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The ball coverings of manifolds

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§ 1. Introduction.

A compact manifold is covered by a finite number of balls. We can define the minimum number of such balls for any compact manifold. This minimum number, of course, is not only related with the dimension of a manifold but also strongly related with the structure of the manifold. Further it is not a homotopy invariant. The authors found out the *ball coverings* (defined in (1.1)) to be a useful tool in topology of manifolds, especially in low dimensions. In the literatures, the authors can find *weak ball coverings* (defined below) in Zeeman's result (§ 2, 2.2) [**12**] and in Glaser's examples (§ 3, 3.5) [**2, 3**]. In the present paper we will make clear some of the usefulness of ball coverings in the PL category. Throughout this paper all manifolds are connected compact PL manifolds and maps are piecewise linear, unless otherwise stated. S^n and B^n mean always a PL n-sphere and a PL n-ball, respectively (in this paper a ball means a closed ball). $\cong$ and $\sim$ mean homeomorphic (or group isomorphic) and homologous, respectively. The closure of a set X is denoted by $Cl(X)$ and $\mathrm{Int}(X)$ and $\mathring{X}$ mean the interiour of X. ∂M is the boundary of a manifold M. $N(X; Y)$ is usually used for a regular neighborhood of X in Y. $\sharp A$ indicates the number of elements (or the number of connected components) of a set A.

1.1. DEFINITION. Let M^n be an n-manifold and $\mathscr{B}=\{B_i\}$ be a set of finite n-balls in M.

(1) $\mathscr{B}$ is called a *weak ball covering* of M if $\cup B_i=M$.

(2) $\mathscr{B}$ is called a *ball covering* of M if $\mathscr{B}$ is a weak one of M and $B_i\cap B_j=\partial B_i\cap\partial B_j$ is an $(n-1)$-manifold (may not be connected) for $B_i, B_j\in\mathscr{B}$ and $i\neq j$. Define

$$\beta(M)=\min.\{\sharp\mathscr{B}\mid\mathscr{B}\text{ is a weak ball covering of } M\}\text{ and}$$

$$b(M)=\min.\{\sharp\mathscr{B}\mid\mathscr{B}\text{ is a ball covering of } M\}.$$

We call a ball covering (or weak one) $\mathscr{B}$ of a manifold M to be minimal if $\sharp\mathscr{B}=b(M)$ (or $\beta(M)$, respectively). Obviously $\beta(M)\leqq b(M)$.

This paper consists of five sections.

In § 2, we discuss the elementary properties of (weak) ball coverings, $\beta(M)$ and $b(M)$ for general dimensional manifold M by the handle decomposition of M. We first show that $b(M_1 \# M_2) \leqq \max.\{b(M_1), b(M_2)\}$ in theorem (2.5). Further, a relation between $b(M)$ and the dimension of M is obtained as a corollary (see 2.6). In § 3, we are interested in the number $b(M)$ for a homology sphere M and a contractible manifold M. We show;

THEOREM (3.2). *If M is a homology sphere and $b(M) \leqq 3$, M should be a sphere.*

Especially we can clarify the difference of $\beta(M)$ and $b(M)$ for some manifold M, that is

COROLLARY (3.6). *There exists a contractible 4-manifold M^4 with $\beta(M)=2$ but $b(M)=3$.*

§ 4 is devoted to study of 3-manifolds. We have a characterization, in (4.2), of a 3-manifold M with $b(M)=2$. Using this result, we are succeeded in getting a complete classification of the closed 3-manifold M with $b(M)=3$. That is;

THEOREM (4.3). *Let M be a closed 3-manifold. $b(M)=3$ if and only if $M \cong k(S^1 \times S^2) \# \varepsilon(S^1 \times_\tau S^2)$ for some $k+\varepsilon \geqq 1$ and $\varepsilon=0$ or 1 according to the orientability of M.*

Further, for large class of 3-manifolds, a special property of intersection of balls are investigated. This shows a relation between the ball coverings and the *Heegaard Splittings* of closed 3-manifolds (4.4, 4.5).

Finally in § 5, two conjectures are stated and interesting relations of the conjectures with the Poincaré and Schoenflies conjectures are discribed (5.4).

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§ 2. Elementary properties of ball coverings.

In this section we will discuss the elementary properties of ball coverings. First, followings are well known results on (weak) ball coverings of closed manifolds.

2.1. PROPOSITION. *Suppose M^n is a closed n-manifold, then*

(1) *$b(M)=2$ if and only if $M^n \cong S^n$,*

(2) *$\beta(M)=2$ if and only if $M^n \cong S^n$ $(n \neq 4)$.*

For $n=4$, (2) of (2.1) is unknown, we have only tnat M^4 is topologically homeomorphic to S^4 if and only if $\beta(M^4)=2$.

2.2. PROPOSITION [Zeeman, **12**, p. 200]. *Suppose M^n is a closed n-manifold and $r \geqq 2$, then $\beta(M) \leqq r$ if*

(1) *M is geometrically $\left[\frac{n}{r}\right]$-connected, or*

(2) *M is* $\left[\frac{n}{r}\right]$*-connected and* $(n, r)\neq(3, 2), (3, 3), (4, 2)$,

where M *is said to be geometrically* q*-connected if any* k*-dimensional subpolyhedron of* M *is contained in an* n*-ball of* M *for any* $k\leqq q$.

2.3. LEMMA. *Suppose there are two balls* B_i *and* B_j *of a (weak) ball covering* $\mathcal{B}=\{B_k\}$ *of a manifold* M *with* $B_i\cap B_j=\emptyset$, *then there exists another (weak) ball covering* $\mathcal{B}'$ *of* M *with* $\sharp\mathcal{B}'<\sharp\mathcal{B}$.

PROOF. Assume $i<j$. There is an arc γ in $\cup\{\partial B_k\,;\,k\neq i, j\}$ so that $\gamma\cap B_m=\partial\gamma\cap\partial B_m=a_m$ is a point, $m=i, j$ (for the weak case, it is not required that $\gamma\subset\cup\{\partial B_k\,;\,k\neq i, j\}$). Let $B_i'=B_i\cup B_j\cup N(\gamma\,;\,M)$, where $N(\gamma\,;\,M)$ is a small regular neighborhood of γ in M, and let $B_k'=B_k$ if $i\neq k<j$ and $B_k'=B_{k+1}$ if $k>j$. Hence $\mathcal{B}'=\{B_k'\}$ is a weak ball covering of M with $\sharp\mathcal{B}'=\sharp\mathcal{B}-1$. Suppose now that $\mathcal{B}$ is a ball covering, let $B_i''=B_i'$, $B_k''=Cl(B_k-B_i')$ if $i\neq k<j$ and $B_k''=Cl(B_{k+1}-B_i')$ if $k>j$. Hence $\mathcal{B}''=\{B_k''\}$ is a ball covering of M with $\sharp\mathcal{B}''=\sharp\mathcal{B}-1$.

2.4. COROLLARY. *If* $\mathcal{B}$ *is a minimal (weak) ball covering of a manifold* M, *then* $B_i\cap B_j\neq\emptyset$ *for any elements* B_i*'s of* $\mathcal{B}$.

Now, from (2.3), we can study a relation among the numbers $b(M_1)$, $b(M_2)$ and $b(M_1\sharp M_2)$ for two n-manifolds M_1, M_2 and their connected sum $M_1\sharp M_2$. The equality in the following will be discussed in § 5.

2.5. THEOREM. $b(M_1\sharp M_2)\leqq\max.\{b(M_1), b(M_2)\}$.

PROOF. Let $\mathcal{B}_i=\{B_{i,1}, B_{i,2}, \cdots, B_{i,b_i}\}$ be a minimal ball covering of the manifold M_i, $i=1, 2$. We may assume $b_1\leqq b_2$ without loss of generality. Choose a point p_i in $\mathrm{Int}(B_{i,1}\cap B_{i,2})=\mathrm{Int}(\partial B_{i,1}\cap\partial B_{i,2})$ and let $N_i=N(p_i\,;\,M_i)$ be a small regular neighborhood of p_i in M_i, $i=1, 2$. Let $h:\partial N_1\to\partial N_2$ be a homeomorphism of two $(n-1)$-spheres ∂N_1 and ∂N_2 so that $h(\partial N_1\cap B_{1,k})=\partial N_2\cap B_{2,k}$, $k=1, 2$. Hence the manifold

$$M=(M_1-\mathring{N}_1)\cup(M_2-\mathring{N}_2)/h$$

is regarded as $M_1\sharp M_2$. Then

$$B_k=(B_{1,k}-\mathring{N}_1)\cup(B_{2,k}-\mathring{N}_2)/(h|\partial N_1\cap B_{1,k})\cong B^n\,,\qquad k=1, 2\,.$$

Hence $\mathcal{B}^{(1)}=\{B_1, B_2, B_{1,3}, \cdots, B_{1,b_1}, B_{2,3}, \cdots, B_{2,b_2}\}$ is a ball covering of M and $\sharp\mathcal{B}^{(1)}=b_1+b_2-2$. Since $B_{1,j}\cap B_{2,k}=\emptyset$ for any j, $k\geqq 3$, we can apply the operation of (2.3) for $B_{1,3}$ and $B_{2,3}$, so that one get an induced ball covering $\mathcal{B}^{(2)}=\{B_1', B_2', B_3', B_{1,4}', \cdots, B_{1,b_1}', B_{2,4}', \cdots, B_{2,b_2}'\}$ with $\sharp\mathcal{B}^{(2)}=b_1+b_2-3$. It is noted that $B_{1,j}'\cap B_{2,k}'=\emptyset$ for any j, $k\geqq 4$. Repeating this process, finally we obtain a ball covering $\mathcal{B}^*=\{B_1^*, B_2^*, \cdots, B_{b_1}^*, B_{2,b_1+1}^*, \cdots, B_{2,b_2}^*\}$ with $\sharp\mathcal{B}^*=b_2=\max.(b_1, b_2)$. This completes the proof.

Following shows a relation between the number $b(M)$ and the dimension of a manifold M.

2.6. Theorem. $b(M^n)\leqq n+1$, *for any n dimensional manifold* M^n.

To calculate $b(M)$ for a manifold M the following is essential, from which (2.6) is obtained.

Let M^n be an n-manifold and

$$M=h^0\cup\bigcup_{k_1} h_{k_1}^{p_1}\cup\cdots\cup\bigcup_{k_m} h_{k_m}^{p_m}$$

be a handle decomposition of M and $h_{k_i}^{p_i}$'s are mutualy disjoint handles of index p_i on ∂M_{i-1}, where $1\leqq p_1<\cdots<p_m$, $M_0=h^0$, a 0-handle, and $M_i=h^0\cup\bigcup h_{k_1}^{p_1}\cup\cdots\cup\bigcup h_{k_i}^{p_i}$, $M=M_m$, $i=1, 2, \cdots, m$.

Hence we have,

2.7. Theorem. *Suppose that an n-manifold* M^n *has a handle decomposition as above, then* $b(M)\leqq m+1$.

Proof. We will prove the theorem by the induction on $i=0, 1, \cdots, m$. Since $M_0=h^0$ is itself an n-ball, $b(M_0)=1$ and the theorem is true for $i=0$. Assume the theorem is true for any $i\leqq t-1$ and now suppose $i=t$. By the assumption, $b(M_{t-1})=r\leqq t$ and let $\mathscr{B}=\{B_1, \cdots, B_r\}$ be a minimal ball covering of M_{t-1}. Let $\{h_1^{p_t}, \cdots, h_s^{p_t}\}$ be a set of all p_t-handles of the decomposition. Choose $(s-1)$ proper arcs $\{\gamma_i\}$ in $Cl\{\bigcup_{k=1}^{r}\partial B_k-\bigcup_j h_j^{p_t}\}$ so that $\gamma_j\cap\gamma_k=\emptyset$ (if $j\neq k$) and $(\bigcup_i\gamma_i\cup\bigcup_j h_j)$ is connected. Then $B_0'=N(\bigcup_{j=1}^{s-1}\gamma_j\,;M_{t-1})\cup\bigcup_{j=1}^{s}h_j^{p_t}$ is an n-ball. Set $B_i'=Cl(B_i-B_0')$, $i=1, 2, \cdots, r$, hence $\mathscr{B}'=\{B_0', B_1', \cdots, B_r'\}$ is a ball covering of $M_t=M_{t-1}\cup\bigcup_{j=1}^{s}h_j^{p_t}$ and $b(M_t)\leqq r+1\leqq t+1$. Hence the proof is completed.

Combining (2.3) with the above, we have,

2.8. Corollary. *Suppose that* M^n *has a handle decomposition as above. If any* p_i*-handle is disjoint from the all* p_j*-handles for some* i, j, $i\neq j$, *then* $b(M)\leqq m$.

2.9. Example. $b(S^p\times S^q)=b(S^1\times_\tau S^q)=3$, for any integers p, $q\geqq 1$, where $S^1\times_\tau S^q$ means a twisted S^q bundle over S^1.

$S^p\times S^q$ is constructed from four handles of indices $0, p, q$ and $p+q$, where one can choose the p-handle disjoint from the q-handle by the general position argument. Hence by (2.8) $b(S^p\times S^q)\leqq 3$. But $b(S^p\times S^q)\nleqq 2$ by (2.1). $b(S^1\times_\tau S^q)=3$ is obtained analogously.

For some manifold M, an upper bound of $b(M)$ can be obtained from the homology groups of M, by (2.7), as follow.

2.10. Corollary [Smale, **9**]. *Suppose* M^n *is a closed* 1*-connected n-manifold,* $n\geqq 6$ *and* $H_k(M)$ *is free for any k, then* $b(M)\leqq\#\{p\mid H_p(M)\neq 0\}$.

2.11. Corollary. *Let* M^n *be an n-manifold.*

(1) *If M has a k-dimensional spine;* $M\searrow K^k$, *then* $b(M)\leqq k+1$.

(2) *If M has the non-empty boundary then* $b(M)\leqq n$.

(3) *If M is closed and $Cl(M-B^n)$ has a k-dimensional spine, then $b(M)\leqq k+2$, where B^n is an n-ball in M.*

PROOF. Suppose $M^n\searrow K^k$, then M has a decomposition in which any handle has index $\leqq k$ [4]. Hence (1) is obtained from (2.7). (2) and (3) are direct consequences of (1).

Now, theorem (2.6) is obtained as a corollary of (1) of (2.11).

REMARK. (2.5) and (2.6) were proved from the weak version by Mielke [**5**].

§ 3. Homology spheres and contractible manifolds.

Homology sphere M has a special property with respect to $b(M)$. To show this property the following lemma is required, which is useful in the later part of this paper.

3.1. LEMMA. *Suppose $\{B_1, B_2\}$ is a ball covering of an n-manifold M^n, then*

(1) *$\pi_1(M)$ is a free group of rank$=\operatorname{rank}\tilde{H}_0(B_1\cap B_2)=\#(B_1\cap B_2)-1$, where $\tilde{H}_*(X)$ means the reduced homology group of X,*

(2) *$H_i(M)\cong H_{i-1}(B_1\cap B_2)=H_{i-1}(\partial B_1\cap\partial B_2)$ for $i=2, 3, \cdots, n$,*

(3) *$H_2(M^n)$ is free abelian for $n\leqq 4$, and the generators of $H_2(M^n)$ are represented by 2-spheres which are the suspension of generators for $H_1(\partial B_1\cap\partial B_2)$.*

PROOF. Let p_i be a center of B_i, then $B_i\searrow p_i*(B_1\cap B_2)=p_i*(\partial B_1\cap\partial B_2)$ ($*$ means "join"), $i=1, 2$. Hence $M=B_1\cup B_2\searrow\Sigma(\partial B_1\cap\partial B_2)=\Sigma(B_1\cap B_2)$, where $\Sigma(X)$ means the suspension of X, and (1) and (2) are proved. For (3) it is noted that any connected compact 3-submanifold X of S^3 is homeomorphic to the complement of the interior of some solid tori in S^3 [**1**]. Hence from the Alexander duality $H_1(\partial B_1\cap\partial B_2)$ is free abelian for $n=4$. This is obvious for $n\leqq 3$. Generators for $H_1(\partial B_1\cap\partial B_2)$ are represented by simple loops in $\operatorname{Int}(\partial B_1\cap\partial B_2)$. These and above observation complete the proof.

3.2. THEOREM. *Suppose M^n is a homology n-sphere and $b(M)\leqq 3$, then $M\cong S^n$.*

PROOF. Since the theorem is obvious for $n=1, 2$, suppose $n\geqq 3$. We may assume $b(M)=3$ by (2.1) and let $\{B_0, B_1, B_2\}$ be a minimal ball covering of M. $H_k(M-\mathring{B}_0)=H_k(B_1\cup B_2)\cong\tilde{H}_{k-1}(B_1\cap B_2)=\tilde{H}_{k-1}(\partial B_1\cap\partial B_2)\cong 0$, by (3.1), for $k\geqq 1$.

Then, from the Poincaré duality, $\partial(B_1\cap B_2)=\partial(\partial B_1\cap\partial B_2)$ is a homology $(n-2)$-sphere. If $n\leqq 4$, $n-2\leqq 2$ and $S^{n-2}\cong\partial(B_1\cap B_2)\subset\partial B_1\cong S^{n-1}$. So, $(B_1\cap B_2)\cong B^{n-1}$ by the theorem of Schoenflies, for $n-1\leqq 3$. Hence $B_1\cup B_2\cong B^n$ and $M^n\cong S^n$ for $n=3, 4$. Since $\pi_1(B_1\cup B_2)\cong\pi_1(\Sigma(B_1\cap B_2))\cong 0$, by (3.1), M^n is a homotopy n-sphere and $M^n\cong S^n$ from the generalized Poincaré theorem for $n\geqq 5$ [**12**].

3.3. COROLLARY. *Suppose M^n is an acyclic n-manifold with $b(M)\leqq 2$ and*

$n \leqq 4$, then $M^n \cong B^n$.

PROOF. It is sufficient to prove for $n=3$ and 4. Assume that $M^n = B_1^n \cup B_2^n$ and $B_1 \cap B_2 = \partial B_1 \cap \partial B_2$ is an $(n-1)$-manifold. Since $n \leqq 4$, by the same arguments as the proof of (3.2), $B_1 \cap B_2 \cong B^{n-1}$. Hence $M^n \cong B^n$.

(3.3) does not hold for $n \geqq 5$ as follow.

3.4. THEOREM. *For any integer $n \geqq 5$, there exists a contractible n-manifold M^n with $b(M)=2$.*

PROOF. First of all we note some homology spheres embedded in the spheres. For any integer $n \geqq 5$, there exists a homology $(n-2)$-sphere H^{n-2} in S^{n-1} which separates S^{n-1} into two manifolds U^{n-1} and V^{n-1} such that $U \cup V = S$, $U \cap V = \partial U = \partial V = H^{n-2}$, $\pi_1(V) \neq 0$ and U is contractible, from Newman [8] for $n \geqq 6$ and from Neuzil [7] for $n=5$.

Let $h_i : S^{n-1} \to \partial B_i^n$ be a homeomorphism and let $h_i(U) = U_i$, $h_i(V) = V_i$, for $i=1, 2$. Then the manifold $M^n = B_1 \cup B_2 / (h_2 h_1^{-1} | U_1)$ is obviously contractible by (3.1). Since $\partial B_1 - \mathring{U}_1 \cong \partial B_2 - \mathring{U}_2 \cong V$ and V_1 and V_2 are attached trivially by $(h_2 h_1^{-1} | \partial V_1)$, ∂M is regarded as the double of V. Hence $\pi_1(\partial M) \neq 0$ and $M^n \neq B^n$. $b(M)=2$ is obvious from the construction of M.

On the other side, in early papers [2, 3], Glaser constructed following examples with respect to weak ball covering.

3.5. PROPOSITION [Glaser, 2, 3]. *For any integer $n \geqq 4$, there is a contractible n-manifold M^n with $\beta(M)=2$.*

For Glaser's example M^n, $n \geqq 5$, it is not difficult to see $b(M)=2$. Most interesting is in 4-dimension as follow.

3.6. COROLLARY. *There exists a contractible 4-manifold M^4 with $\beta(M)=2$ but $b(M)=3$.*

PROOF. Let M^4 be the Glaser's contractible 4-manifold with $\beta(M)=2$, in (3.5), [3]. Since M has a 2-dimensional spine, $b(M) \leqq 3$ by (2.11). If $b(M) \leqq 2$, $M \cong B^4$ from (3.3) and $\beta(M)=1$. Hence $b(M)=3$.

REMARK. Let M^n be a contractible n-manifold. From the handle cancelling argument, we have $b(M^n) \leqq 3$ for $n \geqq 5$.

We will close this section by showing a weak relation between homotopy 4-sphere and ball coverings.

3.7. PROPOSITION. *If M^4 is a homotopy 4-sphere, then $b(M \# k(S^2 \times S^2)) \leqq 3$ for some integer $k \geqq 0$, where $0(S^2 \times S^2) = S^4$ and $k(S^2 \times S^2)$ means the connected sum of k copies of $S^2 \times S^2$ if $k \geqq 1$.*

PROOF. By Munkres [6] M^4 has a differentiable structure and by Wall [10] $M \# k(S^2 \times S^2)$ is diffeomorphic to $k(S^2 \times S^2)$ for some $k \geqq 0$. On the other hand $b(k(S^2 \times S^2)) \leqq 3$ for any $k \geqq 0$ from (2.5) and (2.9).

§ 4. Ball coverings of 3-manifolds.

Let $B_1, B_2, \cdots, B_k$ be a finite number of mutually disjoint 3-balls in the interior of a 3-ball B. A 3-manifold M is called a k-punctured 3-ball if $M \cong Cl(B-\bigcup_{i=1}^{k} B_i)$.

4.1. LEMMA. *Let C_1 and C_2 be punctured 3-balls and F_i^2 be a non-empty 2-manifold (may not be connected), which contains no 2-sphere component, in ∂C_i, $i=1, 2$. Suppose $h: F_1 \to F_2 \subset \partial C_2$ is a homeomorphism, then the 3-manifold $M^3=C_1 \cup C_2/h$ is a punctured 3-ball with 1-handles (possibly non-orientable).*

PROOF. It is noted that $F_1(\cong F_2)$ consists of a finite number of connected bounded 2-manifolds of genus 0 since $F_1 \subset \partial C_1$.

The proof shall proceed by the induction on $\beta_0(F_1)+\beta_1(F_1)$, where $\beta_i(F_1)$ means the i-th betti number of F_1. Since $F_1 \neq \emptyset$, $\beta_0(F_1) \geqq 1$ and M is connected. Suppose $\beta_1(F_1)=0$, then F_1 consists of finite number of disjoint 2-balls and the lemma is trivial. Suppose the lemma is true for $\beta_0(F_1)+\beta_1(F_1) \leqq n-1$, $n \geqq 2$, and assume now $\beta_0(F_1)+\beta_1(F_1)=n$. If one of the components, say D^2, of F_1 is a 2-ball, then let $M^*=C_1 \cup C_2/(h|F_1-D) \subset M$. $\beta_0(F_1-D)+\beta_1(F_1-D)=\beta_0(F_1)-1+\beta_1(F_1)=n-1$. By the inductive hypothesis, M^* is a punctured 3-ball with 1-handles. Since M is obtained from M^* by attaching disjoint 2-balls D and $h(D)$ on ∂M^*, M is also a punctured 3-ball with 1-handles. Now we may assume that no component of F_1 is a 2-ball. Hence at least one component, say S_1^1, of ∂F_1 must bound a 2-ball D_1 in $Cl(\partial C_1-F_1)$. Let $N(D_1; C_1)$ be a regular neighborhood of D_1 in C_1 such that $N(D_1; C_1) \cap \partial C_1=D^*$ is a regular neighborhood of D_1 in ∂C_1. Since $D^*-\mathring{D}_1 \subset F_1$ is an annulus and ∂C_2 consist of 2-spheres, then $C_2'=C_2 \cup N(D_1; C_1)/(h|D^*-\mathring{D}_1)$ is also a punctured 3-ball as C_2. And $C_1'=Cl(C_1-N(D_1; C_1)) \cong C_1$. Let $F_1'=(F_1-\mathring{D}^*) \cup (\partial N(D_1; C_1)-\mathring{D}^*)$ and let $h': F_1' \to \partial C_2'$ be the homeomorphism defined by $h'|F_1-\mathring{D}^*=h$ and $h'|(\partial N(D_1; C_1)-\mathring{D}^*)=1$. Hence $M \cong C_1' \cup C_2'/h'$, $\beta_0(F_1')=\beta_0(F_1)$ and $\beta_1(F_1')=\beta_1(F_1)-1$. From the inductive hypothesis M is a punctured 3-ball with 1-handles.

4.2. COROLLARY. *Let M^3 be a 3-manifold with non-empty boundary. Then M is a punctured 3-ball with 1-handles if and only if $b(M)=2$. Moreover such a manifold M is embeddable into $k(S^1 \times S^2) \# (S^1 \times_\tau S^2)$ for some $k \geqq 0$, where $S^1 \times_\tau S^2$ means the twisted S^2 bundle over S^1.*

PROOF. Suppose M^3 is a 3-manifold with $b(M)=2$ and $\partial M \neq \emptyset$, then there is a ball covering $\{B_1, B_2\}$ of M such that $M=B_1 \cup B_2$ is a punctured 3-ball with 1-handles by (4.1). Conversely let M^3 be a p-punctured 3-ball with q handles of index 1. It is sufficient to complete the proof that we construct a ball covering $\{B_1^*, B_2^*\}$ of M in $k(S^1 \times S^2) \# (S^1 \times_\tau S^2)$ for some $k \geqq 0$.

Let $M_1=B_1^3 \cup B_2^3$ be a 3-manifold homeomorphic to a 3-ball so that $B_1 \cap B_2$

$=\partial B_1 \cap \partial B_2 = D^2$ is a 2-ball and let $x_1, x_2, \cdots, x_p$ be distinct p points in $\mathrm{Int}\, D$. Then $M_2 = Cl(M_1 - \bigcup_{i=1}^{p} N(x_i; M_1))$ is a p-punctured 3-ball, where $N(x_i; M_1)$ is a small regular neighborhood of x_i in M_1 such that $\mathrm{Int}\, D \cap N(x_i; M_1)$ is a 2-ball, $i=1, 2, \cdots, p$. Denote $\partial M_1 = S_0$ and $\partial N(x_i; M_1) = S_i \subset \partial M_2$, $i=1, 2, \cdots, p$.

Let S^2_* be a nontrivial 2-sphere in $S^1 \times_\tau S^2$ such that $S^1 \times_\tau S^2 - S_* \cong \mathring{I} \times S^2$. Choose $p+1$ mutually disjoint 3-balls $C_0, \cdots, C_p$ in $S^1 \times S^2 - S_*$. Let $h: S_0 \cup S_1 \cup \cdots \cup S_p \to \partial C_0 \cup \partial C_1 \cup \cdots \cup \partial C_p$ be an orientation coherent homeomorphism of 2-spheres. Hence $W = M_2 \cup (S^1 \times_\tau S^2 - \bigcup_{i=0}^{p} \mathring{C}_i)/h \cong p(S^1 \times S^2) \# (S^1 \times S^2)$ and $W - S_*$ is orientable. Since M is a p-punctured 3-ball with q handles of index 1, M is obtained from M_2 by attaching 1-handles on ∂M_2. Suppose $h_i(j, k)$ is a 1-handle of M on ∂M_2 from S_j to S_k (may $j=k$), we can take an arc γ_i, proper in $(S^1 \times_\tau S^2 - \bigcup_{i=0}^{p} \mathring{C}_i)$, from a point of $\mathrm{Int}(S_j \cap B_1)$ to a point of $\mathrm{Int}(S_k \cap B_2)$. If $h_i(j, k)$ is an orientable (or non-orientable) handle then we choose γ_i so that $\gamma_i \cap S_* = \emptyset$ (or $\gamma_i \cap S_*$ is just a crossing point, respectively) and $\gamma_i \cap \gamma_j = \emptyset$ if $\gamma_i \neq \gamma_j$, for all 1-handles.

Hence we have $M_2 \cup \bigcup_{i=1}^{q} N(\gamma_i; W) \cong M$, where $N(\gamma_i; W)$ is a small regular neighborhood of γ_i in W. Now set $B_1^* = B_1 \cup \bigcup_{i=1}^{q} N(\gamma_i; W) \cong B^3$ and $B_2^* = B_2$. Then $\{B_1^*, B_2^*\}$ is a ball covering of M, a punctured 3-ball with 1-handles.

Now, from (4.2) we can show a complete classificasion of closed 3-manifold M with $b(M)=3$.

4.3. THEOREM. *Suppose M^3 is a closed 3-manifold, then $b(M)=3$ if and only if $M \cong k(S^1 \times S^2) \# \varepsilon(S^1 \times_\tau S^2)$ for some $k+\varepsilon \geqq 1$ and $\varepsilon=0$ or 1 according to the orientability of M.*

PROOF. Suppose $b(M^3)=3$ and $\{B_0, B_1, B_2\}$ is a ball covering of M. From (4.2) we may assume $M - \mathring{B}_0 = B_1 \cup B_2 \subset W = k'(S^1 \times S^2) \# (S^1 \times_\tau S^2)$ for some sufficiently large k'. Let $\{S_0, S_1, \cdots, S_{k'}\}$ be a set of mutually disjoint 2-spheres in W so that $Cl(W - \bigcup_{i=0}^{k'} N(S_i; W))$ is a $(2k'+1)$-punctured 3-ball, where $N(S_i; W)$ is a regular neighborhood of S_i in W.

If $\partial B_0 \cap \bigcup_{i=0}^{k'} S_i = \emptyset$, cut W along S_i for all $S_i \subset W - (B_1 \cup B_2)$ and paste 3-balls to its boundary 2-spheres. Then one obtains closed 3-manifold $W' = k(S^1 \times S^2) \# \varepsilon(S^1 \times_\tau S^2) \cong M$, for $k+\varepsilon \leqq k'+1$ and $\varepsilon=0$ or 1. Since $b(M)=3$, $k+\varepsilon \geqq 1$ by (2.1). If $\partial B_0 \cap \bigcup_{i=0}^{k'} S_i \neq \emptyset$, we can assume the intersection consists of a finite number of mutually disjoint simple loops. Then one can change S_i's by the standard "cutting and glueing technique" of eliminating the intersection, inductively, starting from an inner-most loop. Hence finally we get a new set $\{S_0', \cdots, S_{k'}'\}$

of 2-spheres in W with $\partial B_0 \cap \bigcup_{i=0}^{k'} S_i' = \emptyset$.

The converse of the theorem is trivial from (2.9) and (2.5).

The rest of the section is devoted to study of a relation between the ball coverings and Heegaard splittings of 3-manifolds. Next is one step for this. Here, it is noted that for a closed 3-manifold M^3, $b(M)=4$ if $M \not\cong k(S^1 \times S^2) \# \varepsilon(S^1 \times_\tau S^2)$ for any $k \geqq 0$ and $\varepsilon = 0, 1$, $(O(S^1 \times S^2) = O(S^1 \times_\tau S^2)$ means a 3-sphere $S^3)$, by (2.1), (2.6) and (4.3). M^3 is said to be trivial if $M \cong S^3$.

4.4. THEOREM. *Let M^3 be a nontrivial orientable closed 3-manifold with $H_2(M)=0$ and let $\{B_1, B_2, B_3, B_4\}$ be a ball covering of M. Then $B_i \cup B_j$ is a solid torus for any $i \neq j$.*

PROOF. First note that $H_2(B_1 \cup B_2 \cup B_3) \cong H_2((B_1 \cup B_2) \cap B_3) \cong 0$, because, $B_1 \cup B_2 \cup B_3 = M - \mathring{B}_4$, $(B_1 \cup B_2) \cap B_3$ consists of finite number of bounded 2-manifolds of genus 0 and $B_i \cap B_j \neq \emptyset$ for any $i, j = 1, \cdots, 4$. Then from the Mayer-Vietoris exact sequence;

$$H_2(B_1 \cup B_2) \cap B_3) \longrightarrow H_2(B_1 \cup B_2) + H_2(B_3) \longrightarrow H_2(B_1 \cup B_2 \cup B_3),$$

it follows that $H_2(B_1 \cup B_2) = 0$. From (3.1) $H_1(B_1 \cap B_2) \cong H_2(B_1 \cup B_2) = 0$, this means that $B_1 \cap B_2 = \partial B_1 \cap \partial B_2$ consists of finite number of 2-balls. Since M is orientable and $B_1 \cup B_2$ is. These arguments are free from the indices of 3-balls $\{B_i\}$. Hence $(B_i \cup B_j)$ is a solid torus for any $i \neq j$, and $(M; B_{i_1} \cup B_{i_2}, B_{i_3} \cup B_{i_4})$ represents a Heegaard Splitting of M if $i_j \neq i_k$ for $j \neq k$.

4.5. THEOREM. *Let M^3 be a nontrivial orientable closed 3-manifold with $\pi_2(M)=0$ and let $\{B_1, B_2, B_3, B_4\}$ be a ball covering of M. Then $B_i \cup B_j$ is a solid torus for any $i \neq j$.*

PROOF. Let $W = M - \mathring{B}_4 = B_1 \cup B_2 \cup B_3$ and $V = Cl(W - B_3) = B_1 \cup B_2$. It is noted $V \cap B_3 = \partial V \cap \partial B_3 \neq \partial B_3$. For, if $V \cap B_3 = \partial B_3$, $B_3 \subset \operatorname{Int} W$ and $B_3 \cap B_4 = \emptyset$, this contradicts (2.4). Then from the diagram,

$$\begin{array}{ccccccc}
0 = H_2(V \cap B_3) & \longrightarrow & H_2(V) & \longrightarrow & H_2(V, V \cap B_3) & \longrightarrow & H_1(V \cap B_3) \\
\downarrow & & \downarrow & & \downarrow \wr\| & & \downarrow \\
0 = H_2(B_3) & \longrightarrow & H_2(W) & \xrightarrow{\cong} & H_2(W, B_3) & \longrightarrow & H_1(B_3) = 0,
\end{array}$$

it follows that $i_* : H_2(V) \to H_2(W)$ is a monomorphism, where i_* is an induced homomorphism from the inclusion $i: V \to W$. From (3.1), $H_2(V) = H_2(B_1 \cup B_2) \cong Z^k = Z + Z + \cdots + Z$, free abelian, $k \geqq 0$ and whose generators can be represented by 2-spheres $S_1, \cdots, S_k$ in Int V.

Now, by $\pi_2^*(W)$ we will indicate $\pi_2(W)$ as a $J\pi_1$-module [**11**], where J is the ring of integers. Since $\pi_2(M) = \pi_2(W \cup B_4) = 0$ and $W \cap B_4 = \partial B_4 = \partial W$, ∂W represents a generator $[\partial W]$ of $\pi_2^*(W)$. Regarding $[S_i]$, represented by $i(S_i) = S_i$ in W, an element of $\pi_2^*(W)$, $[S_i] = \lambda[\partial W]$, where $\lambda \in J\pi_1(W)$. On the other hand $\partial W \sim 0$ in W, so $S_i \sim 0$ in W by the Hurewicz homomorphism. This contradicts for i_* to be injective. Hence $k=0$ and $H_2(V)=H_2(B_1 \cup B_2)=0$. It means that $B_i \cup B_j$ is a solid torus for $i \neq j$, as (4.4).

4.6. COROLLARY. *Suppose M^3 is a nontrivial orientable, closed and irreducible 3-manifold and let $\{B_1, B_2, B_3, B_4\}$ be a ball covering of M. Then $B_i \cup B_j$ is a solid torus for any $i \neq j$.*

§5. Some relations with other problems.

5.1. CONJECTURE $B(n, m)$. *Suppose M^n is a closed n-manifold with $\beta(M) \leqq m \leqq n+1$, then it will be follow that $b(M) \leqq m$.*

5.2. CONJECTURE $C(n)$. *It will be true that $b(M_1 \# M_2)=\max.(b(M_1), b(M_2))$ for any closed n-manifolds M_1 and M_2.*

In (5.1) the closedness is essential, for (3.6) is a counter example for $B(4.2)$. It is trivial that $B(n, 2)$ $(n \neq 4)$, $B(n, n+1)$ and $C(2)$ are true for any n by the definition of (weak) ball covering and by (2.1).

5.3. $C(3)$ *is true.*

PROOF. By (2.5) it is sufficient to show $b(M_1 \# M_2) \nless \max.(b(M_1), b(M_2))$ when $b(M_1 \# M_2)=3$. If $b(M_1 \# M_2)=3$, $M_1 \# M_2 \cong k(S^1 \times S^2) \# \varepsilon(S^1 \times_\tau S^2)$ from (4.4) for some $k+\varepsilon \geqq 1$ and $\varepsilon=0$ or 1. By the same arguments in the proof of (4.4), we get $M_i \cong k_i(S^1 \times S^2) \# \varepsilon_i(S^1 \times_\tau S^2)$, where $k_1+k_2+\varepsilon_1+\varepsilon_2=k+\varepsilon$ and $\varepsilon_i=0$ or 1, $i=1, 2$. Hence from (4.4), $b(M_i) \leqq 3$, $i=1, 2$.

5.4. *Denote by $P(4)$ the Poincaré conjecture of 4-dimension;* any homotopy 4-sphere will be a 4-sphere, *and denote by SC the Schoenflies conjecture;* any $(n-1)$-sphere S^{n-1} will bound an n-ball in S^n for $n \geqq 4$.

Then the following diagram is obtained.

$$\begin{tikzcd}[row sep=small] B(4,3) \arrow[dr, "\text{(i)}"] & & & \\ & P(4) \arrow[r, "\text{(iii)}"'] & SC \arrow[r, "\text{(iv)}"'] & B(4,2). \\ C(4) \arrow[ur, "\text{(ii)}"'] & & & \end{tikzcd}$$

Since (iii) is well known, we will show (i), (ii) and (iv).

(i) Suppose M^4 is a homotopy 4-sphere, $\beta(M) \leqq 3$ from (2.2) for the case $n=4$ and $r=3$. If $B(4, 3)$ is true then $b(M) \leqq 3$ and since M is a homology sphere $M \cong S^4$ by (3.2). Hence (i) was proved.

(ii) Let M^4 be a homotopy 4-sphere, then $M \# k(S^2 \times S^2) \cong k(S^2 \times S^2)$ and $b(M \# k(S^2 \times S^2))=b(k(S^2 \times S^2)) \leqq 3$ for some $k \geqq 0$ from (3.7). If $C(4)$ is true, $b(M) \leqq 3$. Therefore $M \cong S^4$ by (3.2).

(iv) Suppose M^4 is a closed 4-manifold with $\beta(M)=2$ and $\{B_1, B_2\}$ is a weak ball covering of M; $M=B_1 \cup B_2$. We may assume $\partial B_1 \subset \operatorname{Int} B_2$ by the collar of the boundary of B_2. If Sc is true, $Cl(B_2-B_1)=B_0$ is a (PL) 4-ball. Hence $M=B_0 \cup B_1$, $B_0 \cap B_1=\partial B_0=\partial B_1$ and $M \cong S^4$. That is $b(M)=2$. (The authors do not know whether the converse of (iv) is true.)

It is noted that if $B(4, 2)$ is false then there exists a closed 4-manifold M^4 (topologically 4-sphere) such that $\beta(M)=2$ but $b(M)=4$ or 5 by (3.2).

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