J. Math. Soc. Japan
Vol. 28, No. 1, 1976

Convergence of difference approximation of nonlinear evolution equations and generation of semigroups

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(Received Feb. 1, 1975)

In this paper we study the question of existence of solutions (in some generalized sense) of a nonlinear differential equation

(DE) $$(d/dt)u(t) \in Au(t)\,, \qquad 0<t<T$$

in a Banach space X. Here A is a multi-valued dissipative operator in X. The solutions of (DE) will be constructed through difference approximation. Furthermore, we deal with the generation problem of semigroups of nonlinear contractions.

This work is motivated by the recent results in Bénilan [1] and in Kenmochi and Oharu [6]. Among others, Bénilan ([1]) introduces notions of "integral solution" and of "bonne solution" of evolution equation and investigates the properties of bonne solutions. On the other hand, Kenmochi and Oharu ([6]) introduce a notion of "weak solution" of evolution equation and discuss the construction of weak solutions from the viewpoint of difference approximation.

In the present paper we formulate an approximating difference scheme for the Cauchy problem for (DE); the difference scheme is of more general form than that treated in [6]. Our first purpose is then to determine conditions under which the solution of the difference scheme converges in an appropriate sense to some function, which may be regarded as a "solution" of (DE); in fact, it is already proved in [1] that the limit function is a bonne solution of (DE). We shall also see that the conditions given are necessary for the convergence in an analoguous sense to the equivalence theorem in the so-called finite-difference method. The proof of the convergence is based on the technique in [1] and is different from those in [1] and in [6].

In recent years there appeared many works on the generation of semigroups of nonlinear operators, for instance, see Brezis and Pazy [2], Crandall and Liggett [3], Kato [4], [5], Kōmura [7], Martin [8], Miyadera [10] and Oharu [11]. Our second purpose is to construct semigroups of nonlinear contractions by applying the results on the convergence of the difference approximation. We also exhibit some sufficient conditions which assure the construction of semigroups of nonlinear contractions and contain the ones treated in the above cited papers. Thus we see that the results obtained here extend most of the known generation theorems to the case of general Banach space.

The author would like to express his hearty thanks to Professor I. Miyadera for many valuable advices.

§ 1. Preliminaries.

Throughout this paper X denotes a (real) Banach space with the norm $\|\cdot\|$. By an *operator* A *in* X, we mean a multi-valued operator with the domain $D(A)$ and the range $R(A)$ in X, that is, A assigns to each $x\in X$ a subset Ax of X; $D(A)$ is the set $\{x\in X; Ax\neq\emptyset\}$ and $R(A)=\bigcup_{x\in D(A)} Ax$. An operator A in X may be identified with the set $\{[x, y]; x\in D(A), y\in Ax\}$ contained in $X\times X$. For each $x\in D(A)$, we write

$$|||Ax|||=\inf\{\|y\|; y\in Ax\}.$$

We denote by $\langle x, x^*\rangle$ the natural pairing between $x\in X$ and $x^*\in X^*$, where X^* denotes the dual space of X. We use the following notations (see [3]):

$$\langle y, x\rangle_s=\sup\{\langle y, x^*\rangle; x^*\in F(x)\}$$

$$\langle y, x\rangle_i=\inf\{\langle y, x^*\rangle; x^*\in F(x)\}, \qquad x, y\in X,$$

where the *duality map* F from X into X^* is defined by

$$F(x)=\{x^*\in X^*; \langle x, x^*\rangle=\|x\|^2=\|x^*\|^2\}, \qquad x\in X.$$

It is shown in [3] that the function $\langle\cdot, \cdot\rangle_s$ from $X\times X$ into R^1 is upper semi-continuous with respect to the strong topology of $X\times X$. Here R^n denotes n-dimensional euclidean space. For other properties of $\langle\cdot, \cdot\rangle_s$ and $\langle\cdot, \cdot\rangle_i$, we refer to [1] and [3].

An operator A in X is said to be *dissipative* if for every $[x_1, y_1]$, $[x_2, y_2]\in A$,

$$\langle y_1-y_2, x_1-x_2\rangle_i\leqq 0.$$

It is known (see [4]) that A is dissipative if and only if for every $[x_1, y_1]$, $[x_2, y_2]\in A$ and for every $\lambda>0$,

$$\|x_1-x_2\|\leqq\|(x_1-\lambda y_1)-(x_2-\lambda y_2)\|. \tag{1.1}$$

Note that (1.1) implies that for every $\lambda>0$, $J_\lambda=(I-\lambda A)^{-1}$ exists as a function and

$$\|J_\lambda x-J_\lambda y\|\leqq\|x-y\|, \qquad x, y\in R(I-\lambda A).$$

Here I denotes the identity operator in X.

Let A be an operator in X. Then we define the set $D_a(A)$ by $x \in D_a(A)$ if and only if there exists a sequence $\{x_n\} \subset D(A)$ such that $x_n \to x$ as $n \to \infty$ and $|||Ax_n|||$ is bounded. For each $x \in D_a(A)$, we write

$$|Ax| = \inf \{M\,;\ x_n \in D(A),\ \lim_{n\to\infty} x_n = x,\ \overline{\lim_{n\to\infty}}\, |||Ax_n||| \leqq M\} .$$

It is clear that $D(A) \subset D_a(A) \subset \overline{D(A)}$ and $|Ax| \leqq |||Ax|||$ for $x \in D(A)$. Also, we note that A is *almost demiclosed* if and only if $D(A) = D_a(A)$, provided that X is reflexive (see [5]).

Let $C \subset X$. The closure of C is denoted by $\bar{C}$. Let G be a single-valued operator in X. Then G is called a *contraction on* C if for $x, y \in C$,

$$\|Gx - Gy\| \leqq \|x - y\| .$$

A one-parameter family $\{T(t)\,;\ t \geqq 0\}$ of operators from C into itself is called a *semigroup of nonlinear contractions on* C if it has the following properties:

(i) $T(t)$ is a contraction on C for $t \geqq 0$;

(ii) $T(0)x = x$ for $x \in C$ and $T(t+s) = T(t)T(s)$ for $t, s \geqq 0$;

(iii) $\lim_{t\to 0+} T(t)x = x$ for $x \in C$.

§ 2. Cauchy problem and difference approximation.

Let A be a dissipative operator in X and let $T > 0$ be fixed. For a given $x_0 \in X$, we consider the following Cauchy problem, hereafter denoted by (CP), on a finite interval $[0, T]$:

$$(CP) \qquad \begin{cases} (d/dt)u(t) \in Au(t) & \text{a. e. } t \in (0, T) \\ u(0) = x_0 . \end{cases}$$

We shall denote by $(CP: x_0)$ the (CP) wherever we want to specify the initial condition that $u(0) = x_0$.

The following notion of solution of (CP) will be most natural (see [2]).

DEFINITION 2.1. Let $x_0 \in X$. Then an X-valued function $u(t)$ defined on $[0, T]$ is called a *strong solution* of $(CP; x_0)$ if it satisfies

(i) $u(0) = x_0$;

(ii) $u(t)$ is Lipschitz continuous on $[0, T]$;

(iii) $u(t)$ is strongly differentiable a. e. in $(0, T)$, $u(t) \in D(A)$ a. e. $t \in (0, T)$ and $(d/dt)u(t) \in Au(t)$ a. e. $t \in (0, T)$.

We note that if $u(t)$ is a strong solution of (CP), then $u(t) \in \overline{D(A)}$ for every $t \in [0, T]$.

It is known (e. g., see [12]) that the strong solution of (CP) does not necessarily exist for a prescribed $x_0 \in X$. For this reason, we need to consider a notion of solution of (CP) in a certain generalized sense. In this direction, Bénilan introduces in [1] notions of *integral solution* and of *bonne solution*. Also, Kenmochi and Oharu introduce in [6] a notion of *weak solution*. The following notion of solution of (CP) is weaker than that of bonne solution introduced in [1] but will be sufficient for the later arguments in this paper.

DEFINITION 2.2. Let $x_0 \in \overline{D(A)}$. Then an X-valued continuous function $u(t)$ defined on $[0, T]$ is called an *integral solution* of $(CP; x_0)$ if it satisfies

(i) $u(0)=x_0$;

(ii) $u(t) \in \overline{D(A)}$ for $t \in [0, T]$;

(iii) for every s, $t \in [0, T]$ such that $s \leqq t$ and for every $[x, y] \in A$,

$$\frac{1}{2}\|u(t)-x\|^2-\frac{1}{2}\|u(s)-x\|^2 \leqq \int_s^t \langle y, u(\tau)-x\rangle_s d\tau .$$

REMARK. It is shown in [1] that if $u(t)$ is a strong solution of $(CP; x_0)$, $x_0 \in X$, then it is a unique integral solution of $(CP; x_0)$.

We now discuss the problem of the existence of integral solutions of (CP). To this end, we formulate the following type of difference approximation:

$$(DS) \qquad \begin{cases} \dfrac{x_k^n - x_{k-1}^n}{t_k^n - t_{k-1}^n} - \varepsilon_k^n \in A x_k^n \qquad k=1, 2, \cdots, N_n;\ n \geqq 1 \\ x_0^n = x_0 , \end{cases}$$

where $\{t_k^n\}$ represents a partition $\Delta_n=\{0=t_0^n<t_1^n< \cdots <t_{N_n}^n=T\}$ of the interval $[0, T]$ and satisfies the condition

$$|\Delta_n| = \max_{1 \leqq k \leqq N_n} (t_k^n - t_{k-1}^n) \longrightarrow 0 \qquad \text{as } n \to \infty .$$

The term ε_k^n may be referred as an error which occurs at the k-th step of the n-th approximation. In this sense (DS) can be regarded as an approximating difference scheme for (CP). In later arguments, we shall frequently denote by $(DS; x_0)$ the difference scheme (DS) satisfying the condition that $x_0^n = x_0$.

We define the functions $f_n(t)$ and $u_n(t)$ by

$$f_n(t) = \varepsilon_k^n \qquad \text{for } t \in (t_{k-1}^n, t_k^n] , \qquad k=1, 2, \cdots, N_n;\ n \geqq 1$$

$$u_n(t) = \begin{cases} x_0^n & \text{for } t=0 \\ x_k^n & \text{for } t \in (t_{k-1}^n, t_k^n] , \qquad k=1, 2, \cdots, N_n;\ n \geqq 1 . \end{cases}$$

Note that f_n, $u_n \in L^1(0, T; X)$ for each n, where $L^1(0, T; X)$ denotes the usual Banach space.

It is now hoped that the function $u_n(t)$ approximates a solution of (CP). In fact, we obtain the following which is a main result of this paper and will be proved in the next section.

THEOREM I. *Let $x_0 \in X$. For the functions $f_n(t)$ and $u_n(t)$ defined as above,* *we assume that the following conditions are satisfied;*

(C) $f_n \to 0$ *in* $L^1(0, T; X)$ *as* $n \to \infty$,

(S) *there exist a bounded, nondecreasing function* $\omega(r)$ *defined on* $[0, T]$ *and a sequence* $\{\delta_n\}$ *of positive numbers such that* $\omega(r) \to 0$ *as* $r \to 0+$, $\delta_n \to 0$ *as* $n \to \infty$ *and*

$$\|u_n(t)-u_n(s)\| \leqq \omega(|t-s|)+\delta_n \qquad \textit{for } s,\ t \in [0, T] \textit{ and } n \geqq 1.$$

Then there exists an X-valued continuous function $u(t)$ *defined on* $[0, T]$ *such that*

(i) $u_n(t) \to u(t)$ *as* $n \to \infty$ *uniformly for* $t \in [0, T]$,

(ii) *for any integral solution* $v(t)$ *of* (CP), $u(t)$ *satisfies the inequality*

$$\|u(t)-v(t)\| \leqq \|u(s)-v(s)\| \qquad \textit{for } s, t \in [0, T],\ s \leqq t,$$

(iii) $u(t)$ *is a unique integral solution of* (CP; x_0).

Theorem I gives an extension of some results obtained in [**6**]. We also note that the limit function $u(t)$ in Theorem I is necessarily a bonne solution. See [**1**, Theorem 1.1].

REMARK. It will be natural to require the condition (C) for the difference scheme (DS). On the other hand, the condition (S) is necessary for the convergence (i). In fact, let $u(t)$ be a continuous function defined on $[0, T]$ to which $u_n(t)$ converges as $n \to \infty$ uniformly for $t \in [0, T]$. Now, we put

$$\omega(r) = \sup\{\|u(t)-u(s)\|;\ |t-s| \leqq r, \quad t, s \in [0, T]\}, \qquad r \in [0, T]$$

$$\delta_n = 2\sup\{\|u_n(t)-u(t)\|;\ t \in [0, T]\}, \qquad n \geqq 1.$$

Then it is clear that $\omega(r)$ and $\{\delta_n\}$ satisfy the properties in the condition (S). Furthermore,

$$\begin{aligned}\|u_n(t)-u_n(s)\| &\leqq \|u(t)-u(s)\|+\delta_n \\ &\leqq \omega(|t-s|)+\delta_n \qquad \text{for } s, t \in [0, T] \text{ and } n \geqq 1.\end{aligned}$$

Thus the condition (S) is satisfied. Note that this fact corresponds to the equivalence theorem in the so-called finite-difference method.

§3. Proof of Theorem I.

This section is devoted to the proof of Theorem I. The proof is given by several lemmas. The technique used here is due to Bénilan [**1**].

We begin with

LEMMA 3.1. *Let m and n be integers such that* $m \geqq n \geqq 1$. *Let* t_i^n, t_j^n *be the points of the partition* Δ_n *such that* $0 \leqq t_i^n \leqq t_j^n \leqq T$ *and* t_p^m, t_q^m *those of the parti-*

tion Δ_m *such that* $0 \leqq t_p^m \leqq t_q^m \leqq T$. *Then the following inequality holds:*

$$\begin{aligned}(3.1)\qquad &\int_{t_p^m}^{t_q^m} \frac{1}{2}(\|u_n(t_j^n)-u_m(\sigma)\|^2-\|u_n(t_i^n)-u_m(\sigma)\|^2)d\sigma \\ &+\int_{t_i^n}^{t_j^n} \frac{1}{2}(\|u_n(\tau)-u_m(t_q^m)\|^2-\|u_n(\tau)-u_m(t_p^m)\|^2)d\tau \\ &\leqq \int_{t_p^m}^{t_q^m}\int_{t_i^n}^{t_j^n} \langle f_n(\tau)-f_m(\sigma),\ u_n(\tau)-u_m(\sigma)\rangle_s d\tau d\sigma\,.\end{aligned}$$

PROOF. Let $[x, y] \in A$. Since A is dissipative, it follows that

$$(3.2)\qquad \langle (h_k^n)^{-1}(x_k^n-x_{k-1}^n)-\varepsilon_k^n-y,\ x_k^n-x\rangle_i \leqq 0$$

for $k=1, 2, \cdots, N_n$, where $h_k^n=t_k^n-t_{k-1}^n$.
Noting that

$$(3.3)\qquad \frac{1}{2}\|x_k^n-x\|^2-\frac{1}{2}\|x_{k-1}^n-x\|^2 \leqq \langle x_k^n-x_{k-1}^n,\ x_k^n-x\rangle_i\,,$$

we obtain

$$(3.4)\qquad \frac{1}{2}\|x_k^n-x\|^2-\frac{1}{2}\|x_{k-1}^n-x\|^2 \leqq h_k^n\langle \varepsilon_k^n+y,\ x_k^n-x\rangle_s$$

for $k=1, 2, \cdots, N_n$.
Adding the inequality (3.4) with respect to k from $i+1$ to j, we obtain

$$(3.5)\qquad \frac{1}{2}\|u_n(t_j^n)-x\|^2-\frac{1}{2}\|u_n(t_i^n)-x\|^2 \leqq \int_{t_i^n}^{t_j^n} \langle f_n(\tau)+y,\ u_n(\tau)-x\rangle_s d\tau\,.$$

Since $[x, y] \in A$ is arbitrary, taking $x=x_l^m$, $1 \leqq l \leqq N_m$, in (3.5), we have

$$\begin{aligned}(3.6)\qquad &\frac{1}{2}\|u_n(t_j^n)-x_l^m\|^2-\frac{1}{2}\|u_n(t_i^n)-x_l^m\|^2 \\ &\leqq \int_{t_i^n}^{t_j^n} \langle f_n(\tau)+(h_l^m)^{-1}(x_l^m-x_{l-1}^m)-\varepsilon_l^m,\ u_n(\tau)-x_l^m\rangle_s d\tau\,,\end{aligned}$$

where $h_l^m=t_l^m-t_{l-1}^m$.
Therefore, estimating the integrand in (3.6) similarly to (3.4) and then adding the resulting inequality with respect to l from $p+1$ to q, we obtain (3.1).

Q.E.D.

By Lemma 3.1, we easily obtain

LEMMA 3.2. *There exists a sequence* $\{\gamma_k\}$ *of positive numbers such that* $\gamma_k \to 0$ *as* $k \to \infty$ *and such that for every* $s, t, \alpha, \beta \in [0, T]$, $s \leqq t$, $\alpha \leqq \beta$, *the following estimate holds:*

$$\begin{aligned}(3.7)\qquad &\int_\alpha^\beta \frac{1}{2}(\|u_n(t)-u_m(\sigma)\|^2-\|u_n(s)-u_m(\sigma)\|^2)d\sigma \\ &+\int_s^t \frac{1}{2}(\|u_n(\tau)-u_m(\beta)\|^2-\|u_n(\tau)-u_m(\alpha)\|^2)d\tau \\ &\leqq \gamma_n+\gamma_m\,, \qquad n, m \geqq 1\,.\end{aligned}$$

PROOF. In view of the condition (S) in Theorem I, there exists a constant $M>0$ such that $\|u_n(t)\|\leqq M$ for all $t\in[0, T]$ and for all $n\geqq 1$. Then (3.7) is immediately deduced from (3.1). In fact, let $s\in(t_{i-1}^n, t_i^n]$, $t\in(t_{j-1}^n, t_j^n]$, $0\leqq i\leqq j\leqq N_n$, and let $\alpha\in(t_{p-1}^m, t_p^m]$, $\beta\in(t_{q-1}^m, t_q^m]$, $0\leqq p\leqq q\leqq N_m$; if $i=0$, then s is considered to be 0 and the others are similar. By symmetry, it now suffices to see that

$$\begin{aligned}&\int_s^t\frac{1}{2}(\|u_n(\tau)-u_m(\beta)\|^2-\|u_n(\tau)-u_m(\alpha)\|^2)d\tau\\&\qquad-\int_{t_i^n}^{t_j^n}\frac{1}{2}(\|u_n(\tau)-u_m(t_q^m)\|^2-\|u_n(\tau)-u_m(t_p^m)\|^2)d\tau\\&\qquad\leqq 4M^2(|t-t_j^n|+|s-t_i^n|)\\&\qquad\leqq 8M^2|\Delta_n|\end{aligned}$$

and that

$$\begin{aligned}&\int_{t_p^m}^{t_q^m}\int_{t_i^n}^{t_j^n}|\langle f_n(\tau), u_n(\tau)-u_m(\sigma)\rangle_s|d\tau d\sigma\\&\qquad\leqq\int_0^T\int_0^T\|f_n(\tau)\|\|u_n(\tau)-u_m(\sigma)\|d\tau d\sigma\\&\qquad\leqq 2MT\|f_n\|_{L^1},\end{aligned}$$

where $\|\cdot\|_{L^1}$ denotes the norm in the space $L^1(0, T; X)$.
Thus we only put

$$\gamma_k=8M^2|\Delta_k|+2MT\|f_k\|_{L^1},\qquad k\geqq 1.$$

Q. E. D.

Now, we set

$$\phi_{n,m}(\tau, \sigma)=\begin{cases}\frac{1}{2}\|u_n(\tau)-u_m(\sigma)\|^2 & \text{if } (\tau, \sigma)\in[0, T]\times[0, T]\\ 0 & \text{otherwise}\end{cases}$$

and consider the following regularization of $\phi_{n,m}$:

$$\phi_{n,m,\varepsilon}(\tau, \sigma)=(\rho_\varepsilon*\phi_{n,m})(\tau, \sigma)=\iint_{R^2}\rho_\varepsilon(\xi, \eta)\phi_{n,m}(\tau-\xi, \sigma-\eta)d\xi d\eta,$$

where $\rho_\varepsilon(\xi, \eta)=\varepsilon^{-2}\rho\left(\frac{\xi}{\varepsilon}\right)\rho\left(\frac{\eta}{\varepsilon}\right)$, $\varepsilon>0$, and ρ is a molifier such that $\rho\in\mathcal{D}(R^1)$, $\rho\geqq 0$, $\operatorname{supp}[\rho]\subset[-1, 1]$ and $\int_{R^1}\rho(\xi)d\xi=1$.

Let $\varepsilon>0$. In view of Lemma 3.2, we have

$$\begin{aligned}(3.8)\qquad&\int_{\alpha-\eta}^{\beta-\eta}\frac{1}{2}(\|u_n(t-\xi)-u_m(\sigma)\|^2-\|u_n(s-\xi)-u_m(\sigma)\|^2)d\sigma\\&\qquad+\int_{s-\xi}^{t-\xi}\frac{1}{2}(\|u_n(\tau)-u_m(\beta-\eta)\|^2-\|u_n(\tau)-u_m(\alpha-\eta)\|^2)d\tau\\&\qquad\leqq\gamma_n+\gamma_m\end{aligned}$$

for $s, t, \alpha, \beta \in [\varepsilon, T-\varepsilon]$ such that $s \leqq t$, $\alpha \leqq \beta$ and for $|\xi|, |\eta| \leqq \varepsilon$.
Multiplying $\rho_\varepsilon(\xi, \eta)$ to both sides of (3.8) and then integrating with respect to ξ and η over R^2, we obtain

$$\int_\alpha^\beta (\phi_{n,m,\varepsilon}(t, \sigma)-\phi_{n,m,\varepsilon}(s, \sigma))d\sigma+\int_s^t (\phi_{n,m,\varepsilon}(\tau, \beta)-\phi_{n,m,\varepsilon}(\tau, \alpha))d\tau$$
$$\leqq \gamma_n+\gamma_m,$$

or equivalently,

$$\int_\alpha^\beta \int_s^t \left\{\frac{\partial}{\partial\tau}\phi_{n,m,\varepsilon}(\tau, \sigma)+\frac{\partial}{\partial\sigma}\phi_{n,m,\varepsilon}(\tau, \sigma)\right\}d\tau d\sigma \leqq \gamma_n+\gamma_m \tag{3.9}$$

for $s, t, \alpha, \beta \in [\varepsilon, T-\varepsilon]$ such that $s \leqq t$ and $\alpha \leqq \beta$.

By the above consideration, we can prove the following:

LEMMA 3.3. *Let $\varepsilon>0$ and $h>0$ be sufficiently small. Then there exist constants $M_1>0$ and $M_2=M_2(\varepsilon)>0$, both independent of n, m and h, such that*

$$\phi_{n,m}(t, t)-\phi_{n,m}(s, s) \leqq M_1(\omega(\varepsilon)+\delta_n+\delta_m)+M_2Th+Th^{-2}(\gamma_n+\gamma_m) \tag{3.10}$$

for $s, t \in [\varepsilon, T-\varepsilon-h]$, $s \leqq t$, and for $n, m \geqq 1$.

PROOF. Let $h>0$ and put

$$I(r)=\int_r^{r+h}\int_r^{r+h} h^{-2}\left\{\frac{\partial}{\partial\tau}\phi_{n,m,\varepsilon}(\tau, \sigma)-\frac{\partial}{\partial\tau}\phi_{n,m,\varepsilon}(r, r)\right.$$
$$\left.+\frac{\partial}{\partial\sigma}\phi_{n,m,\varepsilon}(\tau, \sigma)-\frac{\partial}{\partial\sigma}\phi_{n,m,\varepsilon}(r, r)\right\}d\tau d\sigma \tag{3.11}$$

for $r \in [\varepsilon, T-\varepsilon-h]$, where $\frac{\partial}{\partial\tau}\phi_{n,m,\varepsilon}(r, r)=\frac{\partial}{\partial\tau}\phi_{n,m,\varepsilon}(\cdot, r)|_{\tau=r}$ and the other is similar. Since $\|u_n(t)\|$ is uniformly bounded for $t \in [0, T]$ and for $n \geqq 1$, applying the mean value theorem to the integrand in (3.11), we see that there exists a constant $M_2=M_2(\varepsilon)>0$, depending only on ε, such that

$$|I(r)| \leqq M_2h \qquad \text{for } r \in [\varepsilon, T-\varepsilon-h]. \tag{3.12}$$

Combining (3.9) with (3.12), we obtain

$$\frac{\partial}{\partial\tau}\phi_{n,m,\varepsilon}(r, r)+\frac{\partial}{\partial\sigma}\phi_{n,m,\varepsilon}(r, r) \leqq M_2h+h^{-2}(\gamma_n+\gamma_m), \quad r \in [\varepsilon, T-\varepsilon-h]. \tag{3.13}$$

Since $\frac{d}{dr}\phi_{n,m,\varepsilon}(r, r)=\frac{\partial}{\partial\tau}\phi_{n,m,\varepsilon}(r, r)+\frac{\partial}{\partial\sigma}\phi_{n,m,\varepsilon}(r, r)$, integrating (3.13) with respect to r, we get

$$\phi_{n,m,\varepsilon}(t, t)-\phi_{n,m,\varepsilon}(s, s) \leqq M_2Th+Th^{-2}(\gamma_n+\gamma_m) \tag{3.14}$$

for $s, t \in [\varepsilon, T-\varepsilon-h]$ such that $s \leqq t$.
Now, by the condition (S), there exists a constant $M_1>0$, independent of n, m,

ε and h, such that

$$(3.15)\qquad \begin{aligned}&|\phi_{n,m,\varepsilon}(r, r)-\phi_{n,m}(r, r)|\\ &\leqq \iint_{R^2}\rho_\varepsilon(\xi, \eta)|\phi_{n,m}(r-\xi, r-\eta)-\phi_{n,m}(r, r)|d\xi d\eta\\ &\leqq \frac{1}{2}M_1(\omega(\varepsilon)+\delta_n+\delta_m)\end{aligned}$$

for $r\in[\varepsilon, T-\varepsilon]$ and for n, $m\geqq 1$.
Therefore, combining (3.14) with (3.15), we obtain (3.10). Q.E.D.

We are now in a position to prove the convergence (i) in Theorem I.

LEMMA 3.4. *For every* $t\in[0, T]$,

$$(3.16)\qquad \lim_{n,m\to\infty}\|u_n(t)-u_m(t)\|=0$$

and the convergence is uniform for $t\in[0, T]$.

PROOF. Since $u_n(0)=u_m(0)=x_0$, it follows from the condition (S) that

$$\begin{aligned}\|u_n(s)-u_m(s)\| &\leqq \|u_n(s)-x_0\|+\|u_m(s)-x_0\|\\ &\leqq 2\omega(s)+\delta_n+\delta_m\end{aligned}$$

for $s\in[0, T]$. Hence, taking $s=\varepsilon$ in (3.10), we obtain

$$\phi_{n,m}(t, t)\leqq M_3(\omega(\varepsilon)+\delta_n+\delta_m)+M_2Th+Th^{-2}(\gamma_n+\gamma_m)$$

for $t\in[\varepsilon, T-\varepsilon-h]$, where M_3 is a positive constant independent of n, m, ε and h. Clearly, this estimate implies that the convergence (3.16) holds uniformly for t on any interval of the form $[\delta, T-\delta]$, $\delta>0$. Taking into account the condition (S) again, we can conclude that the convergence (3.16) holds uniformly for $t\in[0, T]$. Q.E.D.

We define the function $u(t)$ by $u(t)=\lim_{n\to\infty}u_n(t)$ for $t\in[0, T]$. It is then clear that $u(0)=x_0$, $u(t)\in\overline{D(A)}$ for $t\in[0, T]$ and $\|u(t)-u(s)\|\leqq\omega(|t-s|)$ for $s, t\in[0, T]$, and hence that $u(t)$ is continuous on $[0, T]$.

The following now completes the proof of Theorem I.

LEMMA 3.5. *The limit function $u(t)$ is an integral solution of (CP; x_0) and satisfies the property* (ii) *in Theorem I, and consequently, $u(t)$ is a unique integral solution of (CP; x_0).*

PROOF. We note that the inequality (3.5) holds true for any $[x, y]\in A$. Let $[x, y]\in A$ and let $s, t\in[0, T]$, $s\leqq t$. Then choosing t_i^n, t_j^n in (3.5) so that $t_i^n\to s$, $t_j^n\to t$ as $n\to\infty$ and passing to the limit, we obtain

$$\frac{1}{2}\|u(t)-x\|^2-\frac{1}{2}\|u(s)-x\|^2\leqq\int_s^t\langle y, u(\tau)-x\rangle_s d\tau\,.$$

Thus we see that $u(t)$ is an integral solution of $(CP; x_0)$.

Next, let $v(t)$ be any integral solution of (CP), that is, let $v(t)$ satisfy

$$\frac{1}{2}\|v(t)-x\|^2-\frac{1}{2}\|v(s)-x\|^2\leqq\int_s^t\langle y, v(\tau)-x\rangle_s d\tau \tag{3.17}$$

for every $s, t\in[0, T]$ such that $s\leqq t$ and for every $[x, y]\in A$.
Taking $x=x_k^n$ in (3.17) and using the same argument as in the proof of Lemma 3.1, we obtain

$$\begin{aligned}\int_{t_i^n}^{t_j^n}\frac{1}{2}(\|u_n(\sigma)-v(t)\|^2-\|u_n(\sigma)-v(s)\|^2)d\sigma \\ +\int_s^t\frac{1}{2}(\|u_n(t_j^n)-v(\tau)\|^2-\|u_n(t_i^n)-v(\tau)\|^2)d\tau \\ \leqq\int_{t_i^n}^{t_j^n}\int_s^t\langle f_n(\sigma), u_n(\sigma)-v(\tau)\rangle_s d\tau d\sigma,\end{aligned} \tag{3.18}$$

where t_i^n and t_j^n are any points of the partition Δ_n such that $0\leqq t_i^n\leqq t_j^n\leqq T$. Hence, passing to the limit in (3.18) with $t_i^n\to\alpha$, $t_j^n\to\beta$ as $n\to\infty$, $0\leqq\alpha\leqq\beta\leqq T$, we obtain

$$\begin{aligned}\int_\alpha^\beta\frac{1}{2}(\|u(\sigma)-v(t)\|^2-\|u(\sigma)-v(s)\|^2)d\sigma \\ +\int_s^t\frac{1}{2}(\|u(\beta)-v(\tau)\|^2-\|u(\alpha)-v(\tau)\|^2)d\tau\leqq 0\end{aligned} \tag{3.19}$$

for every $s, t, \alpha, \beta\in[0, T]$ such that $s\leqq t$ and $\alpha\leqq\beta$.
Therefore, the proof of this lemma is completed by the following result due to Bénilan [**1**]. Q.E.D.

LEMMA 3.6. *Let $u(t)$ and $v(t)$ be X-valued continuous functions defined on $[0, T]$ and satisfying the inequality* (3.19) *for every $s, t, \alpha, \beta\in[0, T]$ such that $s\leqq t$ and $\alpha\leqq\beta$. Then*

$$\|u(t)-v(t)\|\leqq\|u(s)-v(s)\|$$

for every $s, t\in[0, T]$ such that $s\leqq t$.

We omit here the proof and note only that the argument in this section suggests the method of the proof. For the detailed proof, see [**1**, Lemma 1.2].

§4. Generation of semigroups.

In this section we construct semigroups of nonlinear contractions by applying the preceding result. As will be shown by some examples in the next section, the result obtained extends most of the known theorems on the generation of semigroups of nonlinear contractions.

Let A be a dissipative operator in X. By an integral solution of the

Cauchy problem formulated for A on $[0, \infty)$, denoted by $(CP)_\infty$, we mean an X-valued function $u(t)$ defined on $[0, \infty)$ such that $u(t)$ restricted to any finite interval $[0, T]$ is an integral solution of the Cauchy problem on $[0, T]$.

To simplify the later arguments, we define

DEFINITION 4.1. Let A be a dissipative operator in X. Then we say that A has the *property* $(D)_i$ if for every $x \in \overline{D(A)}$ and for every $T>0$, there exists an approximating difference scheme $(DS; x)$ on $[0, T]$ satisfying the conditions (C) and (S).

In view of Theorem I, we obtain the following type of generation theorem.

THEOREM II. *Let A be a dissipative operator in X having the property $(D)_i$. Then there exists a unique semigroup $\{T(t); t \geqq 0\}$ of nonlinear contractions on $\overline{D(A)}$ such that for each $x \in \overline{D(A)}$, $u(t)=T(t)x$ is a unique integral solution of $(CP)_\infty$ with the initial-valve x.*

PROOF. By Theorem I, for each $x \in \overline{D(A)}$, there exists a unique integral solution $u(t)$ of $(CP)_\infty$ such that $u(0)=x$. We define a family of operators, $T(t)$, $t \geqq 0$, from $\overline{D(A)}$ into itself by setting $T(t)x=u(t)$ for $x \in \overline{D(A)}$ and for $t \geqq 0$. Then for any integral solution $v(t)$ of $(CP)_\infty$,

$$\|T(t)x-v(t)\| \leqq \|x-v(0)\| \qquad \text{for } t \geqq 0 .$$

In particular, we see that

$$\|T(t)x-T(t)y\| \leqq \|x-y\| \qquad \text{for } x,\ y \in \overline{D(A)} \text{ and } t \geqq 0 .$$

Therefore, each $T(t)$ is a contraction on $\overline{D(A)}$. It only remains to prove the semigroup property of $\{T(t); t \geqq 0\}$. To do this, let $x \in \overline{D(A)}$ and let $s \geqq 0$. Then it is clear that $u(t)=T(t+s)x$ and $v(t)=T(t)T(s)x$ are integral solutions of $(CP)_\infty$ such that $u(0)=v(0)=T(s)x$. Hence, by the unicity of the integral solution, $T(t+s)x=T(t)T(s)x$ for $t \geqq 0$. Thus $T(t+s)=T(t)T(s)$ for t, $s \geqq 0$, and hence the proof is completed. Q. E. D.

For some fundamental properties of such semigroups constructed as above, we refer to [**1**]. Also, see [**6**].

Before concluding this section, we give a remark which provides an actual criterion for a dissipative operator to have the property $(D)_i$.

REMARK. Let A be a dissipative operator in X. If there is a set E such that $\bar{E}=\overline{D(A)}$ and such that for every $x \in E$ and for every $T>0$, there exists an approximating difference scheme $(DS; x)$ on $[0, T]$ satisfying the conditions (C) and (S), then A has the property $(D)_i$.

In fact, let $x \in \overline{D(A)}$, $T>0$ and let $\{x_j\} \subset E$ be a sequence such that $x_j \to x$ as $j \to \infty$. By assumption, for each j, there exists an integral solution $u^j(t)$ of $(CP; x_j)$ on $[0, T]$. Then, since

$$\|u^j(t)-u^k(t)\| \leqq \|x_j-x_k\| \qquad \text{for } t\in[0, T] \text{ and for } j,\ k\geqq 1,$$

$u^j(t)$ converges to a continuous function $u(t)$ as $j\to\infty$ uniformly for $t\in[0, T]$. Now, let $f_n^j(t)$ and $u_n^j(t)$ be the step functions associated with $(DS; x_j)$, $j\geqq 1$. Clearly, $f_n^j\to 0$ in $L^1(0, T; X)$ as $n\to\infty$ for each j. Furthermore, if we put

$$\delta_n^j = 2\sup\{\|u_n^j(t)-u^j(t)\|;\ t\in[0, T]\}, \qquad j,\ n\geqq 1$$

$$\gamma_j = 2\sup\{\|u^j(t)-u(t)\|;\ t\in[0, T]\}, \qquad j\geqq 1,$$

then $\delta_n^j\to 0$ as $n\to\infty$ for each j, $\gamma_j\to 0$ as $j\to\infty$ and

$$\begin{aligned}\|u_n^j(t)-u_n^j(s)\| &\leqq \|u^j(t)-u^j(s)\|+\delta_n^j \\ &\leqq \|u(t)-u(s)\|+\gamma_j+\delta_n^j \\ &\leqq \omega(|t-s|)+\gamma_j+\delta_n^j, \qquad s,\ t\in[0, T],\end{aligned}$$

where $\omega(r)=\sup\{\|u(t)-u(s)\|;\ |t-s|\leqq r,\ \ s, t\in[0, T]\}$, $r\in[0, T]$. Therefore, by considering subsequences of $\{f_n^j(t)\}$ and of $\{u_n^j(t)\}$ and noting that $x_j\to x$ as $j\to\infty$, we conclude that there exists an approximating difference scheme $(DS; x)$ on $[0, T]$ satisfying the conditions (C) and (S). Thus A has the property $(D)_i$.

§ 5. A sufficient condition.

In this section we give a certain condition under which the property $(D)_i$ is satisfied. We then see that the condition given contains the ones for the generation of semigroups treated, for instance, in Kōmura [7], Kato [5], Brezis and Pazy [2] and in Crandall and Liggett [3]. Also, see Bénilan [1], Kenmochi and Oharu [6], Martin [8] and Miyadera [10]. For other examples of operators having the property $(D)_i$, we refer to [6].

Let A be a dissipative operator in X and consider the following condition:

(R_a) For any $x\in D_a(A)$ and $\varepsilon>0$, there exist a positive number $h=h(x, \varepsilon)$ with $h\leqq\varepsilon$ and an element $[x_h, y_h]\in A$ such that

$$(a_1) \quad \|x_h-hy_h-x\| \leqq h\varepsilon$$

$$(a_2) \quad |Ax_h| \leqq L(\varepsilon, h, |Ax|),$$

where $L(r, \sigma, \tau)$ is a real-valued function from $[0, \infty)\times[0, \infty)\times[0, \infty)$ into $[0, \infty)$ satisfying the properties

(L_1) L is bounded on bounded (r, σ, τ)-sets;

(L_2) L is nondecreasing with respect to the variables σ and τ;

(L_3) $L(r, \sigma_1, L(r, \sigma_2, \tau))\leqq L(r, \sigma_1+\sigma_2, \tau)$ for every (r, σ_1, τ) and (r, σ_2, τ).

Then we prove

THEOREM III. *Let A be a dissipative operator in X and satisfy the condition (R_a). Then A has the property $(D)_i$.*

To prove Theorem III, we prepare an auxiliary result.

LEMMA 5.1. *Suppose that A satisfies the condition (R_a). Let $x \in D_a(A)$, $T>0$ and let $\varepsilon>0$. Then there exists a Lipschitz continuous function $u(t)$ defined on $[0, T]$, depending on ε, such that*

(i) $u(0)=x$,

(ii) $\|u(t)-u(s)\| \leqq |t-s|(L(\varepsilon, T, |Ax|)+|Ax|+\varepsilon)$ *for* $t, s \in [0, T]$,

(iii) $u(T) \in D_a(A)$ *and* $|Au(T)| \leqq L(\varepsilon, T, |Ax|)$,

(iv) *for any* $\nu>0$, *there are a partition* $\Delta^\nu=\{0=t_0^\nu<t_1^\nu<\cdots<t_{N_\nu}^\nu=T\}$ *of* $[0, T]$ *and a pair* (u^ν, f^ν) *of step functions on* Δ^ν *satisfying the properties*

(a) $|\Delta^\nu|=\max_{1 \leqq k \leqq N_\nu}(t_k^\nu-t_{k-1}^\nu) \leqq \varepsilon$,

(b) $u^\nu(0)=x_0^\nu=x$, $u^\nu(t)=x_k^\nu$ *on* $(t_{k-1}^\nu, t_k^\nu]$ *and* $f^\nu(t)=\varepsilon_k^\nu$ *on* $(t_{k-1}^\nu, t_k^\nu]$, $k=1, 2, \cdots, N_\nu$, *where* $\{x_k^\nu\}_{k=0}^{N_\nu}$ *and* $\{\varepsilon_k^\nu\}_{k=1}^{N_\nu}$ *satisfy the difference scheme:*

$$\frac{x_k^\nu-x_{k-1}^\nu}{t_k^\nu-t_{k-1}^\nu}-\varepsilon_k^\nu \in Ax_k^\nu, \qquad k=1, 2, \cdots, N_\nu,$$

(c) $\|u(t)-u^\nu(t)\| \leqq \varepsilon(L(\varepsilon, T, |Ax|)+|Ax|+\varepsilon)+\nu$ *for* $t \in [0, T]$,

(d) $\|u(T)-u^\nu(T)\|=\|u(T)-x_{N_\nu}^\nu\| \leqq \nu$,

(e) $|Ax_k^\nu| \leqq L(\varepsilon, T, |Ax|)$ *for* $k=1, 2, \cdots, N_\nu$,

(f) $\int_0^T \|f^\nu(t)\| dt \leqq T\varepsilon+\nu$.

PROOF. We use a similar technique to Miyadera [**10**]. See also [8]. Let $\mathcal{P}$ denote the family of all pairs (v, c) where $c \in (0, T]$ and v is an X-valued function defined on $[0, c]$ and satisfying each of properties (i)-(iv) of the lemma with T replaced by c. We then prove this lemma by several steps.

The first step. If $(v_i, c_i) \in \mathcal{P}$, $i=1, 2$, then we write $(v_1, c_1) \leqq (v_2, c_2)$ whenever $c_1 \leqq c_2$ and $v_1(t)=v_2(t)$ for $t \in [0, c_1]$. It is clear that $\mathcal{P}$ is a partially ordered set by the relation "$\leqq$". Now, let $x \in D_a(A)$ and $\varepsilon>0$, $\varepsilon \leqq T$. By the condition (R_a), there exists a triple (h, x_h, y_h) such that $0<h \leqq \varepsilon$, $[x_h, y_h] \in A$ and

$$\|x_h-hy_h-x\| \leqq h\varepsilon, \tag{5.1}$$

$$|Ax_h| \leqq L(\varepsilon, h, |Ax|). \tag{5.2}$$

Since $x \in D_a(A)$, we can choose a sequence $\{[x_n, y_n]\} \subset A$ such that $x_n \to x$ as $n \to \infty$ and $\|y_n\| \leqq |Ax|+\frac{1}{n}$ for $n \geqq 1$. Then

$$\begin{aligned}\|x_h-x_n\| &\leqq \|(x_h-hy_h)-(x_n-hy_n)\| \\ &\leqq h(\varepsilon+\|y_n\|)+\|x_n-x\| \\ &\leqq h\left(\varepsilon+|Ax|+\frac{1}{n}\right)+\|x_n-x\|, \qquad n \geqq 1.\end{aligned}$$

Therefore, letting $n\to\infty$, we obtain that

$$\|x_h-x\|\leqq h(\varepsilon+|Ax|)\,. \tag{5.3}$$

We now define $v(t)=h^{-1}[(h-t)x+tx_h]$ for $t\in[0, h]$. Then it is clear that $v(0)=x$. Also, it follows from (5.2) and (5.3) that $v(t)$ satisfies the properties (ii) and (iii) of the lemma with $T=h$. In order to see the property (iv), let $\nu>0$ and define a partition Δ^ν of $[0, h]$ and a pair (v^ν, g^ν) of step functions on Δ^ν by $\Delta^\nu=\{0=t_0^\nu<t_1^\nu=h\}$, $v^\nu(0)=x$, $v^\nu(t)=x_h$ for $t\in(0, h]$ and $g^\nu(t)=h^{-1}(x_h-x)-y_h$ for $t\in(0, h]$. Then, by (5.1), (5.2) and (5.3), it is easy to see that the triple $(\Delta^\nu, v^\nu, g^\nu)$ satisfies the properties (a)-(f) of (iv) with $T=h$. Thus (v, h) is an element of $\mathcal{P}$ and hence $\mathcal{P}$ is nonempty.

The second step. Let $Q=\{(v_\alpha, c_\alpha)\,;\ \alpha\in\Lambda\}$, Λ some indexing set, be a totally ordered subset of $\mathcal{P}$. Let $c=\sup\{c_\alpha;\ \alpha\in\Lambda\}$. If $c=c_\alpha$ for some $\alpha\in\Lambda$, then (v_α, c_α) is an upper bound for Q. Assume now that $c_\alpha<c$ for all $\alpha\in\Lambda$ and for each $t\in[0, c)$ define $v(t)=v_\alpha(t)$ whenever $t\leqq c_\alpha$. Then v is well-defined on $[0, c)$. Furthermore, by the property (ii), $v(c)=\lim_{t\to c-}v(t)$ exists. Let $\{(v_n, c_n)\}$ be a sequence in Q such that $\lim_{n\to\infty}c_n=c$. Then $v(c)=\lim_{n\to\infty}v(c_n)=\lim_{n\to\infty}v_n(c_n)$. It is clear that $v(t)$ satisfies the properties (i) and (ii) with $T=c$. Since each $v_n(c_n)$ satisfies the property (iii) with $T=c_n$ and $v_n(c_n)\to v(c)$ as $n\to\infty$, it is easy to see that $v(c)\in D_\alpha(A)$ and $|Av(c)|\leqq L(\varepsilon, c, |Ax|)$. Therefore, v satisfies the property (iii) with $T=c$. Next let $\nu>0$ and for each n, let $(\Delta_n^\nu, v_n^\nu, g_n^\nu)$ be a triple satisfying the properties (a)-(f) of (iv) with $T=c_n$ and with ν replaced by $\frac{\nu}{2}$. Fix an integer $m\geqq1$ such that $(c-c_m)(L(\varepsilon, c, |Ax|)+|Ax|+\varepsilon)<\frac{\nu}{2}$ and $(c-c_m)<\varepsilon$. We then define a partition Δ^ν of $[0, c]$ by $\Delta^\nu=\Delta_m^\nu\cup[c_m, c]$ and a pair (v^ν, g^ν) of step functions on Δ^ν by

$$v^\nu(t)=\begin{cases} v_m^\nu(t) & \text{for } t\in[0, c_m] \\ v_m^\nu(c_m) & \text{for } t\in(c_m, c] \end{cases}$$

$$g^\nu(t)=\begin{cases} g_m^\nu(t) & \text{for } t\in(0, c_m] \\ -y^\nu & \text{for } t\in(c_m, c]\,, \end{cases}$$

where y^ν is an element $Av_m^\nu(c_m)$ such that $\|y^\nu\|\leqq L(\varepsilon, c, |Ax|)+\varepsilon$.
It is easy to see that the triple $(\Delta^\nu, v^\nu, g^\nu)$ satisfies the properties (a), (b) and (e) of (iv) with $T=c$. Furthermore, if $t\in[0, c_m]$, then

$$\begin{aligned}\|v(t)-v^\nu(t)\|&=\|v_m(t)-v_m^\nu(t)\|\\ &\leqq\varepsilon(L(\varepsilon, c_m, |Ax|)+|Ax|+\varepsilon)+\frac{\nu}{2}\\ &\leqq\varepsilon(L(\varepsilon, c, |Ax|)+|Ax|+\varepsilon)+\nu\end{aligned}$$

and if $t\in(c_m, c]$, then

$$\begin{aligned}\|v(t)-v^{\nu}(t)\| &\leqq \|v(t)-v(c_m)\|+\|v(c_m)-v_m^{\nu}(c_m)\| \\ &\leqq (t-c_m)(L(\varepsilon, c, |Ax|)+|Ax|+\varepsilon)+\frac{\nu}{2} \\ &\leqq \nu .\end{aligned}$$

Therefore, $(\varDelta^{\nu}, v^{\nu}, g^{\nu})$ satisfies the properties (c) and (d) of (iv) with $T=c$. Finally, we see that

$$\begin{aligned}\int_0^c \|g^{\nu}(t)\|dt &= \int_0^{c_m} \|g_m^{\nu}(t)\|dt+\int_{c_m}^c \|y^{\nu}\|dt \\ &\leqq c_m\varepsilon+\frac{\nu}{2}+(c-c_m)(L(\varepsilon, c, |Ax|)+\varepsilon) \\ &\leqq c\varepsilon+\nu ,\end{aligned}$$

which shows that the triple $(\varDelta^{\nu}, v^{\nu}, g^{\nu})$ satisfies the property (f) of (iv) with $T=c$. Thus (v, c) is an element of $\mathcal{P}$. It is clear that (v, c) is an upper bound for Q. Therefore, by Zorn's lemma, $\mathcal{P}$ has a maximal element.

The third step. Let (u, c) be a maximal element in $\mathcal{P}$. We want to show that $c=T$. Suppose, for contradiction, that $c<T$. Take an $\eta>0$ so that $c+\eta \leqq T$. Since $u(c)\in D_a(A)$, by the condition (R_a), there exist a positive number h with $h\leqq \min\{\varepsilon, \eta\}$ and an element $[x_h, y_h]\in A$ such that

$$\|x_h-hy_h-v(c)\|\leqq h\varepsilon \tag{5.4}$$

$$|Ax_h|\leqq L(\varepsilon, h, |Au(c)|)\leqq L(\varepsilon, h, L(\varepsilon, c, |Ax|))\leqq L(\varepsilon, c+h, |Ax|) . \tag{5.5}$$

Then, similarly to (5.3), we obtain that

$$\|x_h-u(c)\|\leqq h(|Au(c)|+\varepsilon)\leqq h(L(\varepsilon, c, |Ax|)+\varepsilon) . \tag{5.6}$$

Put $d=c+h$ and define a function $v(t)$ on $[0, d]$ by

$$v(t)=\begin{cases} u(t) & \text{for } t\in[0, c] \\ h^{-1}[(d-t)u(c)+(t-c)x_h] & \text{for } t\in(c, d] .\end{cases}$$

Then it is clear that $v(t)$ satisfies the properties (i) and (iii) with $T=d$. Also, by (5.6), we easily see that $v(t)$ satisfies the property (ii) with $T=d$. We next prove the property (iv) with $T=d$. Let $\nu>0$ and let $(\varDelta^{\nu}, u^{\nu}, f^{\nu})$ be a triple, associated with the element (u, c), satisfying the properties (a)-(f) of (iv) with $T=c$ and with ν replaced by $\frac{\nu}{2}$. We define a partition $\varDelta_1^{\nu}$ of $[0, d]$ by $\varDelta_1^{\nu}=\varDelta^{\nu}\cup[c, d]$ and a pair (v^{ν}, g^{ν}) of step functions on $\varDelta_1^{\nu}$ by

$$v^{\nu}(t)=\begin{cases} u^{\nu}(t) & \text{for } t\in[0, c] \\ x_h & \text{for } t\in(c, d]\end{cases}$$

$$g^{\nu}(t)=\begin{cases} f^{\nu}(t) & \text{for } t\in(0, c] \\ h^{-1}(x_h-u^{\nu}(c))-y_h & \text{for } t\in(c, d]. \end{cases}$$

Then it is clear that the properties (a), (b), (d) and (e) of (iv) with $T=d$ are satisfied by $(\varDelta_1^{\nu}, v^{\nu}, g^{\nu})$. The property (c) follows from (5.6) and the property (f) is shown by the following:

$$\begin{aligned}\int_0^d\|g^{\nu}(t)\|dt&=\int_0^c\|f^{\nu}(t)\|dt+\int_c^d\|h^{-1}(x_h-u^{\nu}(c))-y_h\|dt \\ &\leqq c\varepsilon+\frac{\nu}{2}+h\varepsilon+\|u(c)-u^{\nu}(c)\| \\ &\leqq d\varepsilon+\nu .\end{aligned}$$

Therefore, the triple $(\varDelta_1, v^{\nu}, g^{\nu})$ satisfies the property (iv) with $T=d$. Thus (v, d) is an element of $\mathscr{P}$. But $(u, c)\leqq(v, d)$ and $(u, c)\neq(v, d)$, which contradicts to the fact that (u, c) is a maximal element in $\mathscr{P}$. Therefore, it must be true that $c=T$ and $u(t)$ is the desired function. Q. E. D.

It is now easy to prove Theorem III.

PROOF OF THEOREM III. Let $x\in D_a(A)$, $T>0$ and let $\{\varepsilon_n\}$, $\{\nu_n\}$ be sequences of positive numbers such that $\varepsilon_n\to 0$, $\nu_n\to 0$ as $n\to\infty$. Then it follows from Lemma 5.1 that for each n, there exist a partition $\varDelta_n$ of $[0, T]$ and a pair $(u_n(t), f_n(t))$ of step functions on $\varDelta_n$ satisfying the properties (a)-(f) of (iv) with ε_n and ν_n. The property (f) of (iv) implies that $f_n(t)$ satisfies the condition (C) and the properties (ii) and (c) of (iv) imply that $u_n(t)$ satisfies the condition (S). Hence it is proved that there exists an approximating difference scheme $(DS; x)$ on $[0, T]$ satisfying the conditions (C) and (S). Therefore, by the remark after Theorem II, A has the property $(D)_i$. Q. E. D.

REMARK. The condition (R_a) contains the following type of conditions:

(R_1) $R(I-hA)\supset D(A)$ for $h>0$.

(R_2) For any $x\in D_a(A)$, there exists a sequence $\{h_n\}$ of positive numbers such that $h_n\to 0$ as $n\to\infty$ and

$$\bigcap_{n\geqq 1}\overline{R(I-h_nA)}\supset\{x\} .$$

(R_3) For any $x\in D_a(A)$, there exists a sequence $\{h_n\}$ of positive numbers such that $h_n\to 0$ as $n\to\infty$ and

$$d(R(I-h_nA), x)=o(h_n^2) \qquad \text{as } n\to\infty ,$$

where $d(\cdot, \cdot)$ denotes a usual distance.

(R_4) For any $x\in D_a(A)$ and $\varepsilon>0$, there exist a positive number $h=h(x, \varepsilon)$ with $h\leqq\varepsilon$ and an element $[x_h, y_h]\in A$ such that

$$(b_1)\quad \|x_h-hy_h-x\|\leqq h\varepsilon$$

$$(b_2) \quad |Ax_h| \leqq e^{\varepsilon h}|Ax|.$$

In fact, since the conditions (R_1) and (R_2) are the particular cases of the condition (R_3) and the condition (R_4) is not other than the condition (R_a) with $L(r, \sigma, \tau)=e^{r\sigma}\tau$, it suffices to examine only the condition (R_3). Suppose now that the condition (R_3) is satisfied. Let $x \in D_a(A)$ and $\varepsilon>0$. Then there exists a triple (h_n, x_n, y_n) such that $0<h_n \leqq \varepsilon$, $[x_n, y_n] \in A$ and

$$\|x_n - h_n y_n - x\| \leqq \varepsilon h_n^2 .$$

By a similar argument to (5.3), we obtain that

$$\|x_n - x\| \leqq h_n(\varepsilon h_n + |Ax|)$$

and hence

$$\|y_n\| \leqq 2\varepsilon h_n + |Ax|.$$

These estimates show that the condition (R_a) is satisfied with $L(r, \sigma, \tau)=2r\sigma+\tau$.

Finally, we note that the condition (R_1) is the condition treated in Crandall and Liggett [**3**] and the condition (R_2) contains the conditions treated in Kato [**5**] and in Brezis and Pazy [**2**] and also that the condition (R_4) contains the condition treated in Martin [**8**] (see also Miyadera [**10**]). Furthermore, we note that the condition (R_2) may be modified so as to contain the condition that $0 \in S(A ; \hat{D}(A))$ treated in Bénilan [**1**].

REMARK. Recently, Martin treated in [**9**] the problem of existence of solutions to the initial value problem

$$\text{(IVP)} \qquad (d/dt)u(t) = Au(t), \qquad u(0) = x,$$

where A is a single-valued operator in a Banach space X with the closed domain D. To show the existence of solutions to (IVP), he assumed the conditions:

(C1) A is continuous from D into X.

(C2) $\lim_{h \to 0+} h^{-1}d(x+hAx, D)=0$ for each $x \in D$.

(C3) $\langle Ax-Ay, x-y \rangle_s \leqq \omega\|x-y\|^2$ for each x and y in D, where ω is a real number.

Noting that under the condition (C1), the condition (C2) is equivalent to

(C2)′ $\lim_{h \to 0+} h^{-1}d(R(I-hA), x)=0$ for each $x \in D$,

we see that the above problem may be also treated from the viewpoint of the difference approximation. In fact, we can prove the existence of solutions to (IVP) under the conditions (C1), (C2)′ and

(C3)′ $\langle Ax-Ay, x-y \rangle_i \leqq \omega\|x-y\|^2$ for each x and y in D.

Thus the condition (C3) may be replaced by (C3)′ and therefore, we can improve a part of results obtained in [**9**] on the existence of solutions to (IVP) as well as to a nonautonomous differential equation

$$(d/dt)u(t) = A(t, u(t)) .$$

For details, we shall publish elsewhere.

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