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Quadratic fields and Hopf fibrations

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Introduction.

Let k be a quadratic field over $\boldsymbol{Q}$. As usual, we write $k=\boldsymbol{Q}(\mu)$ with $\mu^2=m$ where m is a square free rational integer. By the Cayley-Dickson process we can make a quaternion algebra A out of the vector space $k\times k$ over $\boldsymbol{Q}$. Let A_0 be the subspace of A formed by quaternions of trace zero. By the Hopf map belonging to k, we shall mean the map:

$$h: A \longrightarrow A_0 \qquad \text{defined by } h(z)=\bar{z}\mu z\,,$$

where $\bar{z}$ is the conjugate of z. When we identify A_0 with the space $\boldsymbol{Q}\times k$ using the quaternion units, we obtain another expression of the Hopf map:

$$h(z)=(Nx-Ny,\ 2\mu\bar{x}y)\,, \qquad z=(x, y)\in A\,,$$

where $Nx=\bar{x}x$, the norm in k.

When $k=\boldsymbol{Q}(i)$, the Gaussian field, the map h is the restriction on $\boldsymbol{Q}^4$ of the map $\boldsymbol{R}^4\rightarrow\boldsymbol{R}^3$ which induces the classical Hopf fibration: $S^3\rightarrow S^2$ where each fibre is a great circle of S^3 ([**3**, §5]).

Back to a general quadratic field k, the Hopf map h again maps "3-sphere" to "2-sphere" where each fibre is a "circle". Since one can view the theory of representation of integers by binary quadratic forms as an arithmetic of the "circle", it is natural to ask how one can extend the results of Gauss and Dirichlet to the case of "circle" bundles.

In this paper, we shall obtain a formula for a Hopf map (the formula (3.12)) which can be considered as an analogue of the Dirichlet formulas (2.4), (2.5) on the total number of representations of integers by a complete system of non-equivalent binary quadratic forms of a given discriminant. The case where $k=\boldsymbol{Q}(i)$ has been treated in [**4**].

Notation and conventions.

The symbols $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ denote the set of natural numbers, the ring of rational integers, the field of rational numbers, the field of real numbers, respectively. For an associative ring R with the identity, we denote by $R^\times$ the group of invertible elements of R. For an algebraic number field k of finite degree over $\boldsymbol{Q}$, we denote by $\mathfrak{o}_k$ the ring of integers of k, by I_k the group of fractional ideals of k, by I_k^+ the set of non-zero integral ideals of k; we shall often make the natural identification $I_{\boldsymbol{Q}}^+=\boldsymbol{N}$. For an n-tuple $(a_1, \cdots, a_n)$ of elements $a_i \in k$, $1 \leqq i \leqq n$, where some $a_i \neq 0$, we denote by $id(a_1, \cdots, a_n)$ the ideal in I_k generated by $a_1, \cdots, a_n$. For a non-zero polynomial $f=f(X_1, \cdots, X_m) \in k[X_1, \cdots, X_m]$, we denote by $c(f)$ the content of f, i.e. the ideal of k generated by the coefficients of f. We know that $c(fg)=c(f)c(g)$ ([2, §28]). For a set S, we denote by $[S]$ its cardinality. For a map f, we denote by $\operatorname{Im} f$ the image of f.

§1. Cayley-Dickson process.

Let K be a quadratic field over a field F of characteristic not two. For each $x \in K$, we write its conjugate, trace and norm by $\bar{x}$, $Tx=\bar{x}+x$ and $Nx=\bar{x}x$, respectively. Define a multiplication on the space $A=K^2=K\times K$ by

$$z_1 z_2=(x_1 x_2-y_1\bar{y}_2,\ x_1 y_2+y_1\bar{x}_2), \qquad z_i=(x_i, y_i), \qquad i=1, 2.$$

Then $1_A=(1, 0)$ is the identity of A and we imbed K in A through the identification $1_A=1$. The element $j=(0, 1)$ satisfies $j^2=-1$. Call μ a generator of K over F, $K=F(\mu)$, such that $\mu^2=m\in F^\times$. Then one verifies easily that A is a quaternion algebra over F with quaternion units $1, \mu, j, \mu j$, with the relations $\mu^2=m$, $j^2=-1$, $\mu j+j\mu=0$. For an element $z=(x, y)=x+yj\in A$, its conjugate, trace and norm are defined by $\bar{z}=(\bar{x}, -y)=\bar{x}-yj$, $Tz=\bar{z}+z$ and $Nz=\bar{z}z$, respectively. The map $z\to\bar{z}$ is an involution of the algebra A and we have $Tz=Tx$, $Nz=Nx+Ny$. We shall denote by A_0 the subspace of A consisting of z such that $Tz=0$; A_0 is of rank 3 over F, spanned by $\mu, j, \mu j$. We shall often make the following natural identifications:

$$K=F+F\mu=F^2, \qquad A=K+Kj=K^2=F^4,$$

$$A_0=F\mu+Fj+F\mu j=F^3=F\mu+Kj=F\times K.$$

By the Hopf map, we shall mean the map:

$$h: A \longrightarrow A_0 \qquad \text{defined by } h(z)=\bar{z}\mu z,$$

here we have used that $T(h(z))=0$ for all $z\in A$. The choice of μ in the definition of h is inessential because for any other generator μ' like μ we have $\mu'=a\mu$, $a\in F^\times$. A simple calculation shows that

$$h(z)=(Nx-Ny)\mu+(2\mu\bar{x}y)j=(Nx-Ny,\ 2\mu\bar{x}y)\in A_0=F\times K.$$

For $t\in F$, put

$$S_A(t)=\{z\in A,\ Nz=t\}\,,\qquad S_{A_0}(t)=\{z\in A_0,\ Nz=t\}\,.$$

Since $N(h(z))=(N\mu)(Nz)^2=-m(Nz)^2$, h maps $S_A(t)$ into $S_{A_0}(-mt^2)$.

Along with the map h, we shall need later the maps f, φ defined as follows:

$$f(z)=(Nx,\ \bar{x}y,\ Ny)\,,\qquad \varphi(z)=(Nx,\ T(\bar{x}y),\ Ny)\,,\qquad z=(x,\ y)\,.$$

We put

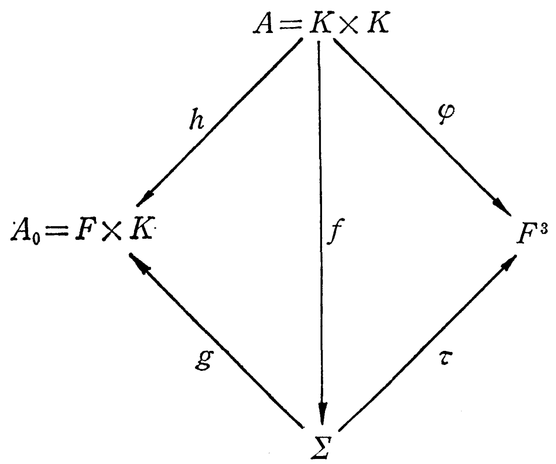
$$\Sigma=\{\sigma=(a,\ \beta,\ c)\in F\times K\times F,\ N\beta=ac\}\,,$$

and define maps g and τ by

$$g(\sigma)=(a-c,\ 2\mu\beta)\,,\qquad \tau(\sigma)=(a,\ T\beta,\ c)\,.$$

It is clear that the diagram (1.1) is commutative.

(1.1)



§ 2. Representation of integers by the norm function.

In order to fix notations and to motivate the method in § 3, we shall collect here some relevant results on binary quadratic forms due to Gauss and Dirichlet ([**1**, Vierter Abschnitt & Fünfter Abschnitt]).

Let k be a quadratic field over $\boldsymbol{Q}$. Denote by P_k, P_k^+ the group of principal ideals and its subgroup of principal ideals $id(\alpha)$ with $N\alpha>0$, respectively. We denote by h, h^+ the order of the groups I_k/P_k, I_k/P_k^+, respectively. Hence, $h^+=h$ if k is imaginary or k is real and contains a unit of norm -1 and $h^+=2h$ otherwise. In this paper, we say that two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent and write $\mathfrak{a}\sim\mathfrak{b}$ only when $\mathfrak{a}$ and $\mathfrak{b}$ belong to the same class modulo P_k^+; this is usually called the strict equivalence of ideals.

For each ideal $\mathfrak{a}\in I_k$, we denote by $N_{\mathfrak{a}}$ the function

$$N_{\mathfrak{a}}:\ \mathfrak{a}\longrightarrow \boldsymbol{Z}\qquad \text{defined by } N_{\mathfrak{a}}x=(N\mathfrak{a})^{-1}Nx\,,\qquad x\in\mathfrak{a}\,.$$

Suppose that $\mathfrak{a}\sim\mathfrak{b}$ with $\mathfrak{b}=id(\rho)\mathfrak{a}$, $N\rho>0$. Since $N(id(\rho))=N\rho$, we have

(2.1) $$N_{\mathfrak{b}}(\rho x)=N_{\mathfrak{a}}x\,,\qquad x\in\mathfrak{a}\,.$$

For $n\in \boldsymbol{N}$, put

$$I_k^+(n)=\{\mathfrak{j}\in I_k^+,\ N\mathfrak{j}=n\}\,,$$

$$X_{\mathfrak{a}}(n)=(N_{\mathfrak{a}})^{-1}(n)=\{x\in\mathfrak{a},\ N_{\mathfrak{a}}x=n\}\,.$$

Call $d_{\mathfrak{a}}$ the map:

$$X_{\mathfrak{a}}(n)\longrightarrow I_k^+(n)\qquad \text{defined by } d_{\mathfrak{a}}(x)=\mathfrak{a}^{-1}id(x)\,.$$

These maps $d_{\mathfrak{a}}$, $\mathfrak{a}\in I_k$, have the following properties:

(d, 1) $$\mathfrak{a}\sim\mathfrak{b}\ \Rightarrow\ \operatorname{Im} d_{\mathfrak{a}}=\operatorname{Im} d_{\mathfrak{b}}\,,$$

(d, 2) $$\mathfrak{a}\not\sim\mathfrak{b}\ \Rightarrow\ \operatorname{Im} d_{\mathfrak{a}}\cap\operatorname{Im} d_{\mathfrak{b}}=\emptyset\,,$$

(d, 3) for any $\mathfrak{j}\in I_k^+(n)$, there is an ideal $\mathfrak{a}\in I_k$ such that $\mathfrak{j}\in\operatorname{Im} d_{\mathfrak{a}}$.

In fact, let $\mathfrak{b}=id(\rho)\mathfrak{a}$, $N\rho>0$. For any $x\in X_{\mathfrak{a}}(n)$, we have $\rho x\in X_{\mathfrak{b}}(n)$ from (2.1) and $d_{\mathfrak{a}}(x)=\mathfrak{a}^{-1}id(x)=\mathfrak{b}^{-1}id(\rho)id(x)=d_{\mathfrak{b}}(\rho x)$, which proves that $\operatorname{Im} d_{\mathfrak{a}}\subseteq\operatorname{Im} d_{\mathfrak{b}}$. Similarly, we have $\operatorname{Im} d_{\mathfrak{b}}\subseteq\operatorname{Im} d_{\mathfrak{a}}$ and (d, 1) is proved. Next, suppose that there is an ideal $\mathfrak{j}$ in $\operatorname{Im} d_{\mathfrak{a}}\cap\operatorname{Im} d_{\mathfrak{b}}$; hence $\mathfrak{j}=\mathfrak{a}^{-1}id(x)=\mathfrak{b}^{-1}id(y)$, $x\in X_{\mathfrak{a}}(n)$, $y\in X_{\mathfrak{b}}(n)$ and so $\mathfrak{b}=id(\rho)\mathfrak{a}$ with $\rho=y/x$, $N\rho>0$, which proves (d, 2). Finally, for a given $\mathfrak{j}\in I_k^+(n)$, put $\mathfrak{a}=\mathfrak{j}^{-1}$. Since $\mathfrak{j}$ is integral, $\mathfrak{a}=\mathfrak{j}^{-1}$ contains 1 and $N_{\mathfrak{a}}1=(N\mathfrak{a})^{-1}=N\mathfrak{j}=n$, which shows that $1\in X_{\mathfrak{a}}(n)$. Now, we have $\mathfrak{j}=\mathfrak{a}^{-1}=d_{\mathfrak{a}}(1)$, and hence $\mathfrak{j}\in\operatorname{Im} d_{\mathfrak{a}}$, which proves (d, 3).

From now on, call $\mathfrak{a}_1,\cdots,\mathfrak{a}_{h^+}$ a complete set of representatives of I_k/P_k^+. From (d, 1), (d, 2), (d, 3), we see that $I_k^+(n)$ is the disjoint union of $\operatorname{Im} d_{\mathfrak{a}_i}$, $1\leqq i\leqq h^+$. Now, put

$$\mathfrak{o}_k^*=\{\varepsilon\in\mathfrak{o}_k^\times,\ N\varepsilon=1\}\,.$$

This group acts on $X_{\mathfrak{a}}(n)$. Let $X_{\mathfrak{a}}^*(n)$ be the quotient of $X_{\mathfrak{a}}(n)$ with respect to the equivalence relation defined by the action of $\mathfrak{o}_k^*$. Since $d_{\mathfrak{a}}(x)=d_{\mathfrak{a}}(x')$ if and only if $x'=\varepsilon x$ for some $\varepsilon\in\mathfrak{o}_k^*$, we have the relation

(2.2) $$[I_k^+(n)]=\sum_{i=1}^{h^+}[X_{\mathfrak{a}_i}^*(n)]\,.$$

On the other hand, from the well-known relation $\zeta_k(s)=\zeta_{\boldsymbol{Q}}(s)L(\chi, s)$, χ being the character of $k/\boldsymbol{Q}$, it follows that

(2.3) $$[I_k^+(n)]=\sum_{d\mid n}\chi(d)\,,$$

and so we have

(2.4) $$\sum_{i=1}^{h^+}[X_{\mathfrak{a}_i}^*(n)]=\sum_{d\mid n}\chi(d)\,.$$

If, in particular, k is imaginary, then $h^+=h$, $\mathfrak{o}_k^*=\mathfrak{o}_k^\times$, $[\mathfrak{o}_k^\times]=2$, 4 or 6, and we obtain the formula

(2.5) $$\sum_{i=1}^{h}[X_{\mathfrak{a}_i}(n)]=[\mathfrak{o}_k^\times]\sum_{d\mid n}\chi(d),$$

which gives the total number of representations of n by a complete system of non-equivalent positive definite binary quadratic forms whose discriminant is the discriminant of k.

§ 3. Representation of integral vectors by the Hopf map.

Let $k=\boldsymbol{Q}(\mu)$ be a quadratic field. We may assume that $m=\mu^2$ is a square free rational integer. We shall maintain notations in § 1 except putting $K=k$, $F=\boldsymbol{Q}$.

Consider the Hopf map h belonging to k:

$$h:\ A=k\times k\longrightarrow A_0=\boldsymbol{Q}\times k,$$

$$h(z)=(Nx-Ny,\ 2\mu\bar{x}y),\qquad z=(x,y)\in A.$$

For an ideal $\mathfrak{a}\in I_k$, we denote by $h_{\mathfrak{a}}$ the map:

$$h_{\mathfrak{a}}:\ \mathfrak{a}\times\mathfrak{a}\longrightarrow \boldsymbol{Z}\times\mathfrak{o}_k$$

defined by

$$h_{\mathfrak{a}}(z)=(N_{\mathfrak{a}}x-N_{\mathfrak{a}}y,\ 2\mu(N\mathfrak{a})^{-1}\bar{x}y),\qquad z=(x,y)\in\mathfrak{a}\times\mathfrak{a}.$$

Notice that $2\mu(N\mathfrak{a})^{-1}\bar{x}y\in id(2\mu)$. For an integer $t\in\boldsymbol{Z}$, $t\neq 0$, put

$$S_A(t)_{\mathfrak{a}}=\{z=(x,y)\in\mathfrak{a}\times\mathfrak{a},\ N_{\mathfrak{a}}x+N_{\mathfrak{a}}y=t\},$$

$$S_{A_0}(t)^*_{\boldsymbol{Z}}=\{w=(u,v)\in\boldsymbol{Z}\times id(2\mu),\ Nw=t\}.$$

Since $N(h_{\mathfrak{a}}(z))=(N\mathfrak{a})^{-2}N(\bar{z}\mu z)=-m((N\mathfrak{a})^{-1}Nz)^2$, the map $h_{\mathfrak{a}}$ induces the map

$$h_{\mathfrak{a},t}:\ S_A(t)_{\mathfrak{a}}\longrightarrow S_{A_0}(-mt^2)^*_{\boldsymbol{Z}}.$$

Our problem is to study the fibres of the map $h_{\mathfrak{a},t}$, $\mathfrak{a}\in I_k$. To do this, we shall first modify the diagram (1.1) as follows. We put

$$f_{\mathfrak{a}}(z)=(N_{\mathfrak{a}}x,\ (N\mathfrak{a})^{-1}\bar{x}y,\ N_{\mathfrak{a}}y),$$

$$\varphi_{\mathfrak{a}}(z)=(N_{\mathfrak{a}}x,\ T_{\mathfrak{a}}(\bar{x}y),\ N_{\mathfrak{a}}y),$$

where $T_{\mathfrak{a}}(\alpha)=(N\mathfrak{a})^{-1}T\alpha$, $\alpha\in k$. Then, (1.1) induces the commutative diagram (3.1) where

$$\Sigma_{\boldsymbol{Z}}=\{\sigma=(a,\beta,c)\in\Sigma\cap(\boldsymbol{Z}\times\mathfrak{o}_k\times\boldsymbol{Z})\}$$

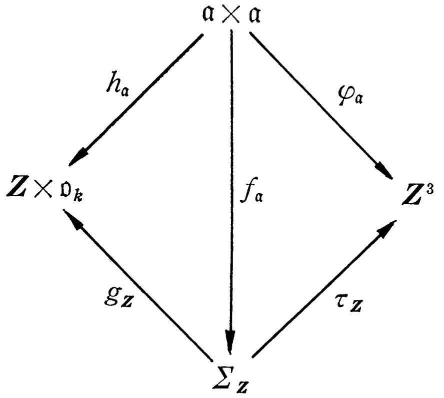
and $g_{\boldsymbol{Z}}$, $\tau_{\boldsymbol{Z}}$ are restrictions of g, τ in (1.1), respectively. Note that $\operatorname{Im} g_{\boldsymbol{Z}}$ is contained in $\boldsymbol{Z}\times id(2\mu)$.

Next, we shall consider the portion of (3.1) corresponding to $t\in\boldsymbol{Z}$, $t\neq 0$, as follows. We put

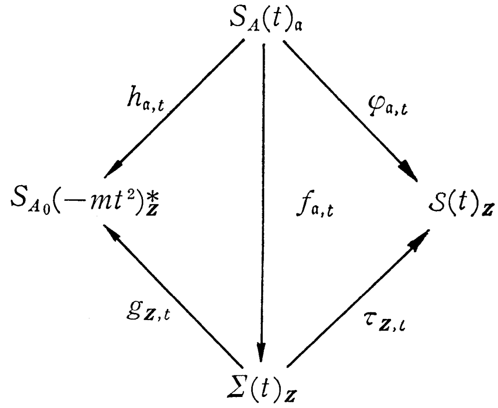
$$\Sigma(t)_{\boldsymbol{Z}}=\{\sigma=(a,\beta,c)\in\Sigma_{\boldsymbol{Z}},\ a+c=t\}\,,$$

$$\mathcal{S}(t)_{\boldsymbol{Z}}=\{s=(a,b,c)\in\boldsymbol{Z}^3,\ a+c=t\}\,.$$

Then, $f_{\mathfrak{a}}$, $\varphi_{\mathfrak{a}}$ induce the maps, $f_{\mathfrak{a},t}$, $\varphi_{\mathfrak{a},t}$, respectively. It is almost trivial to check that the diagram (3.2) is well-defined and commutative. The only non-trivial map is $g_{\boldsymbol{Z},t}$ and we shall focus our attention on this map.



(3.1)



(3.2)

First of all, $g_{\boldsymbol{Z},t}$ is well-defined. In fact, take an element $\sigma=(a,\beta,c)\in\Sigma(t)_{\boldsymbol{Z}}$. Then, $g_{\boldsymbol{Z}}(\sigma)=(a-c,2\mu\beta)=(a-c)\mu+(2\mu\beta)j$. Hence $N(g(\sigma))=N((a-c)\mu)+N(2\mu\beta)$ $=-m(a-c)^2-4mN\beta=-m((a-c)^2+4ac)=-m(a+c)^2=-mt^2$. Now, we shall prove that $g_{\boldsymbol{Z},t}$ is *bijective*. Suppose first that $g_{\boldsymbol{Z},t}(\sigma)=g_{\boldsymbol{Z},t}(\sigma')$ with $\sigma=(a,\beta,c)$, $\sigma'=(a',\beta',c')$. Then we have $\beta=\beta'$ and $a-c=a'-c'$, but since $\sigma,\sigma'\in\Sigma(t)_{\boldsymbol{Z}}$, we have $a+c=a'+c'=t$, and so $\sigma=\sigma'$, i.e. $g_{\boldsymbol{Z},t}$ is injective. Next, take an element $w=(u,v)=u\mu+vj\in S_{A_0}(-mt^2)^*_{\boldsymbol{Z}}$, $u\in\boldsymbol{Z}$, $v\in id(2\mu)$. We have, then

$$-mt^2=Nw=-mu^2+Nv\,. \tag{3.3}$$

Now, put $a=\frac{1}{2}(t+u)$, $\beta=\frac{1}{2}\mu^{-1}v$, $c=\frac{1}{2}(t-u)$. Then, $\beta\in\mathfrak{o}_k$ since $v\in id(2\mu)$. Substituting $v=2\mu\beta$ in (3.3), we get $t^2=u^2+4N\beta$, hence $a,c\in\boldsymbol{Z}$, $a+c=t$ and $N\beta=ac$. Thus, we see that $\sigma=(a,\beta,c)\in\Sigma(t)_{\boldsymbol{Z}}$. Finally, we have $g_{\boldsymbol{Z},t}(\sigma)=$ $(a-c,2\mu\beta)=(u,v)=w$, which proves that $g_{\boldsymbol{Z},t}$ is bijective. Therefore, *the study of the map* $h_{\mathfrak{a},t}$ *is reduced to the study of the map* $f_{\mathfrak{a},t}$.

Here, we can make one more reduction in view of the following:

(3.4)[1] $$f_{\mathfrak{a},t}^{-1}(\sigma)=f_{\mathfrak{a}}^{-1}(\sigma)\,,\qquad \sigma\in\Sigma(t)_{\boldsymbol{Z}}\,.$$

It is enough to show that $f_{\mathfrak{a}}^{-1}(\sigma)\subset f_{\mathfrak{a},t}^{-1}(\sigma)$. So, take $z\in f_{\mathfrak{a}}^{-1}(\sigma)$. Then $f_{\mathfrak{a}}(z)$

1) The similar equality for the map h, i.e. the equality $h_{\mathfrak{a},t}^{-1}(w)=h_{\mathfrak{a}}^{-1}(w)$ for $w\in$ $S_{A_0}(-mt^2)^*_{\boldsymbol{Z}}$ is not true unless k is imaginary.

$=(N_{\mathfrak{a}}x, (N\mathfrak{a})^{-1}\bar{x}y, N_{\mathfrak{a}}y)=\sigma$, hence $N_{\mathfrak{a}}x+N_{\mathfrak{a}}y=a+c=t$, which implies that $z\in f_{\mathfrak{a}}^{-1}(\sigma)\cap S_A(t)_{\mathfrak{a}}=f_{\mathfrak{a},t}^{-1}(\sigma)$ which proves (3.4).

Hence, after having taken a $\sigma\in\Sigma(t)_{Z}$, we can forget about t and *we only have to consider the simpler diagram*

$$\mathfrak{a}\times\mathfrak{a}\xrightarrow{f_{\mathfrak{a}}}\Sigma_{Z}\xrightarrow{\tau_{Z}}Z^3,\qquad \varphi_{\mathfrak{a}}=\tau_Z\circ f_{\mathfrak{a}}:\mathfrak{a}\times\mathfrak{a}\to Z^3 \tag{3.5}$$

and the fibre $f_{\mathfrak{a}}^{-1}(\sigma)$, $\sigma=(a, \beta, c)\in\Sigma_{Z}$ *with* $a+c\neq 0$.

The map $f_{\mathfrak{a}}$ will play the role of $N_{\mathfrak{a}}$ in § 2. In accordance with the notation in § 2, we shall put

$$X_{\mathfrak{a}}(\sigma)=f_{\mathfrak{a}}^{-1}(\sigma)=\{z=(x, y)\in\mathfrak{a}\times\mathfrak{a},\ f_{\mathfrak{a}}(z)=\sigma\}$$

and look for a map $d_{\mathfrak{a}}: X_{\mathfrak{a}}(\sigma)\to I_k^+(n_\sigma)$, where n_σ being determined by σ in a certain way, so that the properties (d, 1), (d, 2), (d, 3) will hold for this map.

For these purposes, let us consider the following diagram

$$\begin{array}{ccc} \mathfrak{a}\times\mathfrak{a}-\{0\} & \xrightarrow{D_{\mathfrak{a}}} & I_k^+ \\ \varphi_{\mathfrak{a}}\big\downarrow & & \big\downarrow N \\ Z^3-\{0\} & \xrightarrow{id} & N \end{array} \tag{3.6}$$

where $\varphi_{\mathfrak{a}}(z)=(N_{\mathfrak{a}}x, T_{\mathfrak{a}}(\bar{x}y), N_{\mathfrak{a}}y)$ for $z=(x, y)$, id is to take the greatest common divisor of three integers, N is the norm of ideals and $D_{\mathfrak{a}}$ is to be defined so that the diagram is commutative. Namely, we put

$$D_{\mathfrak{a}}(z)=\mathfrak{a}^{-1}id(x, y)\qquad\text{for } z=(x, y)\in\mathfrak{a}\times\mathfrak{a}-\{0\}\ .$$

Then, we have

$$\begin{aligned} N(D_{\mathfrak{a}}(z)) &= (N\mathfrak{a})^{-1}id(x, y)id(\bar{x}, \bar{y}) \\ &= (N\mathfrak{a})^{-1}c(xX+yY)c(\bar{x}X+\bar{y}Y) \\ &= (N\mathfrak{a})^{-1}c((xX+yY)(\bar{x}X+\bar{y}Y)) \\ &= (N\mathfrak{a})^{-1}c((Nx)X^2+T(\bar{x}y)XY+(Ny)Y^2) \\ &= (N\mathfrak{a})^{-1}id(Nx, T(\bar{x}y), Ny) \\ &= id(N_{\mathfrak{a}}x, T_{\mathfrak{a}}(\bar{x}y), N_{\mathfrak{a}}y) \\ &= id(\varphi_{\mathfrak{a}}(z))\ , \end{aligned}$$

i. e. (3.6) is commutative. Therefore, if we call n_σ the greatest common divisor of three integers a, $T\beta$, c for $\sigma=(a, \beta, c)\in\Sigma_{Z}$, then, for $z\in X_{\mathfrak{a}}(\sigma)$, we have $N(D_{\mathfrak{a}}(z))=id(\varphi_{\mathfrak{a}}(z))=id(\tau_{Z}f_{\mathfrak{a}}(z))=id(\tau_{Z}(\sigma))=id(a, T\beta, c)=n_\sigma$, and so we can define the map

$$d_{\mathfrak{a}}:\ X_{\mathfrak{a}}(\sigma) \longrightarrow I_k^+(n_\sigma)$$

by restricting $D_{\mathfrak{a}}$ on $X_{\mathfrak{a}}(\sigma)$. We shall now prove the three properties:

(d, 1) $\qquad \mathfrak{a}\sim\mathfrak{b} \ \Rightarrow\ \operatorname{Im} d_{\mathfrak{a}}=\operatorname{Im} d_{\mathfrak{b}}$,

(d, 2) $\qquad \mathfrak{a}\not\sim\mathfrak{b} \ \Rightarrow\ \operatorname{Im} d_{\mathfrak{a}}\cap\operatorname{Im} d_{\mathfrak{b}}=\emptyset$,

(d, 3) $\qquad$ for any $\mathfrak{j}\in I_k^+(n_\sigma)$, there is an ideal $\mathfrak{a}$ such that $\mathfrak{j}\in\operatorname{Im} d_{\mathfrak{a}}$.

In fact, let $\mathfrak{b}=id(\rho)\mathfrak{a}$ with $N\rho>0$. We have then

$$f_{\mathfrak{b}}(\rho z)=f_{\mathfrak{a}}(z) \qquad \text{for } z\in\mathfrak{a}\times\mathfrak{a}.$$

Therefore, for any $z\in X_{\mathfrak{a}}(\sigma)$, we have $\rho z\in X_{\mathfrak{b}}(\sigma)$ and $d_{\mathfrak{b}}(\rho z)=\mathfrak{b}^{-1}id(\rho x, \rho y)=\mathfrak{a}^{-1}id(x, y)=d_{\mathfrak{a}}(z)$, which proves that $\operatorname{Im} d_{\mathfrak{a}}\subseteq\operatorname{Im} d_{\mathfrak{b}}$. Similarly, we have $\operatorname{Im} d_{\mathfrak{b}}\subseteq\operatorname{Im} d_{\mathfrak{a}}$, and (d, 1) is proved. Next, suppose there is an ideal $\mathfrak{j}\in\operatorname{Im} d_{\mathfrak{a}}\cap\operatorname{Im} d_{\mathfrak{b}}$. Hence, $\mathfrak{j}=d_{\mathfrak{a}}(z)=\mathfrak{a}^{-1}id(x, y)=d_{\mathfrak{b}}(\zeta)=\mathfrak{b}^{-1}id(\xi, \eta)$ for some $z=(x, y)\in\mathfrak{a}\times\mathfrak{a}$, $\zeta=(\xi, \eta)\in\mathfrak{b}\times\mathfrak{b}$ with $f_{\mathfrak{a}}(z)=f_{\mathfrak{b}}(\zeta)=(N_{\mathfrak{a}}x, (N\mathfrak{a})^{-1}\bar{x}y, N_{\mathfrak{a}}y)=(N_{\mathfrak{b}}\xi, (N\mathfrak{b})^{-1}\bar{\xi}\eta, N_{\mathfrak{b}}\eta)=\sigma=(a, \beta, c)$. Since $a+c\neq 0$, either $a\neq 0$ or $c\neq 0$. Without loss of generality, we may assume that $a\neq 0$. Then, from the above equalities, we get $y/x=\eta/\xi=\beta/a$. Hence, $id(x, y)=id(x, x\beta a^{-1})=id(x)id(1, \beta a^{-1})$ and $id(\xi, \eta)=id(\xi)id(1, \beta a^{-1})$. Since $\mathfrak{a}^{-1}id(x, y)=\mathfrak{b}^{-1}id(\xi, \eta)$, we have $\mathfrak{a}^{-1}id(x)=\mathfrak{b}^{-1}id(\xi)$, i.e. $\mathfrak{a}\sim\mathfrak{b}$ because $N(x/\xi)>0$ by the equality $N_{\mathfrak{a}}x=N_{\mathfrak{b}}\xi$, which proves (d, 2). Finally, we must prove that for a given $\mathfrak{j}\in I_k^+(n_\sigma)$ there is an ideal $\mathfrak{a}$ and an element $z\in\mathfrak{a}\times\mathfrak{a}$ such that $f_{\mathfrak{a}}(z)=\sigma$ and that $d_{\mathfrak{a}}(z)=\mathfrak{j}$; here the latter condition forces us to put $\mathfrak{a}=\mathfrak{j}^{-1}id(x, y)$ when $z=(x, y)$. Now, for $\sigma=(a, \beta, c)$, we claim that $x=a$, $y=\beta$ satisfy our requirement. In fact, since $1\in\mathfrak{j}^{-1}$, $\mathfrak{a}$ contains a and β, i.e. $z\in\mathfrak{a}\times\mathfrak{a}$. Next, we have $f_{\mathfrak{a}}(z)=(N_{\mathfrak{a}}x, (N\mathfrak{a})^{-1}\bar{x}y, N_{\mathfrak{a}}y)=(N\mathfrak{a})^{-1}(a^2, a\beta, N\beta)=(N\mathfrak{a})^{-1}a(a, \beta, c)=(N\mathfrak{a})^{-1}a\sigma$. However, we have $N\mathfrak{a}=(N\mathfrak{j})^{-1}N(id(a, \beta))=n_\sigma^{-1}c(N(aX+\beta Y))=n_\sigma^{-1}c(a^2X^2+(aT\beta)XY+acY^2)=n_\sigma^{-1}an_\sigma=a$; hence $f_{\mathfrak{a}}(z)=(N\mathfrak{a})^{-1}a\sigma=\sigma$, which completes the proof of (d, 3).

Therefore, if we call $\mathfrak{a}_1, \cdots, \mathfrak{a}_{h^+}$ a complete set of representatives of I_k/P_k^+, we see that $I_k^+(n_\sigma)$ is the disjoint union of $\operatorname{Im} d_{\mathfrak{a}_i}$, $1\leqq i\leqq h^+$. As in § 2, the group $\mathfrak{o}_k^*$ acts on $X_{\mathfrak{a}}(\sigma)$ by $z\to\varepsilon z$. Let $X_{\mathfrak{a}}^*(\sigma)$ be the quotient of $X_{\mathfrak{a}}(\sigma)$ with respect to the equivalence relation defined by the action of $\mathfrak{o}_k^*$. We claim that

$$d_{\mathfrak{a}}(z)=d_{\mathfrak{a}}(z') \ \Leftrightarrow\ z'=\varepsilon z \qquad \text{for some } \varepsilon\in\mathfrak{o}_k^*.$$

In fact, we only have to prove ($\Rightarrow$). Now, the assumption implies that

(3.7) $\qquad id(x, y)=id(x', y') \qquad$ for $z=(x, y)$, $z'=(x', y')\in X_{\mathfrak{a}}(\sigma)$.

Since $f_{\mathfrak{a}}(z)=f_{\mathfrak{a}}(z')$, we have $Nx=Nx'$, $Ny=Ny'$, $\bar{x}y=\bar{x}'y'$. Hence, there is a $\rho\in k^\times$ such that $N\rho=1$ and that $x'=\rho x$, $y'=\rho y$. Substituting these relations in (3.7), we get $\rho\in\mathfrak{o}_k^*$, which proves our assertion.

We have therefore the relation

$$[I_k^+(n_\sigma)]=\sum_{i=1}^{h^+}[X_{\mathfrak{a}_i}^*(\sigma)]\,. \tag{3.8}$$

Using (2.3), we can write (3.8) as

$$\sum_{i=1}^{h^+}[X_{\mathfrak{a}_i}^*(\sigma)]=\sum_{d\mid n_\sigma}\chi(d)\,. \tag{3.9}$$

It is easy to translate all these in terms of the Hopf map h:

$$h(z)=\bar{z}\mu z=(Nx-Ny,\ 2\mu\bar{x}y)\,.$$

For any $w=(u,v)=u\mu+vj=u\mu+v_0j+v_1\mu j\in S_{A_0}(-mt^2)_Z^*$, there is a unique $\sigma\in\Sigma(t)_Z$ such that $g_{Z,t}(\sigma)=w$. In fact, we see that

$$\sigma=\Big(\frac{1}{2}(t+u),\ \frac{1}{2}\mu^{-1}v,\ \frac{1}{2}(t-u)\Big)\,. \tag{3.10}$$

A simple computation shows that

$$T\Big(\frac{1}{2}\mu^{-1}v\Big)=v_1\,. \tag{3.11}$$

Therefore, if we define n_w to be n_σ, we see from (3.10), (3.11) that n_w is the greatest common divisor of $\frac{1}{2}(t+u)$, v_1 and $\frac{1}{2}(t-u)$. Since we have $h_{\mathfrak{a},t}^{-1}(w)=f_{\mathfrak{a},t}^{-1}(\sigma)=f_{\mathfrak{a}}^{-1}(\sigma)=X_{\mathfrak{a}}(\sigma)$, if we denote by $h_{\mathfrak{a},t}^{-1}(w)^*$ the quotient of the fibre $h_{\mathfrak{a},t}^{-1}(w)$ with respect to the equivalence relation under the action of the group $\mathfrak{o}_k^*$, we obtain the equality

$$\sum_{i=1}^{h^+}[h_{\mathfrak{a}_i,t}^{-1}(w)^*]=\sum_{d\mid n_w}\chi(d)\,, \tag{3.12}$$

which can be considered as an analogue for the Hopf map of the Dirichlet formula for binary quadratic forms.

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