_p(n)=1) . \tag{1.3}$$

Fix the population size at N and let

$$P_{jk}^{(N)}=P(Z_1(n+1)=k \mid Z_1(n)=j,\ Z_2(n)=N-j,\ Z_1(n+1)+Z_2(n+1)=N) \tag{1.4}$$

for $j, k=0, 1, \cdots, N$. The Markov chain on $\{0, 1, \cdots, N\}$ with one step transition probability $P_{jk}^{(N)}$ is the induced Markov chain of Karlin and McGregor. They showed that the totality of eigenvalues of the matrix $(P_{jk}^{(N)})$ is

$$1=\lambda_0^{(N)}=\lambda_1^{(N)}\geqq\lambda_2^{(N)}\geqq\cdots\geqq\lambda_N^{(N)}\geqq 0 , \tag{1.5}$$

and gave a simple formula to calculate them. The importance of the eigenvalue $\lambda_2^{(N)}$ is stressed, as it represents the rate of fixation, or the rate of approach to homozygosity, in genetics language. In three simple examples, namely, when the offspring distribution (1.3) is Poisson, binomial, or negative binomial, it is shown in [8] and [10] that

$$(1.6) \qquad 1-\lambda_2^{(N)} \sim \frac{\sigma^2}{N}, \qquad N\to\infty .$$

Here σ^2 is a finite positive constant determined by the offspring distribution, and in Ewens [6], p. 41, it is remarked that if the offspring distribution has mean 1, then σ^2 coincides with the variance.

The purpose of this paper is to investigate asymptotic behavior of the eigenvalues $\lambda_r^{(N)}$ as N becomes large, r being fixed in some cases and varying with N in other cases. We shall prove, under a fairly general condition, that for each ν

$$(1.7) \qquad 1-\lambda_r^{(N)} = \frac{a_{r,1}}{N}+\frac{a_{r,2}}{N^2}+\cdots+\frac{a_{r,\nu}}{N^\nu}+O\Big(\frac{1}{N^{\nu+1}}\Big), \quad N\to\infty,\ r \text{ fixed}.$$

In case r varies with N, one of results we shall give is that

$$(1.8) \qquad \lambda_{[\eta\sqrt{N}]}^{(N)} = e^{-\sigma^2\eta^2/2}\Big(1+O\Big(\frac{1}{\sqrt{N}}\Big)\Big), \qquad N\to\infty ,$$

where η is an arbitrary fixed positive number and $[\eta\sqrt{N}]$ is the greatest integer not exceeding $\eta\sqrt{N}$.

A fundamental result of Karlin and McGregor is that

$$(1.9) \qquad \lambda_r^{(N)} = \frac{\text{coefficient of } s^{N-r} \text{ in } f(s)^{N-r}f'(s)^r}{\text{coefficient of } s^N \text{ in } f(s)^N}, \qquad r=0, 1, \cdots, N .$$

If $c_0c_1c_2>0$, then the transition matrix is diagonalizable and (1.5) is strengthened to

$$(1.10) \qquad 1=\lambda_0^{(N)}=\lambda_1^{(N)}>\lambda_2^{(N)}>\cdots>\lambda_N^{(N)}>0 .$$

The expression of (1.9) suggests that some method in the study of limit theorems on large deviation for sums of independent random variables might be applied. Indeed, we shall see that the method initiated by Cramér [3] and developed by [1], [2], [5], [12], [13] can be applied with due modification. Thus we shall give estimation of the numerator and the denominator of the right-hand side of (1.9) separately. In most cases the main factors of the numerator and the denominator turn out to be identical, and hence it is more delicate factors that are related to the asymptotic behavior of $\lambda_r^{(N)}$.

If we start with a d type direct product branching process $\{Z(n)=(Z_1(n), \cdots, Z_d(n))\,;\ n=0, 1, 2, \cdots\}$, then the induced Markov chain of Karlin and McGregor is the chain with one step transition probability

$$(1.11) \qquad P_{jk}^{(N)}=P(Z(n+1)=k \mid Z(n)=j,\ Z(n+1)\in \boldsymbol{K}^{(N)}), \quad j, k\in \boldsymbol{K}^{(N)},$$

where the state space $\boldsymbol{K}^{(N)}$ is the set of points $j=(j_1, \cdots, j_d)$ such that $j_1, \cdots, j_d$

are nonnegative integers satisfying $j_1+\cdots+j_d=N$. In case all types have a common offspring distribution, they proved that the totality of eigenvalues of matrix $(P_{jk}^{(N)})$ is exactly the set of $\lambda_r^{(N)}$ $(r=0, 1, \cdots, N)$ described in (1.9), each $\lambda_r^{(N)}$ having multiplicity $\binom{r+d-2}{r}$. Note that

$$\sum_{r=0}^{N}\binom{r+d-2}{r}=\binom{N+d-1}{N},$$

which is the cardinality of $\boldsymbol{K}^{(N)}$. Therefore, our results apply also to this induced Markov chain of d types. Relations of our results with chains induced by d type models involving mutation pressure will be discussed at the end of the paper.

In Section 2 we state our assumptions and results. The proofs are given in Sections 3, 4, 5, while Section 6 contains some remarks concerning our results.

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§ 2. Assumptions and results.

Given an offspring distribution $\{c_k\}$, let a be the mean:

$$a=\sum_{k=0}^{\infty} kc_k\,, \tag{2.1}$$

which may be infinite. Let $M(x)$ be the moment generating function of $\{c_k\}$:

$$M(x)=\sum_{k=0}^{\infty} c_k e^{kx}\,, \qquad x \text{ real}\,, \tag{2.2}$$

and let b be the supremum of x such that $M(x)<\infty$. Obviously $b\geqq 0$. Let

$$F(x)=M(x)e^{-x}=\sum_{k=0}^{\infty} c_k e^{(k-1)x}\,, \tag{2.3}$$

$$K(x)=\log M(x)\,. \tag{2.4}$$

$M(x)$, $F(x)$, $K(x)$ are real analytic functions of $x<b$. Since

$$F''(x)=\sum_{k=0}^{\infty} c_k(k-1)^2 e^{(k-1)x}\geqq 0\,, \tag{2.5}$$

$F'(x)$ is non-decreasing for $x<b$. We make the following assumption throughout this paper.

ASSUMPTION 2.1. *c_0 is positive. The maximum span of the distribution $\{c_k\}$ is 1, that is, there is no pair of $\gamma>1$ and δ such that*

$$\sum_n c_{n\gamma+\delta}=1\,.$$

Moreover, one of the following conditions holds:

(i) $1<a\leqq+\infty$;

(ii) $a=1$ *and* $b>0$;

(iii) $a<1$ *and* $\lim_{x\to b-}F'(x)>0$.

REMARK 2.1. In Case (iii), b has to be positive. This is seen from the proof of Lemma 2.1.

REMARK 2.2. If $a<1$, $c_0+c_1<1$ and $b=+\infty$, then (iii) holds. In fact,

$$F'(x)=-c_0e^{-x}+\sum_{k=2}^{\infty}c_k(k-1)e^{(k-1)x}, \tag{2.6}$$

which increases to $+\infty$ as x goes to $+\infty$.

REMARK 2.3. If $a<1$, $0<b<+\infty$, and $\lim_{x\to b-}M(x)=+\infty$, then (iii) holds. This is because

$$F'(x)\geqq -c_0e^{-x}+(M(x)-c_0-c_1e^{x})e^{-x}\to\infty, \qquad x\to b-.$$

EXAMPLES. If the offspring distribution is Poisson, binomial, or negative binomial, then Assumption 2.1 is satisfied. In fact, $b=+\infty$ for Poisson and binomial, $0<b<+\infty$ for negative binomial, and in the case of negative binomial with mean <1, Remark 2.3 applies.

In order to state our results, we need the following lemma.

LEMMA 2.1. *There exists a unique* β *in* $(-\infty, b)$ *such that* $F'(\beta)=0$. *Moreover we have* $K'(\beta)=1$ *and* $K''(\beta)>0$.

PROOF. Since $c_1<1$, $F''(x)$ is positive by (2.5). Hence there is at most one β such that $F'(\beta)=0$. From (2.6) and $c_0>0$ it follows that $F'(x)\to-\infty$ as $x\to-\infty$. It also follows that $\lim_{x\to 0-}F'(x)=a-1$. Hence $F'(\beta)=0$ for some $\beta<0$ in Case (i) and for some $\beta>0$ in Case (iii), while $F'(0)=0$ in Case (ii). Since

$$F(x)=e^{K(x)-x}, \qquad F'(x)=F(x)(K'(x)-1),$$

$$F''(x)=F(x)\{(K'(x)-1)^2+K''(x)\},$$

we have $K'(\beta)=1$ and $K''(\beta)>0$.

Henceforth β denotes that of Lemma 2.1. Also we use the following quantities:

$$\sigma=K''(\beta)^{1/2}, \tag{2.7}$$

$$\kappa_j=\frac{1}{j!}\frac{d^jK(x)}{dx^j}\bigg|_{x=\beta}. \tag{2.8}$$

If we define a distribution $\{\hat{c}_k\}$ by giving mass $\hat{c}_k=c_ke^{k\beta-K(\beta)}$ to each point k, then its moment generating function is $M(x+\beta)/M(\beta)$. Hence $j!\,\kappa_j$ is the

j-th order semi-invariant of this distribution, which is sometimes called a conjugate (or associated) distribution of $\{c_k\}$.

REMARK 2.4. In Case (i), $K(\beta)<\beta<0$. In Case (ii), $\beta=K(\beta)=0$ and σ^2 is the variance of the offspring distribution. In Case (iii), $0<K(\beta)<\beta$. In fact, $K(\beta)<\beta$ (for $\beta\neq 0$) follows from the fact that $F(x)=e^{K(x)-x}$ attains its minimum at $x=\beta$.

THEOREM 2.1. *For each fixed $r>1$, there are constants $a_1, a_2, \cdots$ independent of N such that for each ν*

$$1-\lambda_r^{(N)}=\frac{a_1}{N}+\frac{a_2}{N^2}+\cdots+\frac{a_\nu}{N^\nu}+O\Big(\frac{1}{N^{\nu+1}}\Big), \qquad N\to\infty. \tag{2.9}$$

For each j, a_j is a polynomial of $1/\sigma^2$, σ^2, κ_3, κ_4, $\cdots$, κ_{2j} and r. In particular,

$$a_1=\frac{\sigma^2}{2}r(r-1), \tag{2.10}$$

$$a_2=-\frac{\sigma^4}{8}r(r-1)(r-2)(r-3)-2\kappa_3 r(r-1)(r-2)+\Big(\frac{9\kappa_3^2}{\sigma^4}-\frac{6\kappa_4}{\sigma^2}\Big)r(r-1). \tag{2.11}$$

THEOREM 2.2. *Let $r=r_N$ vary with N in such a way that there are constants $\gamma>0$ and $0<\eta_1<\eta_2<+\infty$ satisfying $\eta_1 N^\gamma\leqq r_N\leqq\eta_2 N^\gamma$. Then, the following formulas hold when $N\to\infty$.*

(i) *If $0<\gamma\leqq 1/3$, then*

$$\lambda_r^{(N)}=1-\frac{r^2}{N}\frac{\sigma^2}{2}+\frac{r}{N}\frac{\sigma^2}{2}+\frac{r^4}{N^2}\frac{\sigma^4}{8}+O\Big(\frac{1}{N^{2-3\gamma}}\Big). \tag{2.12}$$

(ii) *If $1/4\leqq\gamma<1/2$, then*

$$\lambda_r^{(N)}=e^{-\sigma^2 r^2/(2N)}\Big\{1+\frac{r}{N}\frac{\sigma^2}{2}+\frac{r^3}{N^2}\Big(2\kappa_3-\frac{\sigma^4}{2}\Big)+\frac{r^2}{N^2}\Big(\frac{11\sigma^4}{8}-6\kappa_3-\frac{9\kappa_3^2}{\sigma^4}+\frac{6\kappa_4}{\sigma^2}\Big)+R\Big\} \tag{2.13}$$

where

$$R=O\Big(\frac{1}{N^{2-\gamma}}\Big) \quad \textit{for} \quad \frac{1}{4}\leqq\gamma\leqq\frac{1}{3} \tag{2.14}$$

$$R=O\Big(\frac{1}{N^{3-4\gamma}}\Big) \quad \textit{for} \quad \frac{1}{3}\leqq\gamma<\frac{1}{2}. \tag{2.15}$$

(iii) *If $\gamma=1/2$, then*

$$\begin{aligned}\lambda_r^{(N)}=e^{-\sigma^2 r^2/(2N)}\Big[1&+\Big\{\frac{r}{N}\frac{\sigma^2}{2}+\frac{r^3}{N^2}\Big(2\kappa_3-\frac{\sigma^4}{2}\Big)\Big\}\\&+\Big\{\frac{r^2}{N^2}\Big(\frac{11\sigma^4}{8}-6\kappa_3-\frac{9\kappa_3^2}{\sigma^4}+\frac{6\kappa_4}{\sigma^2}\Big)+\frac{r^4}{N^3}\Big(-\frac{13\sigma^6}{12}+7\kappa_3\sigma^2-\frac{9\kappa_3^2}{2\sigma^2}-3\kappa_4\Big)\\&+\frac{r^6}{N^4}\Big(\frac{\sigma^8}{8}+2\kappa_3^2-\kappa_3\sigma^4\Big)\Big\}+O\Big(\frac{1}{N^{3/2}}\Big)\Big].\end{aligned} \tag{2.16}$$

In particular, if $r=r_N=[\eta N^{1/2}]$, then

(2.17) $$\lambda_r^{(N)}=e^{-\sigma^2\gamma^2/2}\Big(1+O\Big(\frac{1}{N^{1/2}}\Big)\Big).$$

REMARK 2.5. The case $1/4\leqq\gamma\leqq 1/3$ is included both in (i) and (ii). But expansion of $e^{-\sigma^2 r^2/(2N)}$ shows that (2.13) is finer estimate than (2.12) in this case.

REMARK 2.6. Theorem 2.2 (iii) can be generalized as follows: If $\gamma=1/2$, then, for each integer $\nu>0$,

$$\lambda_r^{(N)}=e^{-\sigma^2 r^2/(2N)}\Big(1+\sum_{j=1}^{\nu}\frac{r^{\varepsilon_j}}{N^{[(j+1)/2]}}\sum_{k=0}^{n_j}\Big(\frac{r^2}{N}\Big)^k b_{jk}+O\Big(\frac{1}{N^{(\nu+1)/2}}\Big)\Big),$$

where ε_j is 1 or 0 according as j is odd or even, and n_j is a positive integer determined by j. b_{jk} is a polynomial of $1/\sigma^2$, σ^2, κ_3, κ_4, $\cdots$.

REMARK 2.7. In order to define $P_{jk}^{(N)}$ and $\lambda_r^{(N)}$, it is necessary to have

$$P(Z_1(n+1)+Z_2(n+1)=N\mid Z_1(n)=j,\ Z_2(n)=N-j)>0,$$

that is, positivity of the coefficient of s^N in $f(s)^N$. This condition is guaranteed for large N by Assumption 2.1, which will be shown as a consequence of Lemma 4.1.

§ 3. A lemma.

We will prove the above theorems by using the formula (1.9). Let

(3.1) $$A_{N,n}=\text{coefficient of } s^n \text{ in } f(s)^N,$$

(3.2) $$B_{N,n}^{(r)}=\text{coefficient of } s^n \text{ in } f(s)^{N-r}f'(s)^r.$$

Then (1.9) is written as

(3.3) $$\lambda_r^{(N)}=\frac{B_{N,N-r}^{(r)}}{A_{N,N}}.$$

We extend the function $M(x)$ to complex z with $\operatorname{Re} z<b$ by defining

$$M(z)=\sum_{k=0}^{\infty}c_k e^{kz}.$$

We will also use the regular extension $f(z)$ of $f(s)$ to complex z with $|z|<e^b$, and the regular extension $K(z)$ of $K(x)$ in a complex neighborhood of β. We have

$$M(z)^N=\sum_{k=0}^{\infty}A_{N,k}e^{kz},$$

and hence, for each $x<b$,

(3.4) $$A_{N,n}=\frac{1}{2\pi i}\int_{x-i\pi}^{x+i\pi}M(z)^N e^{-nz}dz,$$

where the integral is along the segment from $x-i\pi$ to $x+i\pi$. In order to evaluate $A_{N,N}$, it is convenient to choose $x=\beta$ (saddle point method) and use

$$A_{N,N}=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^N e^{-Nz}dz\,. \tag{3.5}$$

We give the evaluation in a more general form, since evaluation of $B_{N,N-r}^{(r)}$ in (3.3) or $C_{N,N-r-1}^{(r)}$ in (4.7) of Section 4 also reduces to similar integrals.

LEMMA 3.1. *Let*

$$\tilde{A}_N=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N-r}L(z)\,e^{-Nz}dz\,,\qquad N\geqq r\,, \tag{3.6}$$

where r is a fixed integer and $L(z)$ is a bounded measurable function of $z=\beta+iy$, $-\pi<y<\pi$, which is regular in a complex neighborhood of $z=\beta$. Define ρ_j by

$$M(z)^{-r}L(z)=\sum_{j=0}^{\infty}\rho_j(z-\beta)^j\,. \tag{3.7}$$

Then, there are constants $\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots$ independent of N such that, for each ν,

$$\tilde{A}_N=\frac{e^{N(K(\beta)-\beta)}}{\sigma\sqrt{2\pi}\sqrt{N}}\Big\{\rho_0+\frac{\tilde{\alpha}_1}{N}+\frac{\tilde{\alpha}_2}{N^2}+\cdots+\frac{\tilde{\alpha}_\nu}{N^\nu}+O\Big(\frac{1}{N^{\nu+1}}\Big)\Big\},\quad N\to\infty\,. \tag{3.8}$$

For each j, $\tilde{\alpha}_j$ is a polynomial of $1/\sigma^2$, κ_3, $\kappa_4, \cdots, \kappa_{2j+2}$, $\rho_0, \rho_1, \cdots, \rho_{2j}$. In particular,

$$\tilde{\alpha}_1=\rho_0\Big(\frac{3\kappa_4}{\sigma^4}-\frac{15\kappa_3^2}{2\sigma^6}\Big)+\rho_1\frac{3\kappa_3}{\sigma^4}-\rho_2\frac{1}{\sigma^2}\,. \tag{3.9}$$

This lemma is proved by a method adopted by [1], [3], [5]. We give a full proof for completeness.

PROOF. Let $R(z)=M(z)^{-r}L(z)$ in a neighborhood of β. We have, for small $\varepsilon>0$,

$$\tilde{A}_N=\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}e^{N(K(\beta+iy)-(\beta+iy))}R(\beta+iy)dy+J\,, \tag{3.10}$$

$$J=\frac{1}{2\pi i}\Big(\int_{\beta-i\pi}^{\beta-i\varepsilon}+\int_{\beta+i\varepsilon}^{\beta+i\pi}\Big)M(z)^{N-r}L(z)e^{-Nz}dz\,. \tag{3.11}$$

Noting Lemma 2.1, we see

$$\tilde{A}_N=\frac{e^{N(K(\beta)-\beta)}}{2\pi}\int_{-\varepsilon}^{\varepsilon}\exp\big(N\sum_{j=2}^{\infty}\kappa_j(iy)^j\big)\cdot R(\beta+iy)dy+J\,. \tag{3.12}$$

Let

$$\varphi(N)=\frac{\log N}{\sqrt{N}} \tag{3.13}$$

and write the integral in (3.12) as I_1+I_2, where I_1 is the integral over $|y|<\varphi(N)$ and I_2 is the integral over $\varphi(N)<|y|<\varepsilon$. As we shall see, contribution of I_2

and J is negligible compared with that of I_1. Given $\nu>0$, let $p=2\nu+5$. We denote by B any function bounded uniformly in N. We have

$$I_1=\int_{|y|<\varphi(N)} e^{-N\sigma^2y^2/2}\exp\Big(N\sum_{j=3}^{p}\kappa_j(iy)^j\Big)\cdot\Big(1+B\frac{(\log N)^{p+1}}{N^{(p-1)/2}}\Big)\cdot$$
$$\Big(\sum_{j=0}^{p-2}\rho_j(iy)^j+B\frac{(\log N)^{p-1}}{N^{(p-1)/2}}\Big)dy\,,$$

and, by the change of variable $\sqrt{N}\sigma y=u$,

(3.14) $$I_1=\frac{1}{\sigma\sqrt{N}}\int_{|u|<\sigma\log N} e^{-u^2/2}g(u)h(u)du\,,$$

where

(3.15) $$g(u)=\exp\Big(\sum_{j=3}^{p}\frac{\kappa_j}{N^{(j-2)/2}}\Big(\frac{iu}{\sigma}\Big)^j\Big),\quad h(u)=\sum_{j=0}^{p-2}\frac{\rho_j}{N^{j/2}}\Big(\frac{iu}{\sigma}\Big)^j+B\frac{(\log N)^{p+1}}{N^{(p-1)/2}}.$$

Expanding g with respect to the powers of $N^{-1/2}$ and noting that $|u|<\sigma\log N$, we get

(3.16) $$g(u)=1+\frac{1}{N^{1/2}}P_1\Big(\frac{iu}{\sigma}\Big)+\frac{1}{N}P_2\Big(\frac{iu}{\sigma}\Big)+\cdots$$
$$+\frac{1}{N^{(p-2)/2}}P_{p-2}\Big(\frac{iu}{\sigma}\Big)+B\frac{(\log N)^{3(p-1)}}{N^{(p-1)/2}}\,,$$

where $P_j\big(\frac{iu}{\sigma}\big)$ is a polynomial of $\frac{iu}{\sigma}$ with degree $\leqq 3j$. Coefficients in $P_j\big(\frac{iu}{\sigma}\big)$ are polynomials of $\kappa_3, \kappa_4, \cdots, \kappa_{j+2}$. In particular,

$$P_1\Big(\frac{iu}{\sigma}\Big)=\kappa_3\Big(\frac{iu}{\sigma}\Big)^3,\qquad P_2\Big(\frac{iu}{\sigma}\Big)=\kappa_4\Big(\frac{iu}{\sigma}\Big)^4+\frac{\kappa_3^2}{2}\Big(\frac{iu}{\sigma}\Big)^6.$$

Therefore,

$$g(u)h(u)=\rho_0+\frac{1}{N^{1/2}}Q_1\Big(\frac{iu}{\sigma}\Big)+\frac{1}{N}Q_2\Big(\frac{iu}{\sigma}\Big)+\cdots$$
$$+\frac{1}{N^{(p-1)/2}}Q_{p-2}\Big(\frac{iu}{\sigma}\Big)+B\frac{(\log N)^{3(p-1)}}{N^{(p-1)/2}}$$

with

$$Q_j\Big(\frac{iu}{\sigma}\Big)=\sum_{l=0}^{j}\rho_l\Big(\frac{iu}{\sigma}\Big)^l P_{j-l}\Big(\frac{iu}{\sigma}\Big),\qquad P_0\Big(\frac{iu}{\sigma}\Big)=1\,.$$

$Q_j\big(\frac{iu}{\sigma}\big)$ is a polynomial of $\frac{iu}{\sigma}$ with degree $\leqq 3j$, in which coefficients are polynomials of $\kappa_3, \cdots, \kappa_{j+2}$ and $\rho_0, \cdots, \rho_j$. If j is odd (resp. even), then $Q_j\big(\frac{iu}{\sigma}\big)$ has terms only of odd (resp. even) powers of $\frac{iu}{\sigma}$, since $P_j\big(\frac{iu}{\sigma}\big)$, $j=0, 1, \cdots$, have the same property. It follows that $Q_j\big(\frac{iu}{\sigma}\big)$ with odd j contributes nothing to the integral, and hence

$$I_1=\frac{1}{\sigma\sqrt{N}}\int_{|u|<\sigma\log N}e^{-u^2/2}\Big\{\rho_0+\frac{1}{N}Q_2\Big(\frac{iu}{\sigma}\Big)+\frac{1}{N^2}Q_4\Big(\frac{iu}{\sigma}\Big)+\cdots$$
$$+\frac{1}{N^{\nu+1}}Q_{2\nu+2}\Big(\frac{iu}{\sigma}\Big)+B\frac{(\log N)^{6(\nu+2)}}{N^{\nu+2}}\Big\}du\,.$$

If we enlarge the range of integration to the whole line, then the resulting integral is a combination of

$$(3.17)\qquad \int_{-\infty}^{\infty}e^{-u^2/2}u^{2k}du=(2k-1)!!\sqrt{2\pi}\,,\qquad k=0,1,\cdots$$
$$((2k-1)!!=(2k-1)(2k-3)\cdots 3\cdot 1)\,,$$

while the error is estimated as

$$(3.18)\qquad \int_{|u|>\sigma\log N}e^{-u^2/2}u^{2k}du\leqq\int_{|u|>\sigma\log N}e^{-u^2/4}du$$
$$\leqq 4\{\sigma N^{(\sigma^2\log N)/4}\log N\}^{-1}\qquad \text{if}\quad N\ \text{large},\ k\ \text{fixed},$$

by $\int_x^{\infty}e^{-v^2/2}dv\leqq\frac{1}{x}e^{-x^2/2}$. Hence we have

$$(3.19)\qquad I_1=\frac{\sqrt{2\pi}}{\sigma\sqrt{N}}\Big\{\rho_0+\frac{\tilde{\alpha}_1}{N}+\frac{\tilde{\alpha}_2}{N^2}+\cdots+\frac{\tilde{\alpha}_{\nu+1}}{N^{\nu+1}}+O\Big(\frac{(\log N)^{6(\nu+2)}}{N^{\nu+2}}\Big)\Big\}\,,$$

$$(3.20)\qquad \tilde{\alpha}_j=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-u^2/2}Q_{2j}\Big(\frac{iu}{\sigma}\Big)du\,.$$

$\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots$ are independent of N and ν. $\tilde{\alpha}_j$ is a polynomial of $1/\sigma^2, \kappa_3, \cdots, \kappa_{2j+2}, \rho_0, \cdots, \rho_{2j}$. Especially, we get (3.9).

There remains estimation of I_2 and J. Since $R(z)$ is bounded in a neighborhood of β,

$$|I_2|\leqq C_1\int_{\varphi(N)<|y|<\varepsilon}\exp\{\mathrm{Re}\,(N\sum_{j=2}^{\infty}\kappa_j(iy)^j)\}dy$$

for some constant C_1. Hence, if we choose $\varepsilon>0$ small enough, then

$$|I_2|\leqq C_1\int_{|y|>\varphi(N)}e^{-N\sigma^2y^2/4}dy=\frac{C_1}{\sigma\sqrt{N}}\int_{|u|>\sigma\log N}e^{-u^2/4}du\,,$$

which is $o(1/N^n)$ for every n by (3.18). Since it is assumed that the distribution $\{c_k\}$ has maximum span 1,

$$|M(x+iy)|<M(x)\qquad \text{for}\quad x<b,\ 0<|y|\leqq\pi\,.$$

Hence, for each $\varepsilon>0$, there is an $\eta>0$ such that

$$(3.21)\qquad |M(\beta+iy)|\leqq M(\beta)(1-\eta)\qquad \text{for}\quad y\in[-\pi,-\varepsilon]\cup[\varepsilon,\pi]\,.$$

It follows from (3.11) that

(3.22) $$|J| \leqq C_2 M(\beta)^{N-r}(1-\eta)^{N-r} e^{-N\beta} = C_2 e^{N(K(\beta)-\beta)} M(\beta)^{-r}(1-\eta)^{N-r},$$

C_2 being the bound of $|L(\beta+iy)|$. This completes the proof of the lemma.

REMARK 3.1. Explicit expression of $\tilde{\alpha}_j$ is obtained by calculation of (3.20). We write $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$, as they might be useful in other applications.

$$\begin{aligned}\tilde{\alpha}_2 = & \frac{3!!}{\sigma^4}\rho_4 - \frac{5!!}{\sigma^6}(\rho_0\kappa_6+\rho_1\kappa_5+\rho_2\kappa_4+\rho_3\kappa_3) \\ & +\frac{7!!}{2!\,\sigma^8}\{\rho_0(\kappa_4^2+2\kappa_3\kappa_5)+2\rho_1\kappa_3\kappa_4+\rho_2\kappa_3^2\} \\ & -\frac{9!!}{3!\,\sigma^{10}}(3\rho_0\kappa_3^2\kappa_4+\rho_1\kappa_3^3)+\frac{11!!}{4!\,\sigma^{12}}\rho_0\kappa_3^4,\end{aligned}$$

$$\begin{aligned}\tilde{\alpha}_3 = & -\frac{5!!}{\sigma^6}\rho_6+\frac{7!!}{\sigma^8}(\rho_0\kappa_8+\rho_1\kappa_7+\rho_2\kappa_6+\rho_3\kappa_5+\rho_4\kappa_4+\rho_5\kappa_3) \\ & -\frac{9!!}{2!\,\sigma^{10}}\{\rho_0(\kappa_5^2+2\kappa_3\kappa_7+2\kappa_4\kappa_6)+\rho_1(2\kappa_3\kappa_6+2\kappa_4\kappa_5)+\rho_2(\kappa_4^2+2\kappa_3\kappa_5) \\ & +2\rho_3\kappa_3\kappa_4+\rho_4\kappa_3^2\}+\frac{11!!}{3!\,\sigma^{12}}\{\rho_0(\kappa_4^3+3\kappa_3^2\kappa_6+6\kappa_3\kappa_4\kappa_5)+\rho_1(3\kappa_3^2\kappa_5+3\kappa_3\kappa_4^2) \\ & +3\rho_2\kappa_3^2\kappa_4+\rho_3\kappa_3^3\}-\frac{13!!}{4!\,\sigma^{14}}\{\rho_0(4\kappa_3^3\kappa_5+6\kappa_3^2\kappa_4^2)+4\rho_1\kappa_3^3\kappa_4+\rho_2\kappa_3^4\} \\ & +\frac{15!!}{5!\,\sigma^{16}}(5\rho_0\kappa_3^4\kappa_4+\rho_1\kappa_3^5)-\frac{17!!}{6!\,\sigma^{18}}\rho_0\kappa_3^6.\end{aligned}$$

§ 4. Proof of Theorem 2.1.

LEMMA 4.1. *There exist $\alpha_1, \alpha_2, \cdots$ independent of N such that, for each ν,*

(4.1) $$A_{N,N} = \frac{e^{N(K(\beta)-\beta)}}{\sigma\sqrt{2\pi}\sqrt{N}}\left\{1+\frac{\alpha_1}{N}+\frac{\alpha_2}{N^2}+\cdots+\frac{\alpha_\nu}{N^\nu}+O\left(\frac{1}{N^{\nu+1}}\right)\right\}, \quad N\to\infty.$$

For each j, α_j is a polynomial of $1/\sigma^2$, κ_3, κ_4, $\cdots$, κ_{2j+2}. In particular,

(4.2) $$\alpha_1 = \frac{3\kappa_4}{\sigma^4} - \frac{15\kappa_3^2}{2\sigma^6}.$$

PROOF. Apply Lemma 3.1 to the expression (3.5).

LEMMA 4.2. *Let*

(4.3) $$C_{N,n}^{(r)} = \text{coefficient of } s^n \text{ in } f(s)^{N-r}f'(s)^{r-1}f''(s).$$

Then, for any fixed r, there exist $\gamma_1, \gamma_2, \cdots$, independent of N such that, for each ν,

(4.4) $$C_{N,N-r-1}^{(r)} = \frac{e^{N(K(\beta)-\beta)}}{\sigma\sqrt{2\pi}\sqrt{N}}\left\{\sigma^2+\frac{\gamma_1}{N}+\frac{\gamma_2}{N^2}+\cdots+\frac{\gamma_\nu}{N^\nu}+O\left(\frac{1}{N^{\nu+1}}\right)\right\}, \quad N\to\infty.$$

For each j, γ_j is a polynomial of r, $1/\sigma^2$, σ^2, κ_3, κ_4, $\cdots$, κ_{2j+2}. In particular,

$$(4.5)\quad \gamma_1=\sigma^2\Big(\frac{3\kappa_4}{\sigma^4}-\frac{15\kappa_3^2}{2\sigma^6}\Big)-\frac{\sigma^4}{2}(r-1)(r-2)-(\sigma^2+6\kappa_3)(r-1)+\frac{18\kappa_3^2}{\sigma^4}-\sigma^2-\frac{12\kappa_4}{\sigma^2}.$$

PROOF. Since

$$f(e^z)^{N-r}f'(e^z)^{r-1}f''(e^z)=e^{-(r-1)z}M(z)^{N-r}M'(z)^{r-1}(M''(z)-M'(z)),$$

the expression of $C^{(r)}_{N,N-r-1}$ similar to (3.5) is as follows:

$$(4.6)\qquad C^{(r)}_{N,N-r-1}=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N-r}L(z)e^{-Nz}dz,\ L(z)=M'(z)^{r-1}(M''(z)-M'(z)).$$

Hence we can apply Lemma 3.1. Let us find ρ_j of (3.7). We have, from $M(z)=e^{K(z)}$,

$$M(z)^{-r}L(z)=K'(z)^{r-1}(K'(z)^2+K''(z)-K'(z))$$

in a neighborhood of $z=\beta$. Expanding this function around β, we get

$$\rho_0=\sigma^2,\qquad \rho_1=\sigma^4(r-1)+\sigma^2+6\kappa_3,$$

$$\rho_2=\frac{\sigma^4}{2}(r-1)(r-2)+(\sigma^4+9\sigma^2\kappa_3)(r-1)+\sigma^4+3\kappa_3+12\kappa_4.$$

Hence (4.5) follows from (3.9). Since ρ_j is a polynomial of σ^2, κ_3, $\cdots$, κ_{2j+2} and r, the assertion on γ_j follows.

PROOF OF THEOREM 2.1. As is used in [**10**] p. 122 or [8] p. 403, (3.3) can be rewritten as

$$(4.7)\qquad \lambda_r^{(N)}-\lambda_{r+1}^{(N)}=\frac{r}{N-r}\cdot\frac{C^{(r)}_{N,N-r-1}}{A_{N,N}},\qquad r=0,1,\cdots,N-1.$$

Since γ_1, γ_2, $\cdots$ of Lemma 4.2 depends on r, we denote them by $\gamma_{r,1}$, $\gamma_{r,2}$, $\cdots$. It follows from (4.7) and Lemmas 4.1 and 4.2 that

$$\lambda_r^{(N)}-\lambda_{r+1}^{(N)}=\frac{r}{N-r}\cdot\frac{\sigma^2+\gamma_{r,1}N^{-1}+\gamma_{r,2}N^{-2}+\cdots+\gamma_{r,\nu}N^{-\nu}}{1+\alpha_1N^{-1}+\alpha_2N^{-2}+\cdots+\alpha_\nu N^{-\nu}}\Big\{1+O\Big(\frac{1}{N^{\nu+1}}\Big)\Big\}$$

$$=\frac{r}{N-r}\Big\{\sigma^2+\frac{b_{r,1}}{N}+\frac{b_{r,2}}{N^2}+\cdots+\frac{b_{r,\nu}}{N^\nu}+O\Big(\frac{1}{N^{\nu+1}}\Big)\Big\}.$$

Noting that

$$\frac{r}{N-r}=\frac{r}{N}\Big\{1+\frac{r}{N}+\frac{r^2}{N^2}+\cdots+\frac{r^\nu}{N^\nu}+O\Big(\frac{1}{N^{\nu+1}}\Big)\Big\},$$

we obtain

$$\lambda_r^{(N)}-\lambda_{r+1}^{(N)}=\frac{r}{N}\Big\{\sigma^2+\frac{c_{r,1}}{N}+\frac{c_{r,2}}{N^2}+\cdots+\frac{c_{r,\nu}}{N^\nu}+O\Big(\frac{1}{N^{\nu+1}}\Big)\Big\}.$$

Since $b_{r,1}=\gamma_{r,1}-\sigma^2\alpha_1$, we have $c_{r,1}=\gamma_{r,1}-\sigma^2\alpha_1+\sigma^2 r$. In general, $c_{r,j}$ is a polynomial of α_1, $\cdots$, α_j, σ^2, $\gamma_{r,1}$, $\cdots$, $\gamma_{r,j}$ and r. Hence

$$1-\lambda_r^{(N)}=\sum_{j=1}^{r-1}(\lambda_j^{(N)}-\lambda_{j+1}^{(N)})=\frac{a_1}{N}+\frac{a_2}{N^2}+\cdots+\frac{a_\nu}{N^\nu}+O\Big(\frac{1}{N^{\nu+1}}\Big)$$

and it is easy to see that a_j is a polynomial of r, $1/\sigma^2$, σ^2, κ_3, κ_4, $\cdots$, κ_{2j}. In particular,

$$a_1=\sum_{j=1}^{r-1}\sigma^2 j=\frac{\sigma^2}{2}r(r-1)\,,$$

$$a_2=\sum_{j=1}^{r-1}\{(\gamma_{j,1}-\sigma^2\alpha_1)j+\sigma^2 j^2\}$$

$$=\sum_{j=1}^{r-1}\Big\{-\frac{\sigma^4}{2}j(j-1)(j-2)-(\sigma^2+6\kappa_3)j(j-1)+\Big(\frac{18\kappa_3^2}{\sigma^4}-\sigma^2-\frac{12\kappa_4}{\sigma^2}\Big)j+\sigma^2 j^2\Big\}\,,$$

and an algebra shows (2.11). Hence the theorem is proved.

REMARK 4.1. Since $B_{N,N-r}^{(r)}$ can be evaluated in a similar way by the use of Lemma 3.1, we can as well prove Theorem 2.1 directly from (3.3), not via (4.7). But then, in order to get (2.11), we have to use explicit forms of coefficients corresponding to $\tilde{\alpha}_2$ in (3.8), and algebra becomes more complicated.

§5. Proof of Theorem 2.2.

In studying asymptotic behavior of $\lambda_r^{(N)}$ when r varies with N, it seems impossible to use (4.7). So we estimate $B_{N,N-r}^{(r)}$.

LEMMA 5.1. *Let $r=r_N$ be as in Theorem 2.2 with $0<\gamma\leqq 1/3$. Then,*

$$B_{N,N-r}^{(r)}=\frac{e^{N(K(\beta)-\beta)}}{\sigma\sqrt{2\pi}\sqrt{N}}\Big\{1-\frac{r^2}{N}\frac{\sigma^2}{2}+\frac{r}{N}\frac{\sigma^2}{2}+\frac{1}{N}\Big(\frac{3\kappa_4}{\sigma^4}-\frac{15\kappa_3^2}{2\sigma^6}\Big)+\frac{r^4}{N^2}\frac{\sigma^4}{8}+O\Big(\frac{1}{N^{2-3\gamma}}\Big)\Big\}\,. \tag{5.1}$$

PROOF. Since

$$f(e^z)^{N-r}f'(e^z)^r=M(z)^{N-r}(M'(z)e^{-z})^r\,,$$

we have, similarly to (3.5),

$$B_{N,N-r}^{(r)}=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N-r}M'(z)^r e^{-Nz}dz\,. \tag{5.2}$$

Since $K'(\beta)=1$, we can define $\Theta(z)=\log K'(z)$ in a neighborhood of β. For small $\varepsilon>0$ we have

$$B_{N,N-r}^{(r)}=\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}e^{N(K(\beta+iy)-(\beta+iy))+r\Theta(\beta+iy)}dy+J\,, \tag{5.3}$$

$$J=\frac{1}{2\pi i}\Big(\int_{\beta-i\pi}^{\beta-i\varepsilon}+\int_{\beta+i\varepsilon}^{\beta+i\pi}\Big)M(z)^{N-r}M'(z)^r e^{-Nz}dz\,. \tag{5.4}$$

Define θ_j by

$$\Theta(z)=\sum_{j=0}^{\infty}\theta_j(z-\beta)^j .$$

Then,

(5.5) $$\theta_0=0,\quad \theta_1=\sigma^2,\quad \theta_2=3\kappa_3-\frac{\sigma^4}{2},\quad \theta_3=4\kappa_4-3\kappa_3\sigma^2+\frac{\sigma^6}{3}$$

and, in general, θ_j is a polynomial of σ^2, κ_3, κ_4, $\cdots$. Using $\varphi(N)$ of (3.13), we have

(5.6) $$B^{(r)}_{N,N-r}=\frac{e^{N(K(\beta)-\beta)}}{2\pi}(I_1+I_2)+J,$$

where

(5.7) $$I_1=\int_{|y|<\varphi(N)}\exp\{N\sum_{j=2}^{\infty}\kappa_j(iy)^j+r\sum_{j=1}^{\infty}\theta_j(iy)^j\}dy$$

and I_2 is the integral over $\varphi(N)<|y|<\varepsilon$ with the same integrand. Let p, q be sufficiently large integers. Then

(5.8) $$\begin{aligned} I_1&=\int_{|y|<\varphi(N)}e^{-N\sigma^2y^2/2}\exp\Big\{N\sum_{j=3}^{p}\kappa_j(iy)^j+r\sum_{j=1}^{p}\theta_j(iy)^j+B\frac{(\log N)^{p+1}}{N^{(p-1)/2}}\Big\}dy\\ &=\frac{1}{\sigma\sqrt{N}}\int_{|u|<\sigma\log N}e^{-u^2/2}g(u)h(u)\Big(1+B\frac{(\log N)^{p+1}}{N^{(p-1)/2}}\Big)du ,\end{aligned}$$

where $g(u)$ is the same as in (3.15) and

$$h(u)=\exp\Big\{r\sum_{j=1}^{p}\frac{\theta_j}{N^{j/2}}\Big(\frac{iu}{\sigma}\Big)^j\Big\}=1+\sum_{k=1}^{q}\frac{1}{k!}\Big(r\sum_{j=1}^{p}\frac{\theta_j}{N^{j/2}}\Big(\frac{iu}{\sigma}\Big)^j\Big)^k+B\Big(\frac{\log N}{N^{(1-2\gamma)/2}}\Big)^{q+1}.$$

Let us rearrange the terms in $h(u)$ in the decreasing order of powers of N, noting that r is of order N^{γ}. In the below 'odd' (resp. 'even') denotes odd (resp. even) power terms of iu/σ. If $0<\gamma\leqq 1/6$, then

(5.9) $$\begin{aligned} h(u)=1&+\frac{r}{N^{1/2}}\theta_1\frac{iu}{\sigma}+\frac{r^2}{N}\frac{\theta_1^2}{2}\Big(\frac{iu}{\sigma}\Big)^2+\frac{r}{N}\theta_2\Big(\frac{iu}{\sigma}\Big)^2+\frac{r^3}{N^{3/2}}\frac{\theta_1^3}{6}\Big(\frac{iu}{\sigma}\Big)^3\\ &+\frac{r^2}{N^{3/2}}\theta_1\theta_2\Big(\frac{iu}{\sigma}\Big)^3+\frac{r}{N^{3/2}}\theta_3\Big(\frac{iu}{\sigma}\Big)^3+\frac{r^4}{N^2}\frac{\theta_1^4}{24}\Big(\frac{iu}{\sigma}\Big)^4\\ &+\frac{r^3}{N^2}\theta_1^2\theta_2\Big(\frac{iu}{\sigma}\Big)^4+\frac{r^2}{N^2}\ \text{even}+\cdots .\end{aligned}$$

If $1/6\leqq\gamma\leqq 1/4$, then

(5.10) $$\begin{aligned} h(u)=1+\cdots&+\frac{r^2}{N^{3/2}}\theta_1\theta_2\Big(\frac{iu}{\sigma}\Big)^3+\frac{r^4}{N^2}\frac{\theta_1^4}{24}\Big(\frac{iu}{\sigma}\Big)^4+\frac{r}{N^{3/2}}\theta_3\Big(\frac{iu}{\sigma}\Big)^3\\ &+\frac{r^3}{N^2}\theta_1^2\theta_2\Big(\frac{iu}{\sigma}\Big)^4+\frac{r^5}{N^{5/2}}\ \text{odd}+\frac{r^2}{N^2}\ \text{even}+\cdots ,\end{aligned}$$

where the first six terms are common with (5.9). If $1/4\leqq\gamma\leqq 1/3$, then

$$(5.11)\qquad h(u)=1+\frac{r}{N^{1/2}}\theta_1\frac{iu}{\sigma}+\frac{r^2}{N}\frac{\theta_1^2}{2}\Big(\frac{iu}{\sigma}\Big)^2+\frac{r^3}{N^{3/2}}\frac{\theta_1^3}{6}\Big(\frac{iu}{\sigma}\Big)^3+\frac{r}{N}\theta_2\Big(\frac{iu}{\sigma}\Big)^2$$
$$+\frac{r^4}{N^2}\frac{\theta_1^4}{24}\Big(\frac{iu}{\sigma}\Big)^4+\frac{r^2}{N^{3/2}}\theta_1\theta_2\Big(\frac{iu}{\sigma}\Big)^3+\frac{r^5}{N^{5/2}}\text{odd}+\frac{r^3}{N^2}\text{even}+\cdots.$$

From (3.16) and (5.9)-(5.11), expansion of $g(u)h(u)$ in the decreasing order of powers of N is obtained as follows. If $0<\gamma\leqq 1/8$, then

$$(5.12)\quad g(u)h(u)=1+\frac{r}{N^{1/2}}\theta_1\frac{iu}{\sigma}+\frac{1}{N^{1/2}}\kappa_3\Big(\frac{iu}{\sigma}\Big)^3+\frac{r^2}{N}\frac{\theta_1^2}{2}\Big(\frac{iu}{\sigma}\Big)^2$$
$$+\frac{r}{N}\Big\{\theta_2\Big(\frac{iu}{\sigma}\Big)^2+\kappa_3\theta_1\Big(\frac{iu}{\sigma}\Big)^4\Big\}+\frac{1}{N}P_2\Big(\frac{iu}{\sigma}\Big)+\frac{r^3}{N^{3/2}}\frac{\theta_1^3}{6}\Big(\frac{iu}{\sigma}\Big)^3$$
$$+\frac{r^2}{N^{3/2}}\text{odd}+\frac{r}{N^{3/2}}\text{odd}+\frac{1}{N^{3/2}}\text{odd}+\frac{r^4}{N}\frac{\theta_1^4}{24}\Big(\frac{iu}{\sigma}\Big)^4+\frac{r^3}{N^2}\text{even}+\cdots.$$

For different γ, the same terms appear but the order varies. Let us denote the terms in the right-hand side of (5.12) by $U_1, U_2, \cdots$; that is, $U_1=1$, $U_2=\frac{r}{N^{1/2}}\theta_1\frac{iu}{\sigma}, \cdots, U_5=\frac{r}{N}\Big\{\theta_2\Big(\frac{iu}{\sigma}\Big)^2+\kappa_3\theta_1\Big(\frac{iu}{\sigma}\Big)^4\Big\}, \cdots$. If $1/8\leqq\gamma\leqq 1/6$, then we have

$$(5.13)\quad g(u)h(u)=U_1+U_2+U_3+U_4+U_5+U_6+U_7+U_8+U_9+U_{11}+U_{10}+U_{12}+\cdots.$$

If $1/6\leqq\gamma\leqq 1/4$, then

$$(5.14)\qquad g(u)h(u)=U_1+U_2+U_3+U_4+U_5+U_7+U_6+U_8+U_{11}+U_9+U_{12}+\cdots.$$

If $1/4\leqq\gamma\leqq 1/10$, then

$$(5.15)\quad g(u)h(u)=U_1+U_2+U_4+U_3+U_7+U_5+U_{11}+U_8+U_6+\frac{r^5}{N^{5/2}}\text{odd}+U_{12}+\cdots.$$

If $3/10\leqq\gamma\leqq 1/3$, then

$$(5.16)\quad g(u)h(u)=U_1+U_2+U_4+U_3+U_7+U_5+U_{11}+U_8+\frac{r^5}{N^{5/2}}\text{odd}+U_6+U_{12}+\cdots.$$

It follows from (5.8) and (5.12)-(5.16) that, for any γ $(0<\gamma\leqq 1/3)$,

$$(5.17)\qquad I_1=\frac{\sqrt{2\pi}}{\sigma\sqrt{N}}\Big\{1-\frac{r^2}{N}\frac{\theta_1^2}{2\sigma^2}+\frac{r}{N}\Big(-\frac{\theta_2}{\sigma^2}+\frac{3\kappa_3\theta_1}{\sigma^4}\Big)$$
$$+\frac{1}{N}\Big(\frac{3\kappa_4}{\sigma^4}-\frac{15\kappa_3^2}{2\sigma^6}\Big)+\frac{r^4}{N^2}\frac{\theta_1^4}{8\sigma^4}+O\Big(\frac{1}{N^{2-3\gamma}}\Big)\Big\}.$$

Here we have used (3.17) and (3.18).

Let us estimate I_2. Choose $\varepsilon>0$ small enough. Then, for $|y|<\varepsilon$ and large N,

$$\text{Re}\,\{N\sum_{j=2}^{\infty}\kappa_j(iy)^j+r\sum_{j=1}^{\infty}\theta_j(iy)^j\}=-\frac{N\sigma^2y^2}{2}+NBy^4+rBy^2\leqq-\frac{N\sigma^2y^2}{4}.$$

Hence $I_2=o(1/N^n)$ for any n by (3.18). By the same reason as we get (3.22), we have

$$|J| \leqq \text{const}\, e^{N(K(\beta)-\beta)} M(\beta)^{-r}(1-\eta)^{N-r}.$$

Hence, for $0<\eta'<\eta$, we have $|J| \leqq e^{N(K(\beta)-\beta)}(1-\eta')^N$ for large N. Contribution of I_2 and J is thus negligible, and (5.1) follows from (5.5), (5.6) and (5.17).

PROOF OF THEOREM 2.2 (i). In the expression (3.3) of $\lambda_r^{(N)}$, use Lemmas 4.1 and 5.1. Then (2.12) follows immediately.

Although r depends on N, (5.1) of Lemma 5.1 was proved essentially in the same way as the proof of Lemma 3.1. This is because $r=r_N$ was very small compared with N. In order to prove (ii), (iii) of Theorem 2.2, we have to use other estimation. Let us prove (iii) first.

LEMMA 5.2. *Let $r=r_N$ be as in Theorem 2.2 with $\gamma=1/2$. Then,*

$$(5.18)\quad B_{N,N-r}^{(r)} = \frac{e^{N(K(\beta)-\beta)}}{\sigma\sqrt{2\pi}\sqrt{N}} e^{-\sigma^2 r^2/(2N)} \Big[1+\Big\{\frac{r}{N}\frac{\sigma^2}{2}+\frac{r^3}{N^2}\Big(2\kappa_3-\frac{\sigma^4}{2}\Big)\Big\}$$
$$+\Big\{\frac{1}{N}\Big(\frac{3\kappa_4}{\sigma^4}-\frac{15\kappa_3^2}{2\sigma^6}\Big)+\frac{r^2}{N^2}\Big(\frac{11\sigma^4}{8}-6\kappa_3-\frac{9\kappa_3^2}{\sigma^4}+\frac{6\kappa_4}{\sigma^2}\Big)$$
$$+\frac{r^4}{N^3}\Big(-\frac{13\sigma^6}{12}+7\kappa_3\sigma^2-\frac{9\kappa_3^2}{2\sigma^2}-3\kappa_4\Big)+\frac{r^6}{N^4}\Big(\frac{\sigma^8}{8}+2\kappa_3^2-\kappa_3\sigma^4\Big)\Big\}$$
$$+O\Big(\frac{1}{N^{3/2}}\Big)\Big].$$

PROOF. $B_{N,N-r}^{(r)}$ is expressed as (5.6) with I_1 of (5.7) and J of (5.4). Let p, q be sufficiently large integers. We have

$$(5.19)\quad I_1=\int_{|y|<\varphi(n)} e^{-(N\sigma^2y^2/2)+r\sigma^2 iy} \exp\Big\{N\sum_{j=3}^{p}\kappa_j(iy)^j+r\sum_{j=2}^{p}\theta_j(iy)^j+B\frac{(\log N)^{p+1}}{N^{(p-1)/2}}\Big\}dy$$
$$=\frac{1}{\sigma\sqrt{N}}\int_{|u|<\sigma\log N} e^{-(u^2/2)+(r\sigma iu/\sqrt{N})} g(u)l(u)\Big(1+B\frac{(\log N)^{p+1}}{N^{(p-1)/2}}\Big)du,$$

where $g(u)$ is the same as in (3.15) and

$$(5.20)\quad l(u)=\exp\Big\{r\sum_{j=2}^{p}\frac{\theta_j}{N^{j/2}}\Big(\frac{iu}{\sigma}\Big)^j\Big\}=1+\sum_{k=1}^{q}\frac{1}{k!}\Big(r\sum_{j=2}^{p}\frac{\theta_j}{N^{j/2}}\Big(\frac{iu}{\sigma}\Big)^j\Big)^k+B\frac{(\log N)^{2q+2}}{N^{(q+1)/2}}.$$

Rearrangement of $l(u)$ is

$$(5.21)\quad l(u)=1+\frac{r}{N}\theta_2\Big(\frac{iu}{\sigma}\Big)^2+\Big\{\frac{r}{N^{3/2}}\theta_3\Big(\frac{iu}{\sigma}\Big)^3+\frac{r^2}{N^2}\frac{\theta_2^2}{2!}\Big(\frac{iu}{\sigma}\Big)^4\Big\}$$
$$+\Big\{\frac{r}{N^2}\theta_4\Big(\frac{iu}{\sigma}\Big)^4+\frac{r^2}{N^{5/2}}\frac{2\theta_2\theta_3}{2!}\Big(\frac{iu}{\sigma}\Big)^5+\frac{r^3}{N^3}\frac{\theta_2^3}{3!}\Big(\frac{iu}{\sigma}\Big)^6\Big\}+\cdots.$$

Terms within each braces have a common order with respect to N. It follows from (3.16) and (5.21) that

$$(5.22)\quad g(u)l(u)=1+\left\{\frac{1}{N^{1/2}}\kappa_3\left(\frac{iu}{\sigma}\right)^3+\frac{r}{N}\theta_2\left(\frac{iu}{\sigma}\right)^2\right\}+\left\{\frac{1}{N}P_2\left(\frac{iu}{\sigma}\right)\right.$$
$$\left.+\frac{r}{N^{3/2}}\kappa_3\theta_2\left(\frac{iu}{\sigma}\right)^5+\frac{r}{N^{3/2}}\theta_3\left(\frac{iu}{\sigma}\right)^3+\frac{r^2}{N^2}\frac{\theta_2^2}{2}\left(\frac{iu}{\sigma}\right)^4\right\}+B\frac{1}{N^{3/2}}.$$

Let us denote by E the right-hand side of (5.22) with the last term deleted. Use

$$\int_{-\infty}^{\infty}e^{-(u^2/2)+ixu}(iu)^n du=\sqrt{2\pi}\,(-1)^n H_n(x)e^{-x^2/2},$$

where $H_n(x)$ is the Hermite polynomial

$$H_n(x)=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}\,(e^{-x^2/2}).$$

Then we get

$$(5.23)\quad \frac{1}{\sigma\sqrt{N}}\int_{-\infty}^{\infty}e^{-(u^2/2)+(r\sigma iu/\sqrt{N})}E\,du$$
$$=\frac{\sqrt{2\pi}}{\sigma\sqrt{N}}e^{-r^2\sigma^2/(2N)}\left[1+\frac{1}{N^{1/2}}\frac{\kappa_3}{\sigma^3}\left(3\frac{r\sigma}{\sqrt{N}}-\left(\frac{r\sigma}{\sqrt{N}}\right)^3\right)\right.$$
$$+\frac{r}{N}\frac{\theta_2}{\sigma^2}\left(-1+\left(\frac{r\sigma}{\sqrt{N}}\right)^2\right)+\frac{1}{N}\frac{\kappa_4}{\sigma^4}\left(3-6\left(\frac{r\sigma}{\sqrt{N}}\right)^2+\left(\frac{r\sigma}{\sqrt{N}}\right)^4\right)$$
$$+\frac{1}{N}\frac{\kappa_3^2}{2\sigma^6}\left(-15+45\left(\frac{r\sigma}{\sqrt{N}}\right)^2-15\left(\frac{r\sigma}{\sqrt{N}}\right)^4+\left(\frac{r\sigma}{\sqrt{N}}\right)^6\right)$$
$$+\frac{r}{N^{3/2}}\frac{\kappa_3\theta_2}{\sigma^5}\left(-15\frac{r\sigma}{\sqrt{N}}+10\left(\frac{r\sigma}{\sqrt{N}}\right)^3-\left(\frac{r\sigma}{\sqrt{N}}\right)^5\right)$$
$$\left.+\frac{r}{N^{3/2}}\frac{\theta_3}{\sigma^3}\left(3\frac{r\sigma}{\sqrt{N}}-\left(\frac{r\sigma}{\sqrt{N}}\right)^3\right)+\frac{r^2}{N^2}\frac{\theta_2^2}{2\sigma^4}\left(3-6\left(\frac{r\sigma}{\sqrt{N}}\right)^2+\left(\frac{r\sigma}{\sqrt{N}}\right)^4\right)\right].$$

By (5.5) the expression in the brackets in the above formula is shown to coincide with that of (5.18) with the last term $O(1/N^{3/2})$ omitted. By (5.19), the difference between I_1 and (5.23) is

$$-\frac{1}{\sigma\sqrt{N}}\int_{|u|>\sigma\log N}e^{-(u^2/2)+(r\sigma iu/\sqrt{N})}E\,du+\frac{1}{\sigma\sqrt{N}}\int_{|u|<\sigma\log N}e^{-(u^2/2)+(r\sigma iu/\sqrt{N})}B\frac{1}{N^{3/2}}du,$$

and this is $O(1/N^2)$ by virtue of (3.18). The evaluation of I_2 and J is exactly the same as in the proof of Lemma 5.1. Hence Lemma 5.2 is proved.

PROOF OF THEOREM 2.2 (iii). Use Lemmas 4.1 and 5.2 in (3.3).

LEMMA 5.3. *Let $r=r_N$ be as in Theorem 2.2 with $1/4\leqq\gamma<1/2$. Then*

$$(5.24)\quad B_{N,N-r}^{(r)}=\frac{e^{N(K(\beta)-\beta)}}{\sigma\sqrt{2\pi}\sqrt{N}}e^{-r^2\sigma^2/(2N)}\left[1+\frac{r}{N}\frac{\sigma^2}{2}+\frac{r^3}{N^2}\left(2\kappa_3-\frac{\sigma^4}{2}\right)\right.$$
$$\left.+\frac{1}{N}\left(\frac{3\kappa_4}{\sigma^4}-\frac{15\kappa_3^2}{2\sigma^6}\right)+\frac{r^2}{N^2}\left(\frac{11\sigma^4}{8}-6\kappa_3-\frac{9\kappa_3^2}{\sigma^4}+\frac{6\kappa_4}{\sigma^2}\right)+R\right]$$

with R in (2.14) *and* (2.15).

PROOF. We proceed just as in the proof of Lemma 5.2. We get (5.19) with $g(u)$ as in (3.15) and

$$l(u)=1+\frac{r}{N}\theta_2\Big(\frac{iu}{\sigma}\Big)^2+\frac{r}{N^{3/2}}\theta_3\Big(\frac{iu}{\sigma}\Big)^3+\frac{r^2}{N^2}\frac{\theta_2^2}{2}\Big(\frac{iu}{\sigma}\Big)^4+\frac{r}{N^2}\theta_4\Big(\frac{iu}{\sigma}\Big)^4$$

$$+\frac{r^2}{N^{5/2}}\theta_2\theta_3\Big(\frac{iu}{\sigma}\Big)^5+B\frac{(\log N)^6}{N^{3-3\gamma}}.$$

Note that r is of order N^γ. Expansion of $g(u)l(u)$ in the decreasing order of powers of N is

$$g(u)l(u)=1+\frac{1}{N^{1/2}}\kappa_3\Big(\frac{iu}{\sigma}\Big)^3+\frac{r}{N}\theta_2\Big(\frac{iu}{\sigma}\Big)^2+\frac{1}{N}P_2\Big(\frac{iu}{\sigma}\Big) \tag{5.25}$$

$$+\frac{r}{N^{3/2}}\Big\{\kappa_3\theta_2\Big(\frac{iu}{\sigma}\Big)^5+\theta_3\Big(\frac{iu}{\sigma}\Big)^3\Big\}+\frac{r^2}{N^2}\frac{\theta_2^2}{2}\Big(\frac{iu}{\sigma}\Big)^4$$

$$+\frac{1}{N^{3/2}}P_3\Big(\frac{iu}{\sigma}\Big)+\frac{r}{N^2}\text{ even}+B\frac{(\log N)^s}{N^{(5-4\gamma)/2}}$$

where s is some integer. Let us denote by E the right-hand side of (5.25) with the last term deleted. Then,

$$\frac{1}{\sigma\sqrt{N}}\int_{-\infty}^{\infty}e^{-(u^2/2)+(r\sigma iu/\sqrt{N})}E\,du=F+\frac{\sqrt{2\pi}}{\sigma\sqrt{N}}e^{-r^2\sigma^2/(2N)}\cdot O\Big(\frac{1}{N^{2-\gamma}}\Big), \tag{5.26}$$

where F denotes the right-hand side of (5.23). The right-hand side of (5.26) equals

$$\frac{\sqrt{2\pi}}{\sigma\sqrt{N}}e^{-r^2\sigma^2/(2N)}[\cdots],$$

the expression in the brackets being the one in the brackets in (5.24). The rest of the proof is similar to the previous lemma.

PROOF OF THEOREM 2.2 (ii). Use Lemma 5.3 instead of Lemma 5.2. Then, proofs of (ii) and (iii) are quite similar.

§ 6. Concluding remarks.

Let $\{X^{(N)}(n);\ n=0, 1, \cdots\}$ be the induced Markov chain on $\{0, 1, \cdots, N\}$ with one step transition probability (1.4), and let $\{Y^{(N)}(t);\ 0\leqq t<\infty\}$ be a process on the interval $[0, 1]$ with continuous time parameter defined by

$$Y^{(N)}(t)=\frac{1}{N}X^{(N)}(n) \qquad \text{for } t=\frac{n}{N}, \tag{6.1}$$

$$Y^{(N)}(t)=(n+1-Nt)Y^{(N)}\Big(\frac{n}{N}\Big)+(Nt-n)Y^{(N)}\Big(\frac{n+1}{N}\Big) \quad \text{for } \frac{n}{N}\leqq t\leqq\frac{n+1}{N}. \tag{6.2}$$

If the offspring distribution (1.3) is Poisson, then the induced Markov chain is S. Wright's model with

$$P_{jk}^{(N)}=\binom{N}{k}\left(\frac{j}{N}\right)^{k}\left(\frac{N-j}{N}\right)^{N-k}.$$

In this case, calculation of moments of $X^{(N)}(n)$ shows, at least heuristically, that the sequence of the processes $\{Y^{(N)}(t)\}$ converges as $N\to\infty$ to the diffusion process on $[0, 1]$ whose backward Kolmogorov differential equation is

$$\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}x(1-x)\frac{\partial^2 u}{\partial x^2} \tag{6.3}$$

with $\sigma=1$. See the book [4] of Crow and Kimura for detailed analysis. Note that the boundaries 0 and 1 are of pure exit type in Feller's classification. We prove in [14] the convergence to diffusion processes for a wide class of d type induced Markov chains with or without mutation. (In the case $d=2$ with mutation, Karlin and McGregor [9] make a similar assertion without proof.) In particular, if the offspring distribution satisfies Assumption 2.1, then the sequence of the processes $\{Y^{(N)}(t)\}$ converges to the diffusion process on $[0, 1]$ with backward equation (6.3). The eigenfunction expansion of the transition probability of (6.3) is known (see [4] or [11]). The spectrum is discrete and consists of $\sigma^2 r(r-1)/2$, $r=0, 1, 2, \cdots$. This is in accordance with the fact that Theorem 2.1 implies

$$(\lambda_r^{(N)})^{Nt}\longrightarrow \exp\left\{-\frac{\sigma^2 r(r-1)}{2}t\right\} \qquad \text{as } N\to\infty,\ r \text{ fixed}. \tag{6.4}$$

Finer estimation of $\lambda_r^{(N)}$ in Theorem 2.1 gives speed of the convergence (6.4).

If we choose $0<\alpha<1$ and define $Y^{(N)}(t)$, instead of (6.1)-(6.2), by

$$Y^{(N)}(t)=\frac{1}{N^\alpha}X^{(N)}(n) \qquad \text{for } t=\frac{n}{N^\alpha}, \tag{6.5}$$

$$Y^{(N)}(t)=(n+1-N^\alpha t)Y^{(N)}\left(\frac{n}{N^\alpha}\right)+(N^\alpha t-n)Y^{(N)}\left(\frac{n+1}{N^\alpha}\right) \tag{6.6}$$

$$\text{for } \frac{n}{N^\alpha}\leqq t\leqq \frac{n+1}{N^\alpha},$$

then we find in [9] a result suggesting that $\{Y^{(N)}(t)\}$ converges to the diffusion process on $[0, \infty)$ with backward equation

$$\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}x\frac{\partial^2 u}{\partial x^2}. \tag{6.7}$$

(Actually Karlin and McGregor assert, without proof, a similar fact in the case of existence of mutation pressure.) The results (i) and (ii) of Theorem 2.2 imply that

$$(6.8)\qquad (\lambda^{(N)}_{[\gamma N\gamma]})^{N^\alpha t} \longrightarrow \begin{cases} 1 & \text{if } 0\leqq\gamma<(1-\alpha)/2 \\ e^{-\sigma^2\gamma^2 t/2} & \text{if } \gamma=(1-\alpha)/2 \\ 0 & \text{if } \gamma>(1-\alpha)/2 . \end{cases}$$

This should have connection with the fact that the differential operator in (6.7) has point spectrum only at 0 and continuous spectrum on $(0, \infty)$.

Let us consider a d type induced Markov chain with mutation pressure, which is discussed by Karlin and McGregor [8], [9], [**10**]. Underlying quantities are the distribution $\{c_k\}$ of the number of offspring and the probability a_{pq} of mutation from type p to type q $(q\neq p)$. Let $a_{pp}=1-\sum_{q\neq p} a_{pq}$. Let $\hat{a}_1$ $(=1)$, $\hat{a}_2, \cdots, \hat{a}_d$ be the set of eigenvalues of the stochastic matrix (a_{pq}). Then, by Theorem 4 of [**10**], the set of eigenvalues of $(P^{(N)}_{jk})$ of (1.11) consists of $\lambda^{(N)}_0=1$ and

$$(6.9)\qquad \lambda^{(N)}_r\hat{a}_{q_1}\hat{a}_{q_2}\cdots\hat{a}_{q_r}\qquad (2\leqq q_1\leqq q_2\leqq\cdots\leqq q_r\leqq d,\ 1\leqq r\leqq N),$$

where $\lambda^{(N)}_r$ is defined by (1.9). As is indicated by [7] and [**9**], it is natural to assume that a_{pq} depends on N and that

$$a_{pq}=\frac{\alpha_{pq}}{N}\quad (q\neq p),\qquad a_{pp}=1+\frac{\alpha_{pp}}{N},$$

where α_{pq} are independent of N and satisfy $\alpha_{pq}\geqq 0$ $(p\neq q)$, $\alpha_{pp}\leqq 0$, $\sum_{q=1}^{d}\alpha_{pq}=0$. Let $\hat{\alpha}_1$ $(=0)$, $\hat{\alpha}_2, \cdots, \hat{\alpha}_d$ be the eigenvalues of the matrix (α_{pq}). Then we have

$$\hat{a}_p=1+\frac{\hat{\alpha}_p}{N},\qquad p=1, \cdots, d .$$

Hence, the eigenvalue (6.9) is

$$(6.10)\qquad 1-\frac{1}{N}\Big(\frac{\sigma^2 r(r-1)}{2}-\hat{\alpha}_{q_1}-\cdots-\hat{\alpha}_{q_r}\Big)+O\Big(\frac{1}{N^2}\Big)$$

by virtue of Theorem 2.1 if Assumption 2.1 is satisfied. Let $d=2$. It follows that the eigenvalues of $(P^{(N)}_{jk})$ of (1.4) consist of $\mu^{(N)}_0=1$ and

$$(6.11)\qquad \mu^{(N)}_r=1-\frac{1}{N}\Big(\frac{\sigma^2 r(r-1)}{2}+(\alpha_{12}+\alpha_{21})r\Big)+O\Big(\frac{1}{N^2}\Big),\quad r=1, 2, \cdots, N,$$

since $\hat{\alpha}_2=-(\alpha_{12}+\alpha_{21})$. Hence

$$(6.12)\qquad (\mu^{(N)}_r)^{Nt}\longrightarrow \exp\Big\{-\Big(\frac{\sigma^2 r(r-1)}{2}+(\alpha_{12}+\alpha_{21})r\Big)t\Big\}.$$

On the other hand, the limiting diffusion on $[0, 1]$ is the one with backward equation

$$(6.13)\qquad \frac{\partial u}{\partial t}=\frac{\sigma^2}{2}x(1-x)\frac{\partial^2 u}{\partial x^2}+(\alpha_{21}-(\alpha_{12}+\alpha_{21})x)\frac{\partial u}{\partial x}$$

and reflecting barriers at 0 (if $0<\alpha_{21}<\sigma^2/2$) and 1 (if $0<\alpha_{12}<\sigma^2/2$). Note that the boundary 0 is pure exit, regular, or pure entrance according as $\alpha_{21}=0$,

$0<\alpha_{21}<\sigma^2/2$, or $\alpha_{21}\geqq\sigma^2/2$, and the nature of the boundary 1 is determined similarly by α_{12}. The spectrum of this differential operator is discrete and exactly consists of

$$\frac{\sigma^2 r(r-1)}{2}+(\alpha_{12}+\alpha_{21})r, \qquad r=0, 1, 2, \cdots. \tag{6.14}$$

The convergence (6.4), (6.12) for Wright's models was noticed by Feller [7], who pointed out also the connection with the eigenvalues of the diffusion equations.

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