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Scattering theory for elliptic systems

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Abstract. We prove existence and completeness of the wave operators and the invariance principle for first order systems even though the perturbation does not have compact support and no unique continuation property is assumed.

§ 1. Introduction.

The systems considered are of the form

$$Hu=E^{-1}(\sum_{j=1}^{n}A^{j}D_{j}u+Bu)\,, \tag{1.1}$$

where u is an m component vector valued function of $x\in E^{n}$, $A^{j}(x)$, $E(x)$ and $B(x)$ are $m\times m$ matrix valued measurable functions of x and $D_{j}=\partial/i\partial x_{j}$. For the unperturbed system we take

$$H_{0}u=E_{0}{}^{-1}\sum_{j=1}^{n}A_{0}{}^{j}D_{j}u\,, \tag{1.2}$$

where E_{0} and the $A_{0}{}^{j}$ are constant matrices. We make the following assumptions:

1. The matrices E_{0}, $A_{0}{}^{j}$, E, A^{j} are hermitian.
2. H_{0} is elliptic and H is uniformly elliptic.
3. E_{0} is positive definite and E is uniformly positive definite.
4. The A^{j} are bounded and uniformly continuous.
5. E is bounded.
6. The distribution derivatives $D_{j}A^{j}$ satisfy

$$B-B^{*}=\sum_{j=1}^{n}D_{j}A^{j}\,, \tag{1.3}$$

where B^{*} is the hermitian adjoint of B.

7. $B(x)$ is locally square integrable and

$$\sup_{x}\int_{|x-y|<\delta}|B(y)|^{2}|x-y|^{2-n}dy\rightarrow 0 \qquad \text{as } \delta\rightarrow 0\,. \tag{1.4}$$

8. There is an $\alpha>0$ such that

$$\int(\sum|A^j(x)-A_0^j|+|E(x)-E_0|+|B(x)|+|B(x)|^2)\rho(x)^\alpha dx<\infty,$$

where $\rho(x)=1+|x|$.

The domain of H_0 is taken as the set of those $u\in\mathcal{H}=[L^2(E^n)]^m$ such that $A_0(\xi)Fu$ is in $\mathcal{H}$, where $A_0(\xi)=\sum A_0^j\xi_j$ and F denotes the Fourier transform. Let $\mathcal{H}_0$ be $\mathcal{H}$ equipped with the scalar product

$$(u, v)_0=\int v(x)^*E_0u(x)dx. \tag{1.6}$$

It is easily verified that H_0 is self adjoint on $\mathcal{H}_0$. The hypotheses allow us to define H on the same domain. Moreover, if we let $\mathcal{H}_1$ denote $\mathcal{H}$ equipped with the scalar product

$$(u, v)_1=\int v(x)^*E(x)u(x)dx,$$

then (1.3) shows that H is symmetric. In fact we have

LEMMA 1.1. *Under hypotheses* 1-7, *H is self adjoint on $\mathcal{H}_1$.*

Let J be the identification operator $Ju=u$ mapping $\mathcal{H}_0$ onto $\mathcal{H}_1$, and put

$$W(t)=e^{itH}Je^{-itH_0}. \tag{1.7}$$

The wave operators are defined by

$$W_\pm u=\lim_{t\to\pm\infty}W(t)u, \tag{1.8}$$

when these limits exist for each $u\in\mathcal{H}_0$. They are said to be complete if their ranges coincide and equal the absolutely continuous subspace of H. The invariance principle holds if

$$W_\pm u=\lim_{t\to\pm\infty}e^{it\varphi(H)}Je^{-it\varphi(H_0)}u, \tag{1.9}$$

whenever φ satisfies

$$\int_0^\infty\left|\int_\Gamma e^{-i\eta s-it\varphi(s)}ds\right|^2d\eta\to0 \quad \text{as } t\to\infty \tag{1.10}$$

and

$$\int_\Gamma e^{-it\varphi(s)}ds\to0 \quad \text{as } t\to\infty \tag{1.11}$$

for each bounded Borel set Γ. One of our main results is

THEOREM 1.2. *Under hypotheses* 1-8, *the wave operators exist and are complete, and the invariance principle holds.*

Denote the roots of $\det(\lambda E_0-A_0(\xi))=0$ by $\lambda_j(\xi)$. They are continuous algebraic functions (cf. [1]), but in general their multiplicities need not be constant. However, we have

THEOREM 1.3. *If the multiplicities of the roots are constant, then hypothesis* 8 *may be replaced by*

8′. *There are constants* $\alpha>0$ *and* $1\leqq p\leqq\infty$ *satisfying* $\alpha>1-p^{-1}$ *such that*

$$\rho(x)^{\alpha}Z(x)\in L^{p}, \tag{1.12}$$

where

$$Z(x)=\int_{|x-y|<1}\Big(\sum_{j=1}^{n}|A^{j}(y)-A_0^{j}|+|E(y)-E_0|+|B(y)|+|B(y)|^{2}\Big)dy. \tag{1.13}$$

The set of all $\xi\in E^{n}$ such that $\lambda_j(\xi)=1$ for some j is called the *slowness surface* for the system (1.2).

THEOREM 1.4. *Assume in addition that the sheets of the slowness surface have nonvanishing total curvature at each point. Then hypothesis* 8 *can be replaced by*

8″. *There is a* $\beta<(1/2)(n-1)$ *such that*

$$\sup_{x}\int Z(y)\rho(x-y)^{-\beta}dy<\infty \tag{1.14}$$

and

$$Z(x)\rightarrow 0 \quad \text{as } |x|\rightarrow\infty. \tag{1.15}$$

THEOREM 1.5. *Under the same assumptions, hypothesis* 8 *can be replaced by*

8‴. (1.15) *holds and there are constants* $\alpha\geqq 0$ *and* $1\leqq p\leqq\infty$ *such that* $\alpha>1-[2n/(n+1)p]$ *and* (1.12) *holds.*

Scattering for special cases of system (1.1) has been studied by several authors. The case $A^{j}=A_0^{j}$, $B=0$ was dealt with by Schulenberger and Wilcox [**2**, **3**], Birman [**6**], Avila [**5**], LaVita et al. [**4**] and others. Because of the nature of the perturbation in this case they were able to consider nonelliptic systems as well. Systems satisfying the hypotheses of Theorem 1.3 are called uniformly propagative. When $E=E_0=I$, system (1.1) was studied by Lax and Phillips [**8**, **7**]. They required the perturbation to have compact support and that the system have the unique continuation property. Agmon has told the author that he has results for symmetric elliptic systems. We do not know what hypotheses he makes on the coefficients. To the best of our knowledge the present paper is the first to consider systems of the general form (1.1). June 13, 1974.

§ 2. Selfadjointness.

In this section we give a proof of Lemma 1.1. A simple calculation using (1.3) shows that H is symmetric on $\mathscr{H}_1$. The proof that it is actually self adjoint will be carried out by a series of lemmas. We let $H^{s,2}$ denote the set of those $u\in\mathscr{H}$ such that $\rho(\xi)^s Fu\in\mathscr{H}$. Denote its norm by $\|\ \|_{s,2}$. We have

LEMMA 2.1.

$$D(H_0)=H^{1,2}.$$

PROOF. Since H_0 is elliptic, there is a constant C such that

$$C^{-1}|\xi|\,|\omega|\leqq|A_0(\xi)\omega|\leqq C|\xi|\,|\omega| \tag{2.1}$$

for $\xi\in E^n$ and ω a complex vector. If we put $\omega=Fu$ in (2.1) we see that u and $A_0(\xi)Fu$ are in $\mathscr{H}$ if and only if $u\in H^{1,2}$. □

LEMMA 2.2. *For each $\varepsilon>0$ there is a constant K such that*

$$\|Bu\|\leqq\varepsilon\|u\|_{1,2}+K\|u\|, \qquad u\in H^{1,2}. \tag{2.2}$$

PROOF. This follows from hypothesis 7 as in the scalar case (cf. [**9**, p. 140]). □

LEMMA 2.3. *Put $A=EH$. Then there exists a constant C such that*

$$C^{-1}\|u\|_{1,2}\leqq\|Au\|+\|u\|\leqq C\|u\|_{1,2}, \qquad u\in H^{1,2}. \tag{2.3}$$

PROOF. Since the A^j are bounded and uniformly continuous and H is uniformly elliptic, we have

$$C^{-1}\|u\|_{1,2}\leqq\|\sum A^jD_ju\|+\|u\|\leqq C\|u\|_{1,2} \tag{2.4}$$

by the usual coerciveness inequality [**10**]. Now (2.3) follows from (2.4) and (2.2) with ε sufficiently small. □

Let $H^{1,2}_{\mathrm{loc}}$ be the set of those u such that φu is in $H^{1,2}$ for every $\varphi\in C_0^\infty$, the test functions.

LEMMA 2.4. *If $u\in H^{1,2}_{\mathrm{loc}}$ and u, Au are in $\mathscr{H}$, then $u\in H^{1,2}$.*

PROOF. There is a system $\{N_k\}$ of open sets and a sequence $\{\zeta_k\}$ of test functions such that 1) the support of ζ_k is in N_k, 2) $UN_k=E^n$ and

$$\sum_k\|v\|^{(N_k)}\leqq C\|v\|, \qquad v\in L^2,$$

where

$$\|v\|^{(S)}=\left(\int_S|v(x)|^2dx\right)^{1/2},$$

3) $\zeta_k\geqq 0$ and $\sum\zeta_k\equiv 1$ and 4) $\zeta_k+\sum|D_j\zeta_k|\leqq C$. Now $\zeta_k u$ is in $H^{1,2}$ and by (2.3)

$$\|\zeta_k u\|_{1,2}\leqq C(\|A\zeta_k u\|+\|\zeta_k u\|)\leqq C'(\|Au\|^{(N_k)}+\|u\|^{(N_k)}).$$

Thus

$$\begin{aligned}\|u\|_{1,2}&=\|\sum\zeta_k u\|_{1,2}\leqq\sum\|\zeta_k u\|_{1,2}\\&\leqq C'\sum(\|Au\|^{(N_k)}+\|u\|^{(N_k)})\leqq C''(\|Au\|+\|u\|).\end{aligned}$$

Thus $u\in H^{1,2}$. □

LEMMA 2.5. *If, in addition, the coefficients of A are in C^{∞}, then A with $D(A)=H^{1,2}$ is self adjoint.*

PROOF. Suppose u, $f\in\mathcal{H}$ and $(u, Av)=(f, v)$ for every $v\in C_0^{\infty}$. The $u\in H_{\mathrm{loc}}^{1,2}$ by the regularity theory for elliptic systems. Thus $Au=f\in\mathcal{H}$. Apply Lemma 2.4. □

LEMMA 2.6. *Let B be a closed symmetric operator on a Hilbert space $\mathcal{H}$, and suppose there is a sequence $\{B_k\}$ of self adjoint operators such that $D(B_k)=D(B)=D$ and*

$$\|(B_k-B)u\|\leqq\varepsilon_k(\|Bu\|+\|u\|)\,,\qquad u\in D\,,\tag{2.6}$$

where $\varepsilon_k\to 0$ as $k\to\infty$. Then B is self adjoint.

PROOF. Note that (2.6) implies

$$\|(B_k-B)u\|\leqq\delta_k\ (\|B_k u\|+\|u\|)\,,\qquad u\in D\,,\tag{2.7}$$

where $\delta_k\to 0$. Suppose u, $f\in\mathcal{H}$ satisfy

$$(u, Bv)=(f, v)\,,\qquad v\in D\,.\tag{2.8}$$

Put $\Phi(v, B_k v)=(u, (B_k-B)v)$. It is a conjugate linear functional on the graph of B_k. By (2.7)

$$|\Phi(v, B_k v)|\leqq\delta_k\|u\|\ (\|B_k v\|+\|v\|)\,.$$

By the Hahn-Banach theorem it can be extended to the whole of $\mathcal{H}\times\mathcal{H}$ without increasing its norm. Thus there are elements u_k, $f_k\in\mathcal{H}$ such that

$$(u, (B_k-B)v)=(u_k, B_k v)+(f_k, v)\,,\qquad v\in D$$

and

$$\max\,(\|u_k\|,\ \|f_k\|)\leqq\varepsilon_k\|u\|\to 0\qquad\text{as } k\to\infty\,.$$

Thus by (2.8)

$$(u-u_k, B_k v)=(f+f_k, v)\,.$$

Since B_k is self adjoint, $u-u_k\in D$ and $B_k(u-u_k)=f+f_k$. By this and (2.7)

$$\|(B-B_k)(u-u_k)\|\leqq\delta_k\ (\|f+f_k\|+\|u-u_k\|)\to 0\,.$$

Hence $B(u-u_k)\to f$. Since $u-u_k\to u$ and B is a closed operator, we see that $u\in D$ and $Bu=f$. □

LEMMA 2.7. *There exists a sequence $\{A_k\}$ of self adjoint operators on $\mathcal{H}$ such that $D(A_k)=H^{1,2}$ and*

$$\|(A_k-A)u\|\leqq\varepsilon_k\|u\|_{1,2}\,,\tag{2.9}$$

where $\varepsilon_k\to 0$ as $k\to\infty$.

PROOF. Apply the Friedrichs mollifier J_ε to each element of the matrices

A^j and B. Put $\varepsilon=1/k$ and denote the corresponding operator by A_k. The coefficients of the system are infinitely differentiable and satisfy hypotheses 1-7 with constants independent of k for k sufficiently large. By Lemma 2.5, each A_k is self adjoint. Inequality (2.9) follows from standard arguments. □

PROOF OF LEMMA 1.1. H is self adjoint on $\mathcal{H}_1$ if and only if A is self adjoint on $\mathcal{H}$. The latter fact follows from Lemmas 2.6 and 2.7. □

§ 3. **The abstract theory.**

We prove Theorems 1.2-1.5 by verifying that the hypotheses of an abstract theorem are satisfied. We take the theorem from [**11**], which uses ideas of Kato, Kuroda and Birman (cf. [**11**] for references). We state the theorem in a form convenient for the application at hand. Let $\mathcal{H}_0$ (resp. $\mathcal{H}_1$) be a Hilbert space, and let H_0 (resp. H) be a self adjoint operator on it with spectral family $\{E_0(\lambda)\}$ (resp. $\{E(\lambda)\}$). Put $R_0(\zeta)=(\zeta-H_0)^{-1}$ when it exists. Assume

a. There is a linear bijective operator J from $\mathcal{H}_0$ onto $\mathcal{H}_1$ such that $JD(H_0)=D(H_1)$.

b. There are a Hilbert space $\mathcal{K}$ and closed linear operators A, B from $\mathcal{H}_0$ to $\mathcal{K}$ such that A is injective and $D(H_0)\subset D(A)\cap D(B)$.

c. $B\,R(Z)A^*$ is bounded for some Z, $D(H_0)\subset D(B^*A)$ and

$$HJ=JH_0-JB^*A\,. \tag{3.1}$$

d. $BR_0(\zeta)$ is a compact operator for all nonreal ζ.

e. The spectrum of H_0 is absolutely continuous and $[J^*J-I]E_0(\Gamma)$ is a compact operator for each bounded interval Γ.

f. There are functions $M(s)$, $N(s)$ from $\sigma(H_0)$ to $B(\mathcal{K})$ which are locally Hölder continuous such that

$$d(E_0(s)A^*u,\ A^*v)_0/ds=M(s)u,\ v)_{\mathcal{K}} \tag{3.2}$$

$$d(E_0(s)B^*u,\ A^*v)_0/ds=(N(s)u,\ v)_{\mathcal{K}}\,. \tag{3.3}$$

THEOREM 3.1. *Under hypotheses* a-f, *the wave operators* (1.8) *exist and are complete and the invariance principle holds.*

Theorem 3.1 is a special case of a theorem proved in [**11**]. We now use Theorem 3.1 to give the

PROOF OF THEOREM 1.2. We show that hypotheses 1-8 imply a-f. There exist matrix functions $V(x)$, $W(x)$, $S^j(x)$, $T^j(x)$, $L(x)$ and $M(x)$ such that $L(x)$ is bounded and

1) V, W, S^j, T^j are hermitian
2) $W(x)$ is nonsingular for each x
3) $VW=E-E_0$, $S^jT^j=A^j-A_0^j$, $LM=B$

4) $\int(\sum|S^j(x)|^2+\sum|T^j(x)|^2+|V(x)|^2+|W(x)|^2+|L(x)|^2+|M(x)|^2)$
$\rho(x)^\alpha dx<\infty$

5) $\sup_x \int_{|x-y|<1} |M(y)|^2|x-y|^{2-n}dy \leqq C_0$.

We take $\mathcal{K}$ as the direct sum of $n+2$ copies of $\mathcal{H}$ and define

$$Au=(\{T^jD_ju\},\ Mu,\ WH_0u)$$

$$Bu=(\{S^jE^{-1}u\},\ LE^{-1}u,\ VE^{-1}u)$$

with $D(A)=H^{1,2}$ and $D(B)=\mathcal{H}$. This is possible because 5) implies

$$\|Mv\|\leqq C\|v\|_{1,2}$$

(cf. [**9**, p. 138]). Note that

$$B^*(\{v_j\},\ v,\ w)=E^{-1}(\sum S^jv_j+Lv+Vw)\,,$$

and consequently $B^*A=H-H_0$. Now

$$BR_0(\zeta)u=(\{S^jE^{-1}R_0(\zeta)u\},\ LE^{-1}R_0(\zeta)u,\ VE^{-1}R_0(\zeta)u)\,.$$

By 4) this is a compact operator (cf. [**9**, p. 111]). Next note that

$$FE_0(\Gamma)f=\sum X_j(\xi)P_j(\xi)Ff\,, \tag{3.4}$$

where $X_j(\xi)$ is the characteristic function of the set $\lambda_j(\xi)\in\Gamma$ and $P_j(\xi)$ are bounded homogeneous matrices of degree 0 (cf., e. g., [**3**], where explicit formulas are given for the $P_j(\xi)$). Since the λ_j are bounded away from 0, the X_j have compact support if Γ is bounded. Thus there is a $\varphi\in C_0^\infty$ such that $\varphi FE_0(\Gamma)f=FE_0(\Gamma)f$. Let Q be the operator given by $Qu=\bar{F}\varphi*u$, where $\bar{F}$ denotes the inverse Fourier transform. By 4), WQ is a compact operator on $\mathcal{H}$ ([**9**, p. 86]). Since $J^*=E_0^{-1}E$, we have $(J^*J-I)E(\Gamma)=E_0^{-1}VWQE(\Gamma)$. Thus this operator is compact for Γ bounded.

Next note that ([**3**, **12**, **11**])

$$d(E_0(s)f,\ g)_0/ds=(K_s*f,\ g)\,,$$

where

$$K_s(x)=\sum_j\int_{S_{j,s}} e^{ix\xi}E_0P_j(\xi)d\sigma$$

and $S_{j,s}$ is the set $\lambda_j(\xi)=s$. Since the λ_j are continuous and homogeneous of degree 1, the set $S_{j,s}$ are bounded for each s. Thus $K_s(x)$ is a bounded function of x for each s. Note also that $K_s(x)=s^{n-1}K_1(sx)$. This implies that for each $\alpha>0$, $\alpha\leqq 1$,

$$|K_s(x)-K_t(x)|\leqq C|s-t|^\alpha\rho(x)^\alpha\,,$$

where C is independent of x. Thus if $X(x)$, $Y(x)$ are matrix functions

$$|(K_s*(Xu),\ Yv)|\leqq C\|X\|\|Y\|\|u\|\|v\|$$

and

$$|([K_s-K_t]*(Xu),\ Yv)|^2\leqq C|s-t|^\alpha\|u\|^2\|v\|^2$$
$$\times\iint\rho(x-y)^{2\alpha}|X(y)|^2|Y(x)|^2dx\,dy$$
$$\leqq C|s-t|^\alpha\|u\|^2\|v\|^2\|\rho^\alpha X\|^2\|\rho^\alpha Y\|^2 .$$

Thus we see that if $\|\rho^\alpha X\|$ and $\|\rho^\alpha Y\|$ are finite for some $\alpha>0$, then

$$d(E_0(s)Xu,\ Xv)_0/ds=(R(s)u,\ v)\,,$$

where $R(s)$ is a locally Hölder continuous function from the reals to $B(\mathcal{H})$. We now replace X and Y by V, W, S^j, T^j, L, M in the appropriate combinations to see that (3.2) and (3.3) hold. □

PROOF OF THEOREM 1.3. If the multiplicities of the λ_j are constant, then the $S_{j,s}$ are smooth bounded surfaces depending smoothly on s. By Lemma 3.7 of [**11**], condition f will be satisfied if $\rho(x)^\alpha Z(x)$ is bounded for some $\alpha>1$. Employing an interpolation theorem as in [**12, 11**] we obtain hypothesis 8′. □

PROOF OF THEOREMS 1.4 AND 1.5. If the sheets of the slowness surface have nonvanishing total curvature, then it follows from a theorem of Littman [**13**] that

$$|K_s(x)|\leqq C\,\rho(x)^{1/2(1-n)}$$

and

$$|K_s(x)-K_t(x)|\leqq C|s-t|^\alpha\rho(x)^{1/2(1-n)+\alpha}$$

for each $\alpha>0$, $\alpha\leqq1$ (cf. [**11**]). An application of Lemma 7.2 of [**11**] gives Theorem 1.4. Now hypothesis 8‴ implies 8″ for $p<2n/(n+1)$. It implies hypothesis 8′ for $p=\infty$. As in [**12, 11**] another application of the interpolation theorem gives Theorem 1.5. □

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