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Some results on the fix-points and factorization of entire and meromorphic functions

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Introduction.

Let $f(z)$ be a complex-valued function of the complex variable z. A fix-point z_0 of f is a zero of $f(z)-z$. When f is a rational function or an entire function one may define the "iterates" $f_n(z)$ of $f(z)$ by $f_1(z)=f(z)$, $f_n(z)=f_{n-1}(f(z))=f(f_{n-1}(z))$. The existence and distribution of the fix-points play an important role in the theory of iteration of entire functions and the solutions of various functional equations. Regarding the latter subject, we refer the reader to works of Schröeder, Koenigs and others (for a bibliograph cf. [**1**]) which deals with the behavior of the sequence $\{f_n(z)\}$ in the neighborhood of fix-points. As for the former subject, we refer the reader to [**2**] and a book of Kuczma's [**20**]. The theory of iteration of rational functions or entire functions have been extensively studied by Julia [**19**], Rosenbloom [**26**], Fatou [**10**], Myrberg [**21**], Baker [**2**], Cremer [**7, 8**], Töpfer [**27**] and others.

The first interesting result concerning the fix-points and iterates of an entire function was announced by Fatou [**10**] who stated that if any iterate $g_n(z)$ $(n\geqq 2)$ of an entire function g has only a finite number of fix-points, then g is a polynomial. In 1952, Rosenbloom [**25**], based on the technique of Nevanlinna, proved a more general statement that if f and g are entire functions, and if f and $f(g)$ have only a finite number of fix-points, then either f is a polynomial or $g\equiv$constant or $g(z)\equiv z$. In the same paper, the term "prime function" was introduced as an entire function which cannot be represented in the form $f_1(f_2)$, where f_1 and f_2 are non-linear entire functions. From the definition, we see immediately that every non-linear polynomial of prime degree is prime. However, it is, in general, not easy to tell whether a given entire function is prime or not. For instance, the function e^z+z was stated without proof in [**25**] to be a prime function, and it was stated there that the proof was rather complicated. So far, certain classes of prime functions have been obtained through the investigation of Baker [**4**], Gross [**11, 12, 13, 14, 15**],

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Ozawa [**23**], and the author [**14**, **15**, **29**].

Gross [**11**] extended the studies of prime functions to meromorphic functions. According to [**11**], a meromorphic function $h(z)=f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that $f(z)$ is non-linear and meromorphic and $g(z)$ is non-linear and entire (g may be meromorphic when $f(z)$ is rational). h is said to be prime (pseudo-prime) if every factorization of the above form implies that one of the functions f or g is linear (a polynomial unless f is rational). h is said to be E-prime (E-pseudo-prime) if every factorization of the above form into entire factors implies that one of the functions $f(z)$ or $g(z)$ is linear (a polynomial). There are many properties of entire or meromorphic functions which can be used to deal with the factorization problems. For instance, the growth property (maximal modulus for entire functions and Nevanlinna characteristic for meromorphic functions), the geometric distribution of the zeros of the function, the fix-points of the function and so on. In this paper, we are interested in the relationship between the fix-points and the factorizations of a function.

Going back to the example that e^z+z is prime, it is natural to expect more generally that $e^{g(z)}+z$ is prime for any non-constant entire function $g(z)$. This can also be phrased as follows: For any two non-linear entire functions f and g with $f(g)$ being transcendental, the composition $f(g)$ must have infinitely many fix-points. This shows that for a certain class of entire functions the problems of the fix-points and the factorization are related to each other.

In this paper, we introduce some methods used in attacking problems on the fix-points and the factorizations of entire and meromorphic functions and present some results mainly based on a joint paper [**14**] written by Gross and the author. For completeness, we shall repeat some of the proofs here. Our main tools will be Nevanlinna's theory of meromorphic functions, and a special case of the theorem of Tumura-Clunie [see Lemma 4, Sec 4 below]. We shall use the standard notation of Nevanlinna theory, as given in [8; pp. 1-9].

The content of this paper is as follows:

1. Some known results
2. Conjectures
3. Statement of main results
4. Preliminary lemmas
5. Proofs
6. Question.

§ 1. Known results.

THEOREM A (Rosenbloom [**25**]). *Let $p(z)$ be a non-constant polynomial and g be a transcendental entire function. The $p(g)$ has infinitely many fix-points.*

REMARK. Actually a quantitative measure of the number of fix-points was given in [**25**].

THEOREM B (Gross [**11**, p. 211]). *If $F(F^*)$ denotes the family of entire (meromorphic) functions with at most a finite number of fix-points, then* (i) *every entire function has at most one factorization $f(g)$, f transcendental, $f \in F$, g entire;* (ii) *every meromorphic function has at most two distinct factorizations $f_i(g_i)$, f_i meromorphic not rational, $f_i \in F^*$, g_i entire.*

It follows from this that

THEOREM C (Gross [**11**, p. 218]). *If f and g are two transcedental entire functions, then f or $f(g)$ must have infinitely many fix-points.*

THEOREM D (Baker and Gross [**4**]). *Let $p(z)$ be any non-constant polynomial. Then $e^z + p(z)$ is prime.*

§ 2. Conjectures.

Theorem A and Theorem C suggest the following conjecture:

CONJECTURE 1 (Gross [**4**, p. 542]). If f and g are two transcendental entire functions, then $f(g)$ has infinitely many fix-points.

REMARK. If one of f or g is a non-linear polynomial, then the assertion is true according to Theorem A and Lemma 3 (see section 4 below).

CONJECTURE 2 (Baker and Gross [**4**, p. 34]). $e^z + h(z)$ is prime for any non-constant entire function h with order less than one.

§ 3. Statement of the main results.

Our first result gives a partial answer to Conjecture 1 as follows:

THEOREM 1. *If f and g are two non-linear entire functions such that $f(g)$ is transcendental and of finite order, then $f(g)$ has infinitely many fix-points.*

We answer Conjecture 2 completely and more as follows:

THEOREM 2. *Let $p_1(z)$ be a polynomial and not identically zero. Let $h(z)$ be any non-constant entire function of order less than one. Then $p_1(z)e^z + h(z)$ is prime.*

Finally we give a result analogous to Theorem A for meromorphic functions.

THEOREM 3. *Let $p(z)$ be a polynomial of degree $\geqq 3$. Then for any transcendental meromorphic function f, $p(f)$ has infinitely many fix-points.*

REMARK. The argument used in the proof of Theorem A [**10**] can be applied to meromorphic functions f with

$$\overline{\lim_{r \to \infty}} N(r, f)/T(r, f) < 1 .$$

When $n=2$ we prove

THEOREM 4. *Let $p(z)$ be a polynomial of degree 2, and let f be transcendental meromorphic. Then $p(f)$ has finitely many fix-points if and only if f can be expressed as*

$$f(z)=\frac{Q(\sqrt{z-c})e^{\gamma(\sqrt{z-c})}+Q(-\sqrt{z-c})e^{\gamma(-\sqrt{z-c})}}{\dfrac{Q(\sqrt{z-c})e^{\gamma(\sqrt{z-c})}}{\sqrt{z-c}}-\dfrac{Q(-\sqrt{z-c})e^{\gamma(-\sqrt{z-c})}}{\sqrt{z-c}}}+a, \tag{1}$$

where a and c are constants depending on $p(z)$, and $\gamma(z)$ is an entire function.

§ 4. Preliminary Lemmas.

LEMMA 1 (Goldstein [16]). *Let F be an entire function with $\delta(a, F)=1$ for some $a\neq\infty$, where $\delta(a, F)$ denotes the Nevanlinna deficiency. Then $F(z)$ is pseudo-prime.*

We shall need Lemma 2 but not Lemma 1 in our proofs of the main results. However, we shall give an outline of the proof of Lemma 1 in order to establish Lemma 2.

LEMMA 2. *Let $p(z)$ be a non-constant polynomial of degree $k\geqq 1$, and ϕ_1, ϕ_2 be two entire functions with $\phi_1\not\equiv 0$. Suppose that both orders of ϕ_1 and ϕ_2 are less than k. Then $\phi_1 e^{p(z)}+\phi_2$ is pseudo-prime.*

PROOF. We first present a sketch of the proof of Lemma 1.

According to a result of Edrei and Fuchs [9], if F is a meromorphic function with $\sum_{a\neq\infty}\delta(a, F)=1$, then its lower order μ is greater than or equal to 1/2. Furthermore, if p is the integer defined by $p-1/2\leqq\mu<p+1/2$ (hence $p\geqq 1$), then there exist an increasing sequence $\{r_j\}$, $r_j\uparrow\infty$ as $j\to\infty$, a sequence of arcs γ_j of $|z|=r_j$, each of angular measure not less than $2\pi/3p$, and a sequence of segments l_j joining the arcs γ_j, such that on the path $\gamma=l_1+\gamma_2+l_2+\gamma_3+\cdots$, the following estimate holds:

$$\log|F(z)|\leqq\frac{-\pi}{16}T(r, F), \qquad (|z|=r>r_0). \tag{2}$$

Now suppose that

$$F=f(g),$$

where f and g are two transcendental entire functions. Then by a well-known result of Pólya [24], f is of order zero. Hence, by a classical theorem of Wiman [28], there is a sequence $\{R_n\}$ with $R_n\uparrow\infty$ as $n\to\infty$ such that

$$\min_{|z|=R_n}|f(z)|\longrightarrow\infty,$$

as $n\to\infty$.

But by (2), $F(z)\to 0$ as $z\to\infty$ along Γ. So, $\Gamma=g(\gamma)$ must be bounded.

Now suppose that Γ is contained in $|z|\leqq R$, and assume that f has no zeros on $|z|=R$. Then since $f(\Gamma)\to 0$, therefore

$$g(\gamma)\longrightarrow\alpha,$$

where α is a zero of f in $|z|\leqq R$.

Thus, given $\varepsilon>0$, there exists r_0 such that

$$|g(z)-\alpha|\leqq\varepsilon \qquad (z\in\gamma,\ |z|\geqq r_0). \tag{3}$$

On the other hand, assume that f has a zero of order s at α, then there is a constant $A>0$ such that

$$|f(z)|\geqq A|z-\alpha|^s \qquad (|z-\alpha|\leqq\varepsilon). \tag{4}$$

From this and (3) we have

$$F(z)\equiv|f(g(z))|\geqq A|g(z)-\alpha|^s, \qquad (z\in\gamma_j) \tag{5}$$

for $j>j_0$.

So, (2) and (5) yield

$$s\log|g(z)-\alpha|+\log A\leqq -\frac{\pi}{16}T(|z|, F). \tag{6}$$

Hence, for $j\geqq j_0$, we have by Nevanlinna's first fundamental theorem

$$\begin{aligned} T(r_j, g)+O(1) &\geqq m(r_j, \alpha, g)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{g(r_je^{i\theta})-\alpha}\right|d\theta \\ &\geqq \frac{1}{2\pi}\int_{r_j}\log\left|\frac{1}{g(r_je^{i\theta})-\alpha}\right|d\theta \\ &\geqq \frac{1}{3p}\frac{\pi}{16s}T(|\gamma_j|, F)+\frac{\log A}{s}. \end{aligned} \tag{7}$$

So we have for some positive constant B

$$T(r_j, g)\geqq BT(r_j, F) \qquad (j\geqq j_1).$$

Therefore

$$\overline{\lim_{r\to\infty}}\frac{T(r, g)}{T(r, f(g))}\geqq B$$

which contradicts the fact that $\lim\limits_{r\to\infty}T(r, g)/T(r, f(g))=0$, for any two transcendental entire functions f and g (Clunie [**17**, p. 54]). Lemma 1 is thus proved.

We now return to the proof of Lemma 2.

We rewrite $\psi_1e^p+\psi_2$ as $\psi_2\left(\frac{\psi_1}{\psi_2}e^p+1\right)$. Set

$$\psi_1e^p+\psi_2\equiv H=\psi_2F,$$

and

$$F=\frac{\psi_1}{\psi_2}e^p+1\,.$$

Then F is a meromorphic function of regular growth with $\delta(1, F)=1$. Thus F satisfies all the hypotheses of the Edrei-Fuchs result mentioned in Lemma 1. Hence, the estimate (2) holds for F. Now, since ψ_2 is entire of order less than k, the estimate (2) still holds with $\pi/16$ replaced by $\pi/16-\varepsilon$ for some positive constant ε, $\pi/16>\varepsilon>0$.

Suppose that there exist two transcendental entire functions f and g such that

$$H\equiv\psi_2\cdot F=f(g)\,.$$

By an argument exactly the same as that used before, we will get a contradiction. This completes the proof of Lemma 2.

LEMMA 3. *Let f and g be two functions such that $f(g)$ and $g(f)$ are also well-defined functions. Then $f(g)$ has infinitely many fix-points if and only if $g(f)$ does.*

PROOF. Suppose that z_0 is a fix-point of $f(g)$. Then

$$g(f(g)(z_0))=g(z_0)\,.$$

This implies that $g(z_0)$ is a fix-point of $g(f)$. Furthermore, if z_1, z_2 are two distinct fix-points of $f(g)$, then $g(z_1)$ and $g(z_2)$ are two distinct fix-point of $g(f)$.

LEMMA 4. *Let f be a transcendental meromorphic function and $a_i(z)$ ($i=1$, $2, \cdots, n$) be meromorphic functions satisfying*

$$T(r, a_i(z))=o\{T(r, f)\}\,,$$

as $r\to\infty$ for $i=1, 2, \cdots, n$. Assume that

$$f^n(z)+a_1(z)f^{n-1}(z)+a_2(z)f^{n-2}(z)+\cdots+a_n(z)=g \tag{8}$$

and that

$$N(r, f)+N(r, 1/g)=o\{T(r, f)\} \tag{9}$$

as $r\to\infty$ n.e. (The abbreviation "n.e." (Hayman [**18**]) *is used to mean everywhere in $[0, \infty)$ except in a set of finite measure).*

Then

$$g=\left(f+\frac{a_1(z)}{n}\right)^n.$$

REMARK. This is a special case of the Tumura-Clunie theorem [**17**, p. 69]. We present a sketch proof of it as follows.

PROOF. Let

$$f = F - \frac{a_1(z)}{n}. \tag{10}$$

Substituting this into (8), we have

$$F^n + b_2(z)F^{n-2} + b_3(z)F^{n-3} + \cdots + b_n(z) = g, \tag{11}$$

where $b_i(z)$ $(i=2, 3, \cdots, n)$ are meromorphic functions satisfying $T(r, b_i(z)) = o\,T(r, f)$ as $r \to \infty$ n. e.

We now differentiate this and obtain

$$nF^{n-1}F' + H'(z) = g', \tag{12}$$

where $H(z) \equiv b_2(z)F^{n-2} + b_3(z)F^{n-3} + \cdots + b_n(z)$.

We eliminate g' by multiplying g'/g on both sides of (11) and substituting from (12). This gives

$$F^{n-1}\left(nF' - \frac{g'}{g}F\right) + P_{n-2}(F) = 0, \tag{13}$$

where $P_{n-2}(F)$ is a polynomial in F and its derivatives, with coefficients $a(z)$ satisfying

$$T(r, a(z)) = o\,T(r, F) \tag{14}$$

as $r \to \infty$ outside a set of r of finite measure.

We wish to show that $nF' - (g'/g)F \equiv 0$. Suppose that $nF' - (g'/g)F \not\equiv 0$. Then according to the well-known "lemma on the logarithmic derivative" [22] and the fact that $T(r, g) = O\{T(r, F)\}$ as $r \to \infty$ n. e. (see [**30**, p. 199]) we have

$$m\left(r, nF' - \frac{g'}{g}F\right) = o\{T(r, F)\}, \tag{15}$$

as $r \to \infty$ n. e.

Clearly

$$\begin{aligned} N\left(r, nF' - \frac{g'}{g}F\right) &\leqq N(r, F') + N(r, g) + N\left(r, \frac{1}{g}\right) \\ &\leqq N(r, f') + N(r, a_1(z)/n) + N(r, g) + N\left(r, \frac{1}{g}\right) \\ &= o\{T(r, f)\} \\ &= o\{T(r, F)\} \end{aligned} \tag{16}$$

as $n \to \infty$, n. e. by hypothesis (9).

This and (15) give

$$T\left(r, nF' - \frac{g'}{g}F\right) = o\{T(r, F)\} \tag{17}$$

as $r \to \infty$, n. e.

On the other hand, from (13) we have

(18) $$T\left(r, nF' - \frac{g'}{g}F\right) \geqq T\left(r, \frac{1}{F^{n-1}}\right) - T(r, P_{n-2}(F))$$

$$\geqq (1-o(1))T(r, F)$$

as $r \to \infty$ n. e. since $T(r, P_{n-2}(F)) \leqq (n-2+o(1))T(r, F)$ as $r \to \infty$ n. e. (see e. g. [**30**, p. 198]).

This contradicts (17). Hence, we have to conclude that

(19) $$nF' - \frac{g'}{g}F \equiv 0$$

and

(20) $$g = cF^n,$$

where c is a constant, and c must be equal to one. Lemma 4 is thus proved.

LEMMA 5 (Gross [**13**]). *If $F(z)$ is non-periodic, entire, and prime, when only entire factors are considered, then it remains prime even when meromorphic factors are permitted.*

REMARK. This lemma enables us to consider only entire factors in the proof of Theorem 2.

SKETCH PROOF. Suppose that $F(z) = f(g)$, where f is meromorphic. If f is transcendental, then it can have at most one pole, say a, and g must be of the form $e^{\alpha(z)} + a$, where α is entire. Then we can express $F(z)$ by

(21) $$F(z) = h(e^{\alpha(z)}),$$

where h is of the form

$$h(\omega) = h_1(\omega)/\omega^n,$$

and where h_1 is entire and n is an integer. Thus

$$h(e^{\alpha(z)}) = h_1(e^{\alpha(z)})/e^{n\alpha(z)}$$

will have two entire functions $h_1(e^z)/e^{nz}$, and $\alpha(z)$ as its left and right factors respectively. Hence, by hypothesis α must be linear. Then from (21), F would be periodic, contrary to the assumption. For the case where f is rational, the discussion is similar.

LEMMA 6 (Borel [**5**]). *Let $a_i(z)$ be an entire function of order ρ, let $g_i(z)$ also be entire and let $g_j(z) - g_i(z)$ $(i \neq j)$ be a transcendental function or polynomial of degree higher than ρ. Then*

$$\sum_{i=1}^{n} a_i(z) e^{g_i(z)} = a_0(z)$$

holds only when $a_0(z) = a_1(z) = \cdots = a_n(z) = 0$.

§ 5. The proofs of the main results.

PROOF OF THEOREM 1:

Suppose that $f(g)$ has only finitely many fix-points. That is

$$f(g)(z)-z=p(z)e^{Q(z)},$$

where $p(z)$, $Q(z)$ are polynomials.
Then

$$p(z)e^{Q(z)}+z=f(g).$$

This is impossible by Lemma 2.

PROOF OF THEOREM 2. Clearly, according to Lemma 2, the function $p_1(z)e^z+h(z)$ must be pseudo-prime. Now suppose that there exist a non-linear polynomial $p(z)$ of degree k and a transcendental entire function $g(z)$ such that

$$p_1(z)e^z+h(z)=p(g) \qquad (=p_0g^k+p_1g^{k-1}+\cdots+p_k). \tag{22}$$

If we rewrite this as $p(g)-h(z)=p_1(z)e^z$, then all the hypotheses of Lemma 4 are satisfied. Thus we deduce that

$$p_1(z)e^z=p_0\left(g+\frac{p_1}{p_0k}\right)^k. \tag{23}$$

Thus

$$p_0g^k+p_1g^{k-1}+\cdots+p_k-h(z)\equiv p_0\left(g+\frac{p_1}{p_0k}\right)^k. \tag{24}$$

By equating the coefficients of both sides, we have

$$p_k-h(z)\equiv p_0\left(\frac{p_1}{p_0k}\right)^k. \tag{25}$$

Hence

$$h(z)\equiv p_k+p_0\left(\frac{p_1}{p_0k}\right)^k \tag{26}$$

a constant, contrary to the assumption.

Now suppose that there exist a non-linear polynomial p and a transcendental entire function f such that

$$p_1e^z+h=f(p).$$

By examining the geometric distribution of the zeros of p_1e^z+h we find that for any $\varepsilon>0$, the zeros will eventually lie outside the sector $S=\left\{re^{i\theta}\,|\,r\geqq 0,\ -\frac{\pi}{2}+\varepsilon\leqq\theta\leqq\frac{\pi}{2}-\varepsilon\right\}$. If the degree of $p\geqq 3$, then for sufficiently large zero a_i of f at least one zero of $p(z)-a$ will lie in S, a contradiction. Thus we conclude that the degree of $p=2$. Then we may assume that p has the form $p(z)=(z+c)^2+d$. Hence

$$f(p(z))\equiv f(p(-z-2c)).$$

Therefore,

$$p_1(z)e^z+h(z)\equiv p_1(-z-2c)e^{-z-2c}+h(-z-2c).$$

This is impossible, by examining $z\to\infty$, say along the positive real axis.

The theorem is thus proved.

PROOF OF THEOREM 3. Suppose our assertion is false: i. e., $p(f)$ has only finitely many fix-points. Then according to Lemma 3, $f(p)$ also has only finitely many fix-points.

Let

$$f(z)=\frac{f_1(z)}{f_2(z)}, \tag{27}$$

where f_1, f_2 are entire and have no common zeros. Then

$$f(p)-z=\frac{Q_1(z)}{h_1(z)}e^{\alpha_1(z)} \tag{28}$$

or

$$\frac{f_1(p)-zf_2(p)}{f_2(p)}=\frac{Q_1(z)e^{\alpha_1(z)}}{h_1(z)}, \tag{29}$$

where $Q_1(z)$ is a polynomial and h_1, α_1 are entire functions, and h_1, Q_1 have no common zeros. Hence,

$$f_1(p)-zf_2(p)=Q_1(z)e^{\alpha(z)} \tag{30}$$

where α is entire.

Also we have

$$p(f)-z=\frac{Q_2(z)}{h_2(z)}e^{\beta_1(z)}, \tag{31}$$

where Q_2 is a polynomial, β_1, h_2 are entire and Q_2, h_2 have no common zeros. Let

$$p(z)=z^n+A_{n-1}z^{n-1}+A_{n-2}z^{n-2}+\cdots+A_0. \tag{32}$$

Then from this and (31) we have

$$\left(\frac{f_1(z)}{f_2(z)}\right)^n+A_{n-1}\left(\frac{f_1(z)}{f_2(z)}\right)^{n-1}+\cdots+A_0-z=\frac{Q_2(z)}{h_2(z)}e^{\beta_1(z)}. \tag{33}$$

Multiplying both sides of (33) by f_2^n we obtain

$$[f_1(z)]^n+A_{n-1}[f_1(z)]^{n-1}f_2(z)+\cdots+A_0[f_2(z)]^n-z[f_2(z)]^n =Q_2(z)e^{\beta(z)} \tag{34}$$

where β is entire and we note that all the zeros of $h_2(z)$ are zeros of f_2^n.

Replacing z by $p(z)$ in the above identity, we have

$$(35)\qquad [f_1(p)]^n+A_{n-1}[f_1(p)]^{n-1}f_2(p)+\cdots+A_1f_1(p)[f_2(p)]^{n-1} \\ +A_0[f_2(p)]^n-p(z)[f_2(p)]^n=Q_2(p)e^{\beta(p)}.$$

Substituting $f_1(p)=zf_2(p)+Q_1(z)e^{\alpha(z)}$ into the above identity, we have

$$(zf_2(p)+Q_1e^{\alpha})^n+A_{n-1}(zf_2(p)+Q_1e^{\alpha})^{n-1}f_2(p)+\cdots \\ +\cdots+A_1(zf_2(p)+Q_1e^{\alpha})[f_2(p)]^{n-1}+A_0[f_2(p)]^n \\ -p(z)[f_2(p)]^n=Q_2(p)e^{\beta(p)}.$$

Expanding and combining terms, we arrive at

$$(36)\qquad (z^n+A_{n-1}z^{n-1}+\cdots+A_1z+A_0-p(z))[f_2(p)]^n \\ +(nz^{n-1}+(n-1)A_{n-1}z^{n-2}+\cdots)[f_2(p)]^{n-1}[Q_1e^{\alpha}] \\ +\left(\frac{n(n-1)}{2}z^{n-2}+\frac{(n-1)(n-2)}{2}A_{n-1}z^{n-3}+\cdots\right)[f_2(p)]^{n-2}[Q_1e^{\alpha}]^2 \\ +\cdots+[Q_1e^{\alpha}]^n=Q_2(p)e^{\beta(p)}.$$

The left hand side of (36) is a homogeneous polynomial in $f_2(p)$ and Q_1e^{α} with degree n. We note further that the coefficient of $[f_2(p)]^n$ vanishes by (32).

Dividing $[Q_1e^{\alpha}]^n$ on both sides of (36) and setting

$$(37)\qquad F(z)=\frac{f_2(p)}{Q_1e^{\alpha}}$$

we obtain

$$(38)\qquad p'(z)F^{n-1}(z)+\frac{p''(z)}{2}F^{n-2}(z)+\cdots+1=\frac{Q_2(z)}{Q_1(z)^n}e^{\beta(p)-n\alpha},$$

or

$$(39)\qquad F^{n-1}(z)+\frac{p''(z)}{2p'(z)}F^{n-2}(z)+\cdots+\frac{1}{p'(z)}=R(z)e^{\beta(p)-n\alpha},$$

where $R(z)=Q_2(z)/Q_1^n(z)\cdot p'(z)$, a rational function.

The left hand side of (34) is a polynomial in f with rational functions as its coefficients. Thus, Lemma 3 is applicable unless F is a rational function.

Suppose F is a rational function. Then from (37), it follows that

$$f_2(p)=Q_1(z)R_1(z)e^{\alpha},$$

where $R_1(z)$ is a rational function.

Then from (30) we have

$$f_1(p)=R_2(z)e^{\alpha(z)},$$

where $R_2(z)$ is a rational function. Thus

$$\frac{f_1(p)}{f_2(p)}=\frac{Q_1 R}{R_2}=\text{a rational function,}$$

which of course is impossible.

Alternately, we conclude that F is transcendental. We note that F has only finitely many poles. Thus applying Lemma 3 to the identity (39), we have

$$\left[F(z)+\frac{1}{n-1}\frac{p''}{2p'}\right]^{n-1}=R(z)e^{\beta(p)-n\alpha}. \tag{40}$$

Comparing this with (39) and equating the coefficients of both sides of (39) and (40) we obtain

$$\left[\frac{1}{2(n-1)}\frac{p''}{p'}\right]^{n-1}=\frac{1}{p'(z)} \tag{41}$$

or

$$[p'']^{n-1}=[2(n-1)]^{n-1}[p']^{n-2}. \tag{42}$$

By a simple degree argument, we have

$$p'(z)=c(z-a)^{n-1},$$

where c, a are constants.

It follows that

$$p(z)=c_1(z-a)^n+c_2 \tag{43}$$

for some constants c_1, c_2.

Substituting this into equation (31), we get

$$c_1(f-a)^n+c_2-z=\frac{Q_2}{h_2}e^{\beta_1}. \tag{44}$$

Changing variables in (44), by setting $c_2-z=w$, we obtain

$$c_1(f(c_2-w)-a)^n+w=\frac{Q_2(c_2+w)}{h_2(c_2+w)}e^{\beta_1(c_2+w)} \tag{45}$$

or

$$[f_0(w)]^n+w=\frac{Q_3(w)}{h_0(w)}e^{\beta_2(w)}, \tag{46}$$

where $f_0(w)=c_1^{1/n}(f(c_2-w)-a)$, $h_0(w)=h_2(c_2+w)$, $Q_3(w)=Q_2(c_2+w)$, and $\beta_2(w)=\beta_1(c_2+w)$.

Let $w=z^n$. Then

$$f_0^n(z^n)+z^n=\frac{Q_3(z^n)}{h_0(z^n)}e^{\beta_2(z^n)}. \tag{47}$$

Thus when $n\geqq 3$, this implies that

(48) $$\frac{f_0(z^n)}{z}+\rho_i=0 \quad (\rho_1, \rho_2, \cdots, \rho_n \text{ are } n \text{ distinct roots of unity}).$$

has only finitely many zeros, for $i=1, 2, \cdots, n$.

According to Nevanlinna's second fundamental theorem (or Picard's theorem) we conclude at once that $f_0(z^n)/z$ must be a rational function. This will lead to the conclusion that $f(z)$ is rational, contrary to the hypothesis. The theorem is thus proved.

PROOF OF THEOREM 4. We may assume without loss of generality that

$$p(z)=(z-a)^2+c.$$

Then if $p(f)$ has only finitely many fix-points, we will have

(49) $$(f(z)-a)^2+c-z=\frac{Q_4(z)}{h(z)}e^{\beta_0(z)},$$

where Q_4 is a polynomial, $h(z)$ and $\beta_0(z)$ are entire functions. Changing variables by setting $z-c=w$, we get

(50) $$(f(c+w)-a)^2-w=\frac{Q_4(c+w)}{h(c+w)}e^{\beta_0(c+w)}$$

or

(51) $$f_0^2(w)-w=\frac{Q_0(w)e^{\beta_5(w)}}{h_0(w)},$$

where $f_0(w)=(f(c+w)-a)$, $h_0(w)=h(c+w)$, $Q_0(w)=Q_4(c+w)$, and $\beta_5(w)=\beta_0(c+w)$.

Let $f_0(w)=f_1(w)/f_2(w)$. Then we have

(52) $$f_1^2(w)-wf_2^2(w)=Q_0(w)e^{\alpha(w)},$$

where $Q_0(w)$ is a polynomial and $\alpha(w)$ is an entire function. Replacing w by w^2 in the above identity we have

(53) $$f_1^2(w^2)-w^2f_2^2(w^2)=Q_0(w^2)e^{\alpha(w^2)}$$

or

(54) $$[f_1(w^2)+wf_2(w^2)][f_1(w^2)-wf_2(w^2)]=Q_0(w^2)\alpha(w^2).$$

Hence we conclude:

(55) $$f_1(w^2)+wf_2(w^2)=Q(w)e^{\gamma(w)}$$

and

(56) $$f_1(w^2)-wf_2(w^2)=L(w)e^{\beta_4(w)},$$

where Q, L are polynomials, and γ, β_4 are entire functions.

Solving $f_1(w^2)$ and $f_2(w^2)$ from (55) and (56), we obtain

(57) $$f_1(w^2)=\frac{1}{2}(Q(w)e^{\gamma(w)}+L(w)e^{\beta_4(w)})$$

and

$$f_2(w^2)=\frac{1}{2}\left(\frac{Q(w)}{w}e^{\gamma(w)}-\frac{L(w)}{w}e^{\beta_4(w)}\right). \tag{58}$$

Since $f_1(w^2)$ is an even function, from (57) and (58), we have

$$Q(w)e^{\gamma(w)}+L(w)e^{\beta_4(w)}=Q(-w)e^{\gamma(-w)}+L(-w)e^{\beta_4(-w)} \tag{59}$$

and

$$\frac{Q(w)}{w}e^{\gamma(w)}-\frac{L(w)}{w}e^{\beta_4(w)}=\frac{Q(-w)}{-w}e^{\gamma(-w)}-\frac{L(-w)}{-w}e^{\beta_4(-w)}. \tag{60}$$

By Lemma 6, we conclude from (59) that (i) $\gamma(w)=\beta_4(w)+c_1$, (ii) $\gamma(-w)=\gamma(w)+c_2$, or (iii) $\gamma(-w)=\beta_4(w)+c_3$; where c_1, c_2, c_3 are constants.

In case (i), we deduce from (57) and (59) that $f_1(w^2)/f_2(w^2)$ must be rational, a contradiction. Suppose that case (ii) holds. Then (60) and (59) yield

$$e^{c_2}Q(-w)=Q(w) \tag{61}$$

and

$$e^{c_2}\frac{Q(-w)}{-w}=\frac{Q(w)}{w}, \tag{62}$$

which are inconsistent. Thus, we conclude that only case (iii) can be true.

Hence, (59) and (60) yield

$$e^{c_3}Q(-w)=L(w) \tag{63}$$

and

$$e^{c_3}\frac{Q(-w)}{-w}=\frac{L(w)}{w}, \tag{64}$$

and so we deduce

$$L(w)=Q(-w). \tag{65}$$

Thus

$$f_1(w^2)=\frac{1}{2}(Q(w)e^{\gamma(w)}+Q(-w)e^{\gamma(-w)-c_3}). \tag{66}$$

By the evenness of the left hand side of (66), again we conclude by Lemma 6 that $e^{-c_3}=1$. Hence

$$f_1(w^2)=\frac{1}{2}(Q(w)e^{\gamma(w)}+Q(-w)e^{\gamma(-w)}), \tag{67}$$

or

$$f_1(w)=\frac{1}{2}(Q(\sqrt{w})e^{\gamma(\sqrt{w})}+Q(-\sqrt{w})e^{\gamma(-\sqrt{w})}). \tag{68}$$

Similarly, we conclude that

$$f_2(w)=\frac{1}{2}\Big(\frac{Q(\sqrt{w})}{\sqrt{w}}e^{\gamma(-\sqrt{w})}-\frac{Q(-\sqrt{w})}{\sqrt{w}}e^{\gamma(-\sqrt{w})}\Big). \tag{69}$$

The expression (1) will follow from (68) and (69). It is easy to see that when $p(z)=(z-a)^2+c$ and f has the form (1), $f(p)$ has only finitely many fix-points, and so does $p(f)$.

§ 6. Question.

We note that if f is a transcendental meromorphic function satisfying either $N(r, f)=o\{T(r, f)\}$ or $N\Big(r, \dfrac{1}{f}\Big)=o\{T(r, f)\}$ as $r\to\infty$, then by adopting the argument used in the proof of Theorem 3 one can show that Theorem 3 remains true if p is replaced by a rational function of weight greater than 2.

For arbitrary meromorphic function f, we ask the following question: Does Theorem 3 remain valid if p is replaced by a rational function of weight greater than 2?

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