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Reduction theorems for characters of finite groups of Lie type*

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Introduction.

The first main result of this paper is a formula for the values of irreducible complex characters of finite Chevalley groups, of normal or twisted type, on elements whose centralizers are contained in Levi factors of parabolic subgroups. The result is based on the organization, due to Harish-Chandra [**13**] and Springer ([**18**], [**19**]), of the character theory of these groups from the point of view of cusp forms, and can be stated as follows.

THEOREM A. *Let G be a finite group with a split (B, N)-pair of characteristic p, and let (W, R) be the Coxeter system of G. Let ζ be an irreducible complex-valued character of G, such that $(\zeta, \tilde{\varphi}^G)\neq 0$, where φ is an irreducible cuspidal character of some Levi factor L_J of a parabolic subgroup P_J (for $J\subseteqq R$), and $\tilde{\varphi}$ is the extension of φ to P_J with $O_p(P_J)\leqq \ker\tilde{\varphi}$. Let $x\in G$ be an element such that $C_G(x)\leqq L_{J'}$, for some Levi factor $L_{J'}$ of another parabolic subgroup $P_{J'}$. Then $\zeta(x)=0$ unless there exists a subset $J''\subseteqq J'$ such that L_J and $L_{J''}$ are conjugate by an element of the Coxeter group W. If this occurs, then the value $\zeta(x)$ is given by*

$$\zeta(x)=\sum(\zeta, \tilde{\lambda}^G)\lambda(x)\,,$$

where the sum is taken over irreducible characters λ of $L_{J'}$, such that $\lambda\in\tilde{\eta}^{L_{J'}}$, for an irreducible cuspidal character η of $L_{J''}$, with $J''\subseteqq J'$, and $L_{J''}$ conjugate to L_J by an element of W.

A sharper version of Theorem A gives the value of the character $\zeta(x)$ on an element x whose semisimple, or p-regular, part x_s has $L_{J'}$ as its centralizer, in terms of certain decomposition numbers and the values of the characters λ in Theorem A on the unipotent part x_u of x. More precisely,

$$\zeta(x)=\sum\alpha_{\zeta,\lambda}^{x_s}\lambda(x_u)\,,$$

for certain algebraic integers $\alpha_{\zeta,\lambda}^{x_s}$, corresponding to x_s, ζ, and the characters

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$\{\lambda\}$ as in the statement of Theorem A. This theorem is somewhat analogous in appearance to Brauer's Second Main Theorem ([**15**]), where the value of an irreducible character belonging to a given p-block, on an element x, is given in terms of values of characters belonging to corresponding blocks of the centralizer of the p-part of x. In our situation, the block theory is replaced by the description of the characters due to Harish-Chandra and Springer.

The second main result is a reduction formula for characters in 1_B^G, which can be stated as follows.

THEOREM B. *Let $\{G(q)\}$ be a system of finite groups with split (B, N)-pairs of characteristic $p|q$, as in* [**1**], *with $\{q\}$ a set of prime powers containing all primes, and let (W, R) be the common Coxeter system of the groups $\{G(q)\}$. For each q, the irreducible characters in $1_{B(q)}^{G(q)}$ are in a natural bijective correspondence with the characters of W. Let $\zeta_{\varphi,q}\in 1_{B(q)}^{G(q)}$ be the irreducible character of $G(q)$ corresponding to a character φ of W. Let $x\in G(q)$ be an element whose centralizer $C_{G(q)}(x)\leqq L_J(q)$ for some $J\subseteqq R$, with $L_J(q)$ a Levi factor of the parabolic subgroup $P_J(q)$ of $G(q)$. Then for each irreducible character $\zeta_{\varphi,q}\in 1_{G(q)}^{G(q)}$,*

$$\zeta_{\varphi,q}(x)=\sum(\zeta_{\varphi,q}, \tilde{\eta}_{\psi,q}^{G(q)})\eta_{\psi,q}(x)\,,$$

where the $\tilde{\eta}_{\psi,q}$ are characters in $1_{B_J(q)}^{L_J(q)}$, corresponding to characters $\{\psi\}$ of W_J, extended to $P_J(q)$ as in Theorem A. The multiplicities $(\zeta_{\varphi,q}, \tilde{\eta}_{\psi,q}^{G(q)})$ are independent of q, and are equal to the multiplicities (φ, ψ^W) of the corresponding characters of the Coxeter groups.

It follows from Theorem B that the values of the characters in 1_B^G on p-regular elements whose centralizers are Levi factors of parabolic subgroups, are given generically, as polynomials in q. This result, in turn, implies that if $\mathfrak{C}$ is the conjugacy class of such an element, then $|\mathfrak{C}\cap BwB|$ is given generically, as a polynomial in q. The formulas for the class intersection numbers $|\mathfrak{C}\cap BwB|$ can be used to show that the values of the irreducible characters in 1_B^G on certain semi-simple elements not necessarily conjugate to elements in the split torus, are given generically, as polynomials in q (see [**10**]).

The paper is organized into two chapters. The first contains an exposition of the work of Harish-Chandra and Springer in the setting of finite groups with split (B, N)-pairs of characteristic p. This approach starts from an axiomatic description of the Chevalley groups and their twisted analogues, in the language of finite groups. The results in § 1-3 are known, for finite groups of K-rational points on reductive algebraic groups defined over finite fields ([**13**], [**18**], [**19**]). The first main result appears in § 4. The second chapter is concerned with a reduction theorem for 1_B^G and generic character values on elements of the standard torus $T=B\cap N$. The appropriate set-up for this topic is furnished by a system of (B, N)-pairs, of type (W, R), where (W, R)

is a given Coxeter system ([1], [7]).

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Chapter I. Reduction theorems for finite groups with split (B, N)-pairs

1. Preliminary results. The Levi decomposition.

Standard notation from finite group theory will be used. In particular $H \leqq G$ means that H is a subgroup of G, while $X \subseteqq G$ means X is a subset. It will be convenient to write $a^{b^{-1}} = {}^{b}a = bab^{-1}$, for $a, b \in G$. Similarly $X^{b^{-1}} = {}^{b}X = bXb^{-1}$, for $X \subseteqq G$. (A, B) denotes the set of commutators $(a, b) = a^{-1}b^{-1}ab$, for $a \in A$, $b \in B$. $O_p(G)$ is the unique maximal normal p-subgroup of G, for a prime p.

We begin by recalling some facts about *finite groups with split* (B, N)-*pairs of characteristic* p, for some prime p, ([16], [5]). Such a group G has, first of all, a (B, N)-pair $\{B, N\}$, associated with a Coxeter system $\{W, R\}$, where $R = \{w_1, \cdots, w_n\}$ is the set of distinguished generators of the finite Coxeter group W. We shall set $T = B \cap N$; then $T \trianglelefteq N$, and $N/T \cong W$. It is assumed that the following conditions hold.

(1.1) The group B is the semidirect product $B = UT$, with $U = O_p(B)$, and $U \cap T = \{1\}$.

(1.2) T is an abelian p'-group.

(1.3) $T = \bigcap_{n \in N} B^n$.

Included in the definition is the special case of an abelian p'-group T, viewed as a split (B, N)-pair of characteristic p, with $B = N = T$, $W = \{1\}$, $R = \emptyset$, and $U = \{1\}$.

Let $\{G, B, N, W, R\}$ denote a finite group with a split (B, N)-pair of characteristic p. There exists a root system Δ in euclidean space E^n, such that (W, R) can be identified with the Weyl group of Δ. This means that $R = \{w_1, \cdots, w_n\}$ is identified with the reflections corresponding to a set of fundamental roots $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ in Δ. We denote by $l(w)$ the length of w as an element of the Coxeter system (W, R). The set of positive roots determined by Π is denoted by Δ_+; Δ_- denotes the negative roots. For each

$w \in W$, put $\Delta_w^- = \Delta_+ \cap w^{-1}(\Delta_-)$, $\Delta_w^+ = \Delta_+ \cap w^{-1}(\Delta_+)$. Let w_R denote the unique element of W of maximal length.

Let $\{n_w\}$ be a fixed set of coset representatives of T in N, such that $n_w T$ corresponds to the element $w \in W$. We write H^w instead of H^{n_w}, for subgroups H of G normalized by T.

We let $U^- = U^{w_R}$, and put

$$U_w^+ = U \cap U^w, \quad U_w^- = U \cap U^{w_R w}, \quad w \in W;$$

and

$$U_{\alpha_i} = U \cap U^{w_R w_i}, \qquad 1 \leqq i \leqq n.$$

(1.4) PROPOSITION. a) $U^- \cap B = \{1\}$.

b) $U = U_w^- U_w^+$, *and* $U_w^- \cap U_w^+ = \{1\}$ *for all* $w \in W$.

c) *There exists a bijection* $\alpha \to U_\alpha$ *between the roots* $\{\alpha\}$ *in* Δ *and the set of N-conjugates of* $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$, *such that for all* $w \in W$ *and* $\alpha \in \Delta$, ${}^w U_\alpha = U_{w(\alpha)}$.

d) *For each* $w \in W$, *there exists an ordering* $\{\beta_1, \beta_2, \cdots\}$ *of the roots in* Δ_w^-, *such that* $U_w^- = U_{\beta_1} U_{\beta_2} \cdots$, *and similarly for* U_w^+.

e) (Strong form of the Bruhat decomposition.) *Each* (B, B)*-double coset* $BwB = Bn_w U_w^-$. *Moreover,* $G = \bigcup_{w \in W} Bn_w U_w^-$, *and each element* $x \in G$ *can be expressed uniquely in the form* $x = bn_w u$, *with* $b \in B$, $w \in W$, *and* $u \in U_w^-$.

f) (Commutator Relations.) *If* $\{\alpha, \beta\}$ *are independent roots in* Δ, *then*

$$(U_\alpha, U_\beta) \subseteqq \prod_{i,j>0} U_{i\alpha + j\beta},$$

where the product is taken over all roots of the form $i\alpha + j\beta$, *with* $i, j > 0$, *in some order.*

The proofs of statements a)-e) are given in [**16**] and [**5**]. The commutator relations f) are proved in [**12**].

Let J be a subset of the set of distinguished generators R of W. We denote by W_J the parabolic subgroup of W generated by J, and by P_J the corresponding parabolic subgroup of G, given by $P_J = BW_J B$. We let Π_J be the set of fundamental roots corresponding to J, and Δ_J the root system generated by Π_J. Put $\Delta_{J,+} = \Delta_+ \cap \Delta_J$, $\Delta_{J,-} = \Delta_- \cap \Delta_J$. Let w_J denote the element of W_J of maximal length. The pair $\{W_J, J\}$ is also a Coxeter system.

(1.5) PROPOSITION. *Let* $J \neq \emptyset$, *and put* $L_J = \langle T, U_\alpha; \alpha \in \Delta_J \rangle$, $V_J = \langle U_\alpha; \alpha \in \Delta_+ - \Delta_{J,+} \rangle$. *Then the following statements hold.*

a) $V_J = O_p(P_J)$.

b) $P_J = L_J V_J$, *and* $L_J \cap V_J = \{1\}$.

c) $P_J = N_G(V_J)$.

d) *The subgroup* L_J *is a finite group with a split* (B, N)*-pair of characteristic* p, $\{B_J, N_J, U_J, W_J, J\}$, *with* $B_J = B \cap L_J$, $N_J = N \cap L_J$, $U_J = U \cap L_J = U_{w_J}^-$, *and Coxeter system* $\{W_J, J\}$.

PROOF. The proof is similar to the proof given in [6], § 8.5. As $\Delta_{J,+}=\Delta_{w_J}^-$, $V_J\leqq U_{w_J}^+$. By Proposition (1.4) (d), $U_{w_J}^+\leqq V_J$, and hence $V_J=U_{w_J}^+$. Let $\alpha\in\Delta_{w_J}^+$ and $\beta\in\Delta_J$. For all (i, j) such that $i>0$, $j>0$, and $i\alpha+j\beta\in\Delta$, we have $i\alpha+j\beta\in\Delta_{w_J}^+$. The commutator relations (1.4) (f) imply that $L_J\leqq N_G(V_J)$.

By the strong form of the Bruhat Theorem (Proposition (1.4) (e)), $P_J=\bigcup_{w\in W_J} UTn_wU_w^-$. By (1.4) (d), $U_w^-\leqq L_J$, for all $w\in W_J$. We require now the facts, first communicated to the author by Richen, and proved in [5], that for each $\alpha_i\in\Pi$, $U_{\alpha_i}T\cup U_{\alpha_i}Tn_{w_i}U_{\alpha_i}$ is a subgroup of G, and that $n_{w_i}T\cap U_{\alpha_i}U_{-\alpha_i}U_{\alpha_i}\neq\emptyset$. It follows that $n_w\in L_J$ for all $w\in W_J$. Apply (1.4) (b) to obtain $P_J=\bigcup_{w\in W_J} U_{w_J}^+U_{w_J}^-Tn_wU_w^-$. As $V_J=U_{w_J}^+$, and $U_{w_J}^-Tn_wU_w^-\subseteqq L_J$ for all $w\in W_J$, it follows that $P_J=V_JL_J$. Using the commutator relations and the fact that $U_{\alpha_i}T\cup U_{\alpha_i}Tn_{w_i}U_{\alpha_i}$ is a subgroup of G, it can be proved in the usual way (cf. [6], Chapter 8) that L_J has a (B, N)-pair (B_J, N_J) with Borel subgroup $B_J=U_{w_J}^-T$, and $N_J=\bigcup_{w\in W_J} n_wT$. Therefore $B\cap L_J=B_J$, and $V_J\cap L_J\leqq B_J\cap V_J=U_{w_J}^-T\cap U_{w_J}^+=\{1\}$, by the uniqueness part of Proposition (1.4) (b).

As $P_J=V_JL_J$, and L_J normalizes V_J, $V_J\leqq O_p(P_J)$. By (1.4) (e), U is a p-Sylow subgroup of P_J, and hence $O_p(P_J)\leqq U$. Suppose $x\in O_p(P_J)$. Then $x=vv'$, $v\in V_J$, $v'\in U_{w_J}^-$, and $x^{n_{w_J}}=v^{n_{w_J}}v'^{n_{w_J}}\in U$. Therefore $v'^{n_{w_J}}\in U\cap U^-$, and hence $v'=1$ by (1.4) (a). This completes the proof of parts a) and b) of Proposition (1.5).

For part c), we have $P_J\leqq N_G(V_J)$. If the inclusion is proper, then $N_G(V_J)=P_{J'}$, for some subset J' of R such that $J'\supset J$. For some $\alpha_i\in\Pi_{J'}-\Pi_J$, $U_{\alpha_i}\leqq V_J$, and $n_{w_i}\in P_{J'}=N_G(V_J)$. Then $U_{\alpha_i}^{w_i}\leqq V_J\cap U^-$, which is impossible. Therefore c) holds.

It has been noted that L_J has a (B, N)-pair with Borel subgroup $B_J=U_{w_J}^-\cdot T$. Evidently $U_{w_J}^-=O_p(B_J)$. Moreover, $\bigcap_{w\in W_J} B_J^w\leqq U_{w_J}^-T\cap(U_{w_J}^-T)^{w_J}=T$. It follows that L_J has the required split (B, N)-pair, and Proposition (1.5) is proved.

(1.6) DEFINITION. The subgroup L_J defined in Proposition (1.5), for $J\neq\emptyset$, is called a *standard Levi factor* of the parabolic subgroup P_J. The standard Levi factor $L_\emptyset$ of the parabolic subgroup $B=P_\emptyset$ is defined to be T. The parabolic subgroups $P_J=BW_JB$ containing the given Borel subgroup B are called *standard parabolic subgroups* of G. The factorization $P_J=L_JV_J$ is called a *Levi decomposition* of P_J.

(1.7) PROPOSITION. *Let $P_J=L_JV_J$ be a standard parabolic subgroup of G. For $J'\subset J$, let $P_{J,J'}=L_J\cap P_{J'}$, and $V_{J,J'}=L_J\cap V_{J'}$. Then $P_{J,J'}$ is a standard parabolic subgroup of L_J containing the Borel subgroup $B_J=B\cap L_J$, and has the Levi decomposition $P_{J,J'}=L_{J'}V_{J,J'}$. The map $P_{J'}\mapsto P_{J,J'}$ is a bijection of the set of all standard parabolic subgroups contained in P_J with the set of standard parabolic subgroups of L_J.*

PROOF. $L_J \cap P_{J'}$ contains B_J, and is a standard parabolic subgroup of L_J. Because $V_J \leqq V_{J'}$, $V_{J'} = V_J V_{J,J'}$ (semi-direct) and $P_{J'} = L_{J'} V_{J'} = (L_{J'} V_{J,J'}) V_J$. Moreover $L_{J'} V_{J,J'} \leqq L_J$. It follows that $P_{J,J'} = L_{J'} V_{J,J'}$ is a Levi decomposition of $P_{J,J'}$. If $P_{J,J'} = P_{J,J''}$ for $J', J'' \subseteqq J$, then $V_{J,J'} = V_{J,J''}$, and $V_{J'} = V_{J''}$. By Proposition (1.5) (c), we have $P_{J'} = P_{J''}$. Finally, if $\tilde{P}$ is a standard parabolic subgroup of L_J, then $\tilde{P} V_J$ is a subgroup containing B, and $\tilde{P} V_J \cap L_J = \tilde{P}$. This completes the proof.

2. Intersections of parabolic subgroups.

In § 3, it will be necessary to compute the scalar product (φ^G, ψ^G), for characters φ and ψ of parabolic subgroups P_J and $P_{J'}$. By Mackey's Theorem ([**11**], (9.8)), this problem involves subgroups of the form $P_J \cap P_{J'}^x$, where x is a $(P_J, P_{J'})$-double coset representative. This section contains results on these intersections, in finite groups with split (B, N)-pairs, which correspond to results proved by Harish-Chandra [**13**] and Springer ([**18**], [**19**]) in the context of algebraic groups.

Let $\{G, B, N, W, R\}$ be as in § 1. Let $J_1, J_2 \subseteqq R$. Then

(2.1) $$G = \bigcup_{w \in W_{J_1, J_2}} P_{J_1} w P_{J_2} = \bigcup_{w \in W_{J_1, J_2}} B W_{J_1} w W_{J_2} B$$

where W_{J_1,J_2} is the set of distinguished (W_{J_1}, W_{J_2})-double coset representatives of the subgroups W_{J_1} and W_{J_2} of W ([**4**], Example 3, p. 37).

(2.2) PROPOSITION (Kilmoyer [**14**]). *Let $J_1, J_2 \subseteqq R$, and let $w \in W_{J_1,J_2}$. Then $W_{J_1} \cap {}^w W_{J_2} = W_K$, where $K = J_1 \cap {}^w J_2$.*

All the results in this section are based on the preceding result.

(2.3) COROLLARY. *Let $J_1, J_2 \subseteqq R$. Then*

$$\Pi_{J_1} \cap w(\Pi_{J_2}) = \Pi_K , \qquad \Delta_{J_1} \cap w(\Delta_{J_2}) = \Delta_K .$$

For the rest of the section, J_1, J_2, $w \in W_{J_1,J_2}$, and K will be as in Proposition (2.2).

(2.4) PROPOSITION. $P_K = (P_{J_1} \cap {}^w P_{J_2}) V_{J_1}$.

PROOF. As $l(w_j w) \geqq l(w)$, for all $w_j \in J_1$, we have $w^{-1}(\Delta_{J_1,+}) \subseteqq \Delta_+$. Therefore $\Delta_{J_1,+} \subseteqq w(\Delta_+)$, $B_{J_1} \subseteqq {}^w P_{J_2} \cap P_{J_1}$, and $B \subseteqq (P_{J_1} \cap {}^w P_{J_2}) V_{J_1}$, so that $(P_{J_1} \cap {}^w P_{J_2}) V_{J_1} = P_J$, for some $J \subseteqq R$. Suppose $n_w b n_{w_2} b' = b_1 n_{w_1} b_1' n_w$, for $w_1 \in W_{J_1}$, $w_2 \in W_{J_2}$, $b, b', \cdots \in B$. Then $B w w_2 B \cap B w_1 w B \neq \emptyset$, because $w \in W_{J_1,J_2}$, and $l(w w_2) = l(w) + l(w_2)$, etc. Then $w w_2 = w_1 w$, and hence

$$(P_{J_1} \cap {}^w P_{J_2}) V_{J_1} \leqq B(W_{J_1} \cap {}^w W_{J_2}) B = P_K ,$$

by Proposition (2.2). The reverse inclusion is clear.

(2.5) PROPOSITION. a) $V_K = V_{J_1}(P_{J_1} \cap {}^w V_{J_2})$.

b) $P_{J_1} \cap {}^w V_{J_2} = (L_{J_1} \cap {}^w V_{J_2})(V_{J_1} \cap {}^w V_{J_2})$.

c) $L_{J_1} \cap {}^w P_{J_2}$ *is a standard parabolic subgroup of* L_{J_1}; *in fact,* $L_{J_1} \cap {}^w P_{J_2} = P_K \cap L_{J_1}$.

d) $O_p(L_{J_1} \cap {}^w P_{J_2}) = L_{J_1} \cap {}^w V_{J_2}$, *and a Levi decomposition of* $L_{J_1} \cap {}^w V_{J_2}$ *is given by* $L_{J_1} \cap {}^w P_{J_2} = L_K(L_{J_1} \cap {}^w V_{J_2})$.

PROOF. a) We have $U \geqq V_K \geqq V_{J_1}$, and

$$V_K = \langle U_\alpha : \alpha \in \Delta_{wK}^+ \rangle .$$

Suppose some U_α such that $\alpha \in \Delta_{wK}^+$, is not contained in V_{J_1}. Then $\alpha \in \Delta_{wJ_1}^- \subseteqq \Delta_{J_1}$, and, as $\alpha \notin \Delta_K$, $w^{-1}(\alpha) \notin \Delta_{J_2}$ by Corollary (2.3). Moreover $w^{-1}(\alpha) \in \Delta_+$ because $w \in W_{J_1,J_2}$ and $\alpha \in \Delta_{J_1,+}$, so that $U_{w^{-1}(\alpha)} \leqq V_{J_2}$ by the definition of V_{J_2} in Proposition (1.5). Therefore $U_\alpha \leqq {}^w V_{J_2} \cap P_{J_1}$, and as V_{J_1} is normalized by $P_{J_1} \cap {}^w V_{J_2}$ so that $V_{J_1}(P_{J_1} \cap {}^w V_{J_2})$ is a group, we have $V_K \leqq V_{J_1}(P_{J_1} \cap {}^w V_{J_2})$. As the right side belongs to $O_p(P_K)$, we have a), by Proposition (1.5) (a).

b) By the proof of part a), each U_α, $\alpha \in \Delta_{wK}^+$, which is not contained in V_{J_1} is contained in $P_{J_1} \cap {}^w V_{J_2}$. Moreover, $(U_\alpha, U_\beta) \subseteqq V_{J_1}$ for $U_\alpha \leqq V_{J_1}$, $U_\beta \leqq V_K$ because $V_{J_1} \trianglelefteq P_K$, by Proposition (2.4). Let $x \in P_{J_1} \cap {}^w V_{J_2}$; as $x \in V_K$, we have $x = \prod x_\alpha$, $x_\alpha \in U_\alpha$, $\alpha \in \Delta_{wK}^+$. Using the commutator formulas and the above remarks, we can rearrange the factors to obtain $x = x_{J_1} \cdot x'$, with $x_{J_1} \in V_{J_1}$, and $x' = \prod x_\beta$, with $\beta \in \Delta_{wK}^+$, $U_\beta \leqq P_{J_1} \cap {}^w V_{J_2}$, and $\beta \notin \Delta_{wJ_1}^+$. Then each $x_\beta \in L_{J_1} \cap {}^w V_{J_2}$, and hence $x_{J_1} \in V_{J_1} \cap {}^w V_{J_2}$, and $P_{J_1} \cap {}^w V_{J_2} \leqq (L_{J_1} \cap {}^w V_{J_2})(V_{J_1} \cap {}^w V_{J_2})$. The reverse inclusion is clear.

c) and d). By the proof of Proposition (2.4), $B_{J_1} \leqq {}^w P_{J_2}$, so that $L_{J_1} \cap {}^w P_{J_2}$ is a standard parabolic subgroup of L_{J_1}. Evidently $L_{J_1} \cap {}^w P_{J_2} \leqq P_K$. On the other hand, $P_K \cap L_{J_1}$ is a standard parabolic subgroup of L_{J_1}, with Levi factor L_K, and $O_p(P_K \cap L_{J_1}) = V_K \cap L_{J_1}$, by Proposition (1.7). By parts a) and b), $V_K \cap L_{J_1} = L_{J_1} \cap {}^w V_{J_2}$. It follows that $P_K \cap L_{J_1} = L_K(L_{J_1} \cap {}^w V_{J_2}) \leqq L_{J_1} \cap {}^w P_{J_2}$. These statements, taken together, establish c) and d).

The next two propositions are the important ones for the applications to character theory.

(2.6) PROPOSITION. *The following statements are equivalent.*

a) $P_K = P_{J_1}$,

b) $P_{J_1} \cap {}^w V_{J_2} \leqq V_{J_1}$,

c) $L_{J_1} \leqq {}^w L_{J_2}$.

PROOF. a) implies b), by part a) of Proposition (2.5). Conversely, b) implies $V_K = V_{J_1}$, and hence $P_K = P_{J_1}$ by Proposition (1.5) (c). Next, a) implies $W_K = W_{J_1}$, hence $\Delta_K = \Delta_{J_1}$. As $\Delta_K = \Delta_{J_1} \cap w(\Delta_{J_2})$ by Corollary (2.3), it follows that $L_{J_1} \leqq {}^w L_{J_2}$. Conversely, $L_{J_1} \leqq {}^w L_{J_2}$ implies $L_{J_1} \cap {}^w V_{J_2} = \{1\}$, and $P_{J_1} \cap {}^w V_{J_2} = V_{J_1} \cap {}^w V_{J_2} \leqq V_{J_1}$, by Proposition (2.5) (b). Thus c) implies b), and the proposition is proved.

(2.7) PROPOSITION. $P_{J_1}\cap{}^wP_{J_2}=L_K(P_{J_1}\cap{}^wV_{J_2})(V_{J_1}\cap{}^wP_{J_2})$.

PROOF. As $P_K=L_KV_K$, $x\in P_{J_1}\cap{}^wP_{J_2}$ can be expressed in the form

$$x=lv_1v_2\,,$$

for some $l\in L_K$, $v_1\in P_{J_1}\cap{}^wV_{J_2}$, and $v_2\in V_{J_1}$, by Proposition (2.5) (a). Then $v_2\in V_{J_1}\cap{}^wP_{J_2}$, and we have the inclusion one way. For the reverse inclusion, $L_K=\langle U_\alpha,\ T:\ \alpha\in\Delta_K\rangle\leqq P_{J_1}\cap{}^wP_{J_2}$, because $\Delta_K=\Delta_{J_1}\cap{}^w\Delta_{J_2}$ by Corollary (2.3). The rest is clear.

3. Cuspidal characters.

The discussion in this section is taken from [**19**], with the changes necessary to adapt the material to the context of finite groups with split (B, N)-pairs.

All characters and representations are taken in the complex field $\boldsymbol{C}$. The set of all irreducible complex characters of a group G will be denoted by $\mathcal{E}(G)$.

(3.1) DEFINITIONS. Let G be a finite group with a split (B, N)-pair of characteristic p, and Coxeter system (W, R). An irreducible character ζ of G is called *cuspidal* if for all $J\subseteqq R$, $J\neq R$,

$$\zeta_{P_J}(x)=\frac{1}{|V_J|}\sum_{v\in V_J}\zeta(xv)=0$$

for all $x\in G$. The set of irreducible cuspidal characters will be denoted by ${}^\circ\mathcal{E}(G)$. All characters of an abelian p'-group T are said to be cuspidal, so that ${}^\circ\mathcal{E}(T)=\mathcal{E}(T)$.

(3.2) PROPOSITION. *An irreducible character $\zeta\in\mathcal{E}(G)$ is cuspidal if and only if $(\zeta, 1_{V_J}^G)=0$, for all proper subsets $J\subset R$.*

PROOF. In case G is an abelian p'-group, the proposition is true because all characters are cuspidal, and the condition is vacuously satisfied. Assume now that G has Coxeter system (W, R), $R\neq\emptyset$. If ζ is cuspidal, then $\zeta_{P_J}(1)=0$, for $J\subset R$, implies $(\zeta|_{V_J}, 1_{V_J})=0$, and hence $(\zeta, 1_{V_J}^G)=0$ by Frobenius reciprocity. Now let $\zeta\in\mathcal{E}(G)$ satisfy the hypothesis of the Proposition, and let Z be an irreducible representation of $\boldsymbol{C}G$ affording ζ. For $J\subsetneqq R$, let $e_J=|V_J|^{-1}\sum_{v\in V_J}v$. Then $(\zeta, 1_{V_J}^G)=0$ implies $Z(e_J)=0$, therefore, for all $x\in G$, $Z(xe_J)=0$. Taking the trace, we obtain $|V_J|^{-1}\sum_{v\in V_J}\zeta(xv)=0$. This completes the proof.

(3.3) PROPOSITION. *Let $\zeta\in\mathcal{E}(G)$. There exists a standard parabolic subgroup P_J of G, with $\emptyset\subseteqq J\subseteqq R$, and an irreducible character $\varphi\in{}^\circ\mathcal{E}(L_J)$, such that $(\zeta, \tilde\varphi^G)\neq 0$, where $\tilde\varphi$ is the irreducible character of P_J defined by $\tilde\varphi(mv)=\varphi(m)$, $m\in L_J$, $v\in V_J$.*

PROOF (Springer [**19**]). Let M be an irreducible $\boldsymbol{C}G$-module affording ζ.

There exist subsets $J \subseteqq R$ such that $(\zeta, 1_{V_J}^G) \neq 0$, for example $J = R$, since $V_R = O_p(G) = \{1\}$. Let $J \subseteqq R$ be a subset which is minimal with respect to this property. Let $N \leqq M$ be the subspace of M affording the representation 1_{V_J}; then $N \neq 0$. Since P_J normalizes V_J, N is a CP_J-module. Let θ be the character of P_J afforded by N; then $V_J \leqq \ker \theta$, and $\theta = \sum \theta_i$, where the $\{\theta_i\}$ are irreducible characters of L_J. Moreover, they are all cuspidal characters of L_J, because if, in the language of Proposition (1.7), for some proper parabolic subgroup $P_{J,J'}$ of L_J, $(\theta_i | L_{J'}, 1_{V_{J,J'}}) \neq 0$, there exists a vector $w \neq 0$ in N fixed by $V_{J,J'} V_J = V_{J'}$, for $J' \subset J$, contrary to the minimality of J. We have also $(\zeta, \theta^G) \neq 0$, and as $\theta = \sum \tilde{\theta}_i$, where $\{\tilde{\theta}_i\}$ are the characters of P_J defined by $\theta_i(mv) = \theta_i(m)$, $m \in L_J$, $v \in V_J$, we have $(\zeta, \tilde{\theta}_i^G) \neq 0$ for some i, and $\theta_i \in {}^\circ\mathcal{E}(L_J)$, as required.

(3.4) DEFINITION. The notation $\tilde{\theta}$ will be used consistently for the character $\tilde{\theta}$ of a parabolic subgroup P_J obtained from a character θ of L_J by setting $\tilde{\theta}(mv) = \theta(m)$, $m \in L_J$, $v \in V_J$.

For the next result, introduce the subgroup $N_W(L_J) = \{w \in W : {}^wL_J = L_J\}$. Note that $W_J \trianglelefteq N_W(L_J)$, for each subset $J \subseteqq R$, because the inverse image N_J of W_J in N satisfies $N_J = N \cap L_J$, and the inverse image N' of $N_W(L_J)$ in N clearly normalizes N_J, so that $N'/N_J \cong N_W(L_J)/W_J$.

(3.5) THEOREM. *Let G be a finite group with a split (B, N)-pair of characteristic p, and Coxeter system (W, R). Let $J_1, J_2 \subseteqq R$, and let $\varphi_i \in {}^\circ\mathcal{E}(L_{J_i})$, $i = 1, 2$. Then $(\tilde{\varphi}_1^G, \tilde{\varphi}_2^G) = 0$, unless $L_{J_1} = {}^wL_{J_2}$ and $\varphi_1 = {}^w\varphi_2$, for some $w \in W$. If these conditions are satisfied, then $\tilde{\varphi}_1^G = \tilde{\varphi}_2^G$, and*

$$(\tilde{\varphi}_1^G, \tilde{\varphi}_1^G) = \sum_{w \in N_W(L_{J_1})/W_{J_1}} (\varphi_1, {}^w\varphi_1) .$$

PROOF (Springer [19], § 5.2). By Mackey's Theorem ([11], (9.8)),

$$(\tilde{\varphi}_1^G, \tilde{\varphi}_2^G) = \sum_{w \in W_{J_1, J_2}} (\tilde{\varphi}_1, {}^w\tilde{\varphi}_2)_{P_{J_1} \cap {}^wP_{J_2}} .$$

For a fixed $w \in W_{J_1, J_2}$, the subgroup $P_{J_1} \cap {}^wP_{J_2}$ can be factored, according to Proposition (2.7) and (2.5) (b), with uniqueness of expression, as

$$P_{J_1} \cap {}^wP_{J_2} = L_K(L_{J_1} \cap {}^wV_{J_2})(V_{J_1} \cap {}^wL_{J_2})(V_{J_1} \cap {}^wV_{J_2}) .$$

Note that Proposition (2.5) applies to both $P_{J_1} \cap {}^wV_{J_2}$ and $V_{J_1} \cap {}^wP_{J_2}$, because $w^{-1} \in W_{J_2, J_1}$.

We prove next that $(\tilde{\varphi}_1, {}^w\tilde{\varphi}_2)_{P_{J_1} \cap {}^wP_{J_2}} \neq 0$ implies $L_{J_1} = {}^wL_{J_2}$, and $\varphi_1 = {}^w\varphi_2$. The scalar product

$$(\tilde{\varphi}_1, {}^w\tilde{\varphi}_2) = |P_{J_1} \cap {}^wP_{J_2}|^{-1} \sum \tilde{\varphi}_1(xyzv) {}^w\tilde{\varphi}_2((xyzv)^{-1}) ,$$

where the sum is taken over $x \in L_K$, $y \in L_{J_1} \cap {}^wV_{J_2}$, $z \in V_{J_1} \cap {}^wL_{J_2}$, and $v \in$

$V_{J_1}\cap{}^wV_{J_2}$. As $V_{J_1}\cap{}^wV_{J_2}$ is contained in the kernels of both characters involved, the scalar product is equal to a multiple of

$$\sum_{x,y,z}\tilde{\varphi}_1(xyz)^w\tilde{\varphi}_2((xyz)^{-1})=\sum_{x,y,z}\tilde{\varphi}_1(xyz)^w\tilde{\varphi}_2((xzz^{-1}yz)^{-1}),$$

which in turn is a multiple of

$$\sum_{x,y,z}\varphi_1(xy)^w\varphi_2((xz)^{-1}). \tag{3.6}$$

By Proposition (2.5)(d), $L_{J_1}\cap{}^wV_{J_2}$ and $V_{J_1}\cap{}^wL_{J_2}$ are the O_p-subgroups of parabolic subgroups of L_{J_1} and ${}^wL_{J_2}$, respectively. Therefore, if either subgroup is different from $\{1\}$, and the expression (3.6) is different from zero, we contradict the assumption that $\varphi_1\in{}^\circ\mathcal{E}(L_{J_1})$ and $\varphi_2\in{}^\circ\mathcal{E}(L_{J_2})$. Therefore both subgroups are $\{1\}$, and by Proposition (2.5)(b), $P_{J_1}\cap{}^wV_{J_2}\leqq V_{J_1}$ and ${}^{w^{-1}}V_{J_1}\cap P_{J_2}\leqq V_{J_2}$. By Proposition (2.6), it follows that $L_{J_1}\leqq{}^wL_{J_2}$ and $L_{J_2}\leqq{}^{w^{-1}}L_{J_1}$. Hence $L_{J_1}={}^wL_{J_2}$, $\varphi_1={}^w\varphi_2$, and the proof of the first part of the theorem is completed.

In order to derive the expression for $(\tilde{\varphi}_1^G, \tilde{\varphi}_2^G)$, fix $w\in W_{J_1,J_2}$ such that $L_{J_1}={}^wL_{J_2}$ and $\varphi_1={}^w\varphi_2$. Then $W_{J_1}={}^wW_{J_2}$. We now introduce the terminology (A, B)*-transversal* for subgroups A, B of H to denote a set of (A, B)-double coset representatives in H. It follows that right multiplication by w induces a bijection of the (W_{J_1}, W_{J_1})-transversals with the (W_{J_1}, W_{J_2})-transversals. Using the bijection between $W_{J_1}\backslash W/W_{J_2}$ and $P_{J_1}\backslash G/P_{J_2}$, it follows that the (W_{J_1}, W_{J_1})-transversals, are among the $(P_{J_1}, {}^wP_{J_2})$-transversals. We have $(\tilde{\varphi}_1^G, \tilde{\varphi}_2^G)=(\tilde{\varphi}_1^G, ({}^w\tilde{\varphi}_2)^G)$, where ${}^w\tilde{\varphi}_2$ is the conjugate character of $\tilde{\varphi}_2$ on ${}^wP_{J_2}$. The multiplicity $(\tilde{\varphi}_1^G, ({}^w\tilde{\varphi}_2)^G)$ is given, by Mackey's formula, by

$$\sum_{w'}(\tilde{\varphi}_1, {}^{w'w}\tilde{\varphi}_2)_{P_{J_1}\cap{}^{w'w}P_{J_2}},$$

where $\{w'\}$ is a (W_{J_1}, W_{J_1})-transversal in W, because of the discussion above, chosen in such a way that $\{w'w\}=W_{J_1,J_2}$. From the proof of the first part of the theorem, all summands are zero except those for which $L_{J_1}={}^{w'w}L_{J_2}$, or $w'\in N_W(L_{J_1})$. Because $W_{J_1}\trianglelefteq N_W(L_{J_1})$ the elements of a (W_{J_1}, W_{J_1})-transversal belonging to $N_W(L_{J_1})$ correspond to distinct elements of $N_W(L_{J_1})/W_{J_1}$, and we have

$$(\tilde{\varphi}_1^G, \tilde{\varphi}_2^G)=\sum_{w'\in N_W(L_{J_1})/W_{J_1}}(\varphi_1, {}^{w'}\varphi_1).$$

The same formula holds also for $(\tilde{\varphi}_1^G, \tilde{\varphi}_1^G)$, and hence, by symmetry, for $(\tilde{\varphi}_2^G, \tilde{\varphi}_2^G)$. It follows that $\tilde{\varphi}_1^G=\tilde{\varphi}_2^G$, and the theorem is proved.

4. First reduction theorem.

Throughout this section $\{G, B, N, W, R, \text{etc.}\}$ denotes a finite group with a split (B, N)-pair of characteristic p, as in § 1-3.

(4.1) DEFINITIONS. For each $J \subseteqq R$, the *J-series* of G, $\mathcal{E}_J(G)$, is defined to be the set of irreducible characters $\zeta \in \mathcal{E}(G)$ such that $(\zeta, \tilde{\varphi}^G) \neq 0$ for some $\varphi \in {}^\circ\mathcal{E}(L_J)$. We set $\operatorname{char}(G) = \sum_{\zeta \in \mathcal{E}(G)} Z\zeta$, and $\mathcal{M}_J(G) = \sum_{\zeta \in \mathcal{E}_J(G)} Z\zeta$.

Note that $\mathcal{E}_R(G) = {}^\circ\mathcal{E}(G)$. Irreducible characters belonging to ${}^\circ\mathcal{E}(G)$, and $\mathcal{E}_\emptyset(G)$, are sometimes said to belong to the *discrete series*, and the *principal series*, respectively, of G.

Theorem (3.5) asserts that $\mathcal{M}_J(G) \perp \mathcal{M}_{J'}(G)$ (where $\perp$ means orthogonality with respect to the usual scalar product of $\operatorname{char}(G)$), for $J, J' \subseteqq R$, unless $L_J = {}^wL_{J'}$ for some $w \in W_{J,J'}$.

(4.2) DEFINITION. Let $J, J' \subseteqq R$. The subsets J and J' are called *equivalent, with respect to* $W = W_R$, if $L_J = {}^wL_{J'}$ for some $w \in W$, and the notation $J \sim_R J'$ will be used to denote this occurrence.

We remark that $\sim_R$ is, in fact, an equivalence relation. Moreover $J \sim_R J'$ if and only if $L_J = {}^wL_{J'}$ for some distinguished double coset representative $w \in W_{J,J'}$, since the subgroups W_J and $W_{J'}$, and T, normalize the subgroups L_J and $L_{J'}$, respectively. Suppose now that $w \in W_{J,J'}$, $L_J = {}^wL_{J'}$, and φ' is a cuspidal character of $L_{J'}$. We wish to show that $\varphi = {}^w\varphi'$ is a cuspidal character of L_J. Let $J'' \subset J'$, and $V_{J',J''} = O_p(P_{J''} \cap L_{J'})$. As ${}^wJ' = J$, ${}^wJ'' \subset J$, and consideration of the root subgroups involved shows that ${}^wV_{J',J''} = V_{J,wJ''} = O_p(P_{wJ''} \cap L_{J'})$. It follows that $\varphi \in {}^\circ\mathcal{E}(L_J)$, and by Theorem (3.5), $\tilde{\varphi}^G = \tilde{\varphi}'^G$. This discussion, combined with Proposition (3.3), proves the following result.

(4.3) PROPOSITION. *The character ring of* G, $\operatorname{char}(G)$, *is the orthogonal direct sum of the Z-submodules* $\{\mathcal{M}_{J_i}(G)\}$, *where* $\{J_i\}$ *is a set of representatives of the equivalence classes of subsets of* R. *In particular,* $J \sim_R J'$ *implies* $\mathcal{E}_J(G) = \mathcal{E}_{J'}(G)$.

For $J \subseteqq R$, the preceding discussion can be applied to the group L_J, so that $\operatorname{char}(L_J) = \sum_{J' \subseteqq J} \mathcal{M}_{J'}(L_J)$, etc. The equivalence relation $\sim$ has to be used with care in this situation, because for subsets $J', J'' \subseteqq J \subseteqq R$, it may happen that $J' \sim_R J''$, but $J' \nsim_J J''$. For example, if (W, R) is of type A_3, with $R = \{w_1, w_2, w_3\}$, let $J' = \{w_1\}$, $J'' = \{w_3\}$, $J = \{w_1, w_3\}$, and the notation arranged so that $w_1w_3 = w_3w_1$. Then $J' \sim_R J''$, but $J' \nsim_J J''$.

(4.4) FIRST REDUCTION THEOREM. *Let* $\zeta \in \mathcal{E}_J(G)$, *and let* $x \in G$ *be such that* $C_G(x) \leqq L_{J'}$, *for some subsets* $J, J' \subseteqq R$. *Then* $\zeta(x) = 0$ *unless there exists at least one subset* $J'' \subseteqq J'$ *such that* $J'' \sim_R J$. *If this occurs, let* $J'' \subseteqq J'$ *be a representative of the R-equivalence class of* J; *then*

$$\zeta(x)=\sum_{\substack{\lambda\in\mathcal{E}_{J''}(L_{J'})\\ J''\sim_R J}}(\zeta,\tilde{\lambda}^G)\lambda(x)\,.$$

PROOF. By Frobenius reciprocity, it follows that

$$\zeta(x)=\sum_{\lambda\in\mathcal{E}(L_{J'})}(\zeta,\tilde{\lambda}^G)\lambda(x)+\sum_{\substack{\xi\in\mathcal{E}(P_{J'})\\ V_{J'}\not\leq\ker\xi}}(\lambda,\xi^G)\xi(x)\,.$$

By ([**11**], (6.8)), $\xi(x)=0$ for all characters in the second sum. As $\mathcal{E}(L_{J'})=\bigcup_{J''\subseteqq J'}\mathcal{E}_{J''}(L_{J'})$, it suffices to prove that, for $\lambda\in\mathcal{E}_{J''}(L_{J'})$, $(\zeta,\tilde{\lambda}^G)\neq 0$ implies $J''\sim_R J$. The proof is based on the following Lemma.

(4.5) LEMMA. *Let $J''\subseteqq J'\subseteqq R$. Then $P_{J''}=L_{J''}\cdot V_{J',J''}V_{J'}$, where $V_{J',J''}=L_{J'}\cap V_{J''}$; and $P_{J'}=L_{J'}V_{J'}$. Let $\mu\in\mathcal{E}(L_{J''})$; then $\tilde{\mu}^{P_{J'}}$ has $V_{J'}$ in its kernel, and $\tilde{\mu}^{P_{J'}}\mid L_{J'}=\hat{\mu}^{L_{J'}}$, where $\hat{\mu}$ is the character of $P_{J',J''}=L_{J''}\cdot V_{J',J''}$ defined by $\hat{\mu}(lv)=\mu(l)$, $l\in L_{J''}$, $v\in V_{J',J''}$.*

PROOF. The facts about the Levi decompositions of $P_{J''}$ and $P_{J'}$ were proved in § 1 (Proposition (1.5) and (1.7)). As $V_{J'}\leqq\ker\tilde{\mu}$, and is normal in $P_{J'}$, $V_{J'}\leqq\ker(\tilde{\mu}^{P_{J'}})$. From Mackey's Subgroup Theorem, as there is only one $(P_{J''},L_{J'})$-double coset in $P_{J'}$, we obtain

$$\tilde{\mu}^{P_{J'}}\mid L_{J'}=\tilde{\mu}\,|_{P_{J''}\cap L_{J'}}{}^{L_{J'}}=\hat{\mu}^{L_{J'}}\,,$$

as required.

The proof of Theorem (4.4) is now completed as follows. Suppose $(\zeta,\tilde{\lambda}^G)\neq 0$, for $\lambda\in\mathcal{E}_{J''}(L_{J'})$, and some subset $J''\subseteqq J'$. By Lemma (4.5), $(\tilde{\lambda},\tilde{\mu}^{P_{J'}})\neq 0$, for some irreducible character $\mu\in{}^\circ\mathcal{E}(L_{J''})$. Therefore $\tilde{\lambda}^G$ is a component of $\tilde{\mu}^G$, and hence $(\zeta,\tilde{\mu}^G)\neq 0$. It follows that $\zeta\in\mathcal{E}_J(G)\cap\mathcal{E}_{J''}(G)$, and hence by Theorem (3.5), $J\sim_R J''$. This completes the proof of the theorem.

The first Corollary is related to conjectures about characters in ${}^\circ\mathcal{E}(G)$, due to MacDonald ([**18**], (6.7)), and was proved in a different way by Springer ([**18**], Proposition (6.8)), for semi-simple elements in reductive algebraic groups.

(4.6) COROLLARY. *Let $\zeta\in{}^\circ\mathcal{E}(G)$, and let $x\in G$ be such that $C_G(x)\leqq L_J$, for some proper subset $J\subset R$. Then $\zeta(x)=0$.*

(4.7) DEFINITION. An element $t\in T$ is said to have a *parabolic centralizer* if $C_G(t)=L_J$, for some $J\subseteqq R$.

We note that the concept of parabolic centralizer is a rather natural generalization of the familiar concept of a *regular element* $t\in T$, which satisfies the condition $C_G(t)=T$.

(4.8) COROLLARY. *Let $\zeta\in\mathcal{E}_J(G)$, for some $J\subseteqq R$. Let $t\in T$ have a parabolic centralizer $L_{J'}$. There exist algebraic integers $\{\alpha^t_{\zeta,\lambda}\}$ depending on t, ζ and $\lambda\in\mathcal{E}_{J''}(L_{J'})$, such that, all $\alpha^t_{\zeta,\lambda}=0$ unless $J''\sim_R J$ for some $J''\subseteqq J'$, and in that case,*

$$\zeta(tu)=\sum_{\substack{\lambda\in\mathcal{E}_{J''}(L_{J'})\\ J''\sim_R J}}\alpha^t_{\zeta,\lambda}\lambda(u)\,,$$

for all p-elements $u \in C_G(t)$.

PROOF. For each p-element $u \in C_G(t)$, we have $C_G(tu) \leqq C_G(t) = L_{J'}$. By Theorem (4.4),

$$\zeta(tu) = \sum (\zeta, \tilde{\lambda}^G)\lambda(tu)$$

with $\lambda \in \mathcal{E}_{J'}(J')$, and $J'' \sim_R J$, or $\zeta(tu)=0$. As $t \in Z(C_G(t))$, $\lambda(tu)=\lambda(t)\lambda(1)^{-1}\lambda(u)$, with $\lambda(t)\lambda(1)^{-1} \in$ alg. int. $\{C\}$. Setting $\alpha^t_{\zeta,\lambda} = (\zeta, \tilde{\lambda}^G)\lambda(t)\lambda(1)^{-1}$, the corollary follows.

As noted in the introduction, there is analogy between the form of Corollary (4.8) and Brauer's Second Main Theorem ([**15**]). Following this analogy, we shall call the algebraic integers defined in (4.8) *decomposition numbers*. For characters $\zeta \in 1_B^G$, it is shown in Chapter II that the decomposition numbers $\{\alpha^t_{\zeta,\lambda}\}$ are computable in terms of the Coxeter group. Further information about the decomposition numbers $\{\alpha^t_{\zeta,\lambda}\}$ seems to be lacking in the general case, as is information about values of irreducible characters $\zeta \in \mathcal{E}(G)$ on p-elements, to which Corollary (4.8) reduces us in certain cases.

Chapter II. Reduction theorems for characters in 1_B^G

5. Second reduction theorem.

The main result of this section is a version of Theorem (4.4) for characters in 1_B^G, which is sharper in two ways: the decomposition numbers $\{\alpha^t_{\zeta,\lambda}\}$ are rational integers computable in terms of the Coxeter group W; and the only characters λ of $L_{J'}$ for which $(\zeta, \tilde{\lambda}^G) \neq 0$, are components $\lambda \in 1_{B_{J'}}^{L_{J'}}$.

Throughout Chapter II, the following assumptions will be in force. We shall consider a system $\mathcal{S}$ of finite groups with (B, N)-pairs of type (W, R) ([7], [**1**]). We assume the set $\{q\}$ of prime powers associated with $\mathcal{S}$ contains almost all primes. For each q, it is assumed that the corresponding group with a (B, N)-pair $G(q) \in \mathcal{S}$ is a finite group with a split (B, N)-pair of characteristic $p|q$. There exist positive integers $\{c_i\}$ corresponding to the distinguished generators $R=\{w_1, \cdots, w_n\}$, such that $c_i = c_j$ if w_i and w_j are conjugate in W. For each q, it is assumed that $|B(q)w_iB(q)/B(q)| = q^{c_i}$, $1 \leqq i \leqq n$, where $B(q)$ denotes the Borel subgroup of $G(q)$. Let $N(q)$, $T(q)$, $P_J(q)$, $L_J(q)$, $V_J(q)$, etc. denote the subgroups of $G(q)$ considered in Chapter I. We let $B_J(q) = B(q) \cap L_J(q)$, and view $B_J(q)$ as a standard minimal parabolic subgroup of $L_J(q)$.

Let A be the *generic ring* of $\mathcal{S}$, over the polynomial ring in one indeterminate $\mathfrak{o} = Q[X]$, as in [**1**], p. 252. Let $K = Q(X)$, and $\bar{K}$ a finite extension field of K, such that $\bar{K}$ is a splitting field for A. Let $\mathfrak{o}^*$ denote the integral closure of $\mathfrak{o}$ in $\bar{K}$. For each q, there is a bijection $\varphi \to \zeta_{\varphi,q}$ between the irreducible characters $\varphi \in \mathcal{E}(W)$, and the irreducible characters $\zeta_{\varphi,q} \in 1_{B(q)}^{G(q)}$. The map $\varphi \to \zeta_{\varphi,q}$ is defined in terms of the irreducible characters $\{\chi_\varphi\}_{\varphi \in \mathcal{E}(W)}$ of $A^{\bar{K}}$, and depends on the choice of homomorphisms $f^*: \mathfrak{o}^* \to \bar{Q}$, extending the homomorphisms $f: \mathfrak{o} \to Q$ given by $X \to q$, for each q. For each $J \subseteqq R$, $\{P_J(q)\}$ is a system of type (W_J, J), with generic ring $A_J = \sum_{w \in W_J} \mathfrak{o} a_w$. We assume that the correspondences between characters $\{\phi\}$ of W_J and characters $\mu_{\phi,q} \in 1_{B(q)}^{P_J(q)}$, for each $J \subseteqq R$, are defined by a fixed extension $f^*: \mathfrak{o}^* \to \bar{Q}$. Finally, for each $J \subseteqq R$, $L_J(q)$ is also a system of (B, N)-pairs of type (W_J, J), and the lift $\eta \to \tilde{\eta}$, defined in (3.4), sets up a bijection between the characters of $1_{B_J(q)}^{L_J(q)}$ and the characters in $1_{B(q)}^{P_J(q)}$.

Let IND: $A \to \mathfrak{o}$ be the homomorphism given by $\mathrm{IND}(a_{w_i}) = X^{c_i}$, $1 \leqq i \leqq n$, where $\{a_w\}_{w \in W}$ is the basis of A over $\mathfrak{o}$ defined in [1], p. 252. We shall also need the *Poincare polynomial* $P(X)$ of (W, R), given by $P(X) = \sum_{w \in W} \mathrm{IND}(a_w)$.

Because of the assumption that $\{q\}$ contains almost all primes, Theorem (2.6) of [1] can be applied. By that theorem, for each $\varphi \in \mathcal{E}(W)$, there exists a *generic degree* $d_\varphi(X) \in \mathfrak{o}$, such that $d_\varphi(q) = \zeta_{\varphi,q}(1)$, for all $q \in \{q\}$, and $d_\varphi(1) = \varphi(1)$.

All these considerations apply also to the systems of groups $\{L_J(q)\}$ of type (W_J, J), for each $J \subseteqq R$.

(5.1) Second Reduction Theorem. *Let $J \subseteqq R$, $\varphi \in \mathcal{E}(W)$, and $q \in \{q\}$. Let $x \in G(q)$ be an element such that $C_{G(q)}(x) \leqq L_J(q)$. Then*

$$\zeta_{\varphi,q}(x) = \sum_{\phi \in \mathcal{E}(W_J)} (\zeta_{\varphi,q}, \tilde{\eta}_{\phi,q}^{G(q)}) \eta_{\phi,q}(x), \tag{5.2}$$

where $\{\eta_{\phi,q}\}$ are the characters in $1_{B_J(q)}^{L_J(q)}$, whose extensions $\{\tilde{\eta}_{\phi,q}\}$ to $P_J(q)$ are the characters in $1_{B(q)}^{P_J(q)}$ corresponding to the characters ϕ of W_J. The multiplicities $(\zeta_{\varphi,q}, \tilde{\eta}_{\phi,q}^{G(q)})$ are independent of q, and are given by

$$(\zeta_{\varphi,q}, \tilde{\eta}_{\phi,q}) = (\varphi, \phi^W), \tag{5.3}$$

for all q, and $\varphi \in \mathcal{E}(W)$, $\phi \in \mathcal{E}(W_J)$.

The proof requires several lemmas.

(5.4) Lemma. *Let $\eta \in \mathcal{E}(L_J(q))$ be such that $(\eta, 1_{B_J(q)}^{L_J(q)}) = 0$. Then $(\tilde{\eta}^G, 1_{B(q)}^{G(q)}) = 0$ (where $\tilde{\eta}$ is defined according to Definition* (3.4)).

Proof. Suppose $(\tilde{\eta}^G, 1_{B(q)}^{G(q)}) \neq 0$. Then there exists a distinguished double coset representative $w \in W_{J,\emptyset}$ such that (by Mackey's Theorem)

$$(\tilde{\eta}, {}^w 1_{B(q)})_{P_J(q) \cap {}^w B(q)} \neq 0. \tag{5.5}$$

By Proposition (2.5), $L_J(q) \cap {}^w B(q)$ is a standard parabolic subgroup of $L_J(q)$, and hence contains $B_J(q)$. Let M be a $\boldsymbol{C}P_J(q)$-module affording $\tilde{\eta}$; then $V_J(q)$ acts trivially on M, and M can be viewed an $L_J(q)$-module affording η. By (5.5), M contains a non-zero vector fixed by the elements in $P_J(q) \cap {}^w B(q)$ and hence by the elements in $B_J(q)$. Therefore $(\eta|_{B(q)}, 1_{B_J(q)}) \neq 0$, and by Frobenius reciprocity $(\eta, 1_{B_J(q)}^{L_J(q)})\neq 0$. This completes the proof of the Lemma.

PROOF OF (5.2). As in the proof of Theorem (4.4), we have

$$\zeta_{\varphi,q}(x)=\sum_{\eta\in\mathcal{E}(L_J(q))}(\zeta_{\varphi,q}, \tilde{\eta}^{G(q)})\eta(x)+\sum_{\substack{\xi\in\mathcal{E}(P_J(q))\\ V_J(q)\not\leq\ker\xi}}(\zeta_{\varphi,q}, \xi^{G(q)})\xi(x).$$

As $C_{G(q)}(x)\cap V_J(q)=\{1\}$, by the hypothesis that $C_{G(q)}(x)\leqq L_J(q)$, $\xi(x)=0$ for all ξ in the second summand, by [**11**], (6.8). By Lemma (5.4), all multiplicities in the first sum vanish, except for those associated with characters $\eta\in 1_{B_J(q)}^{L_J(q)}$. This completes the proof of (5.2).

An alternative proof of (5.2) was communicated to the author by R. Kilmoyer, for the case of a regular element $x\in T(q)$, i.e. for x such that $C_G(x)=T(q)$. The following proof of (5.2) in the general case resulted from an attempt to generalize Kilmoyer's idea. Let $\eta\in\mathcal{E}(L_J(q))$; then

$$(5.6)\qquad (\zeta_{\varphi,q}, \tilde{\eta}^G)=|P_J(q)|^{-1}\sum_{s\in P_J(q)}\zeta_{\varphi,q}(s)\tilde{\eta}(s^{-1})$$

$$=|L_J(q)|^{-1}\sum_{l\in L_J(q)}(|V_J(q)|^{-1}\sum_{v\in V_J(q)}\zeta_{\varphi,q}(lv))\eta(l^{-1}),$$

using the Levi decomposition $P_J(q)=L_J(q)V_J(q)$, and the fact that $\tilde{\eta}(lv)=\eta(l)$, $l\in L_J(q)$, $v\in V_J(q)$. The function θ defined by

$$(5.7)\qquad \theta(x)=|V_J(q)|^{-1}\sum_{v\in V_J(q)}\zeta_{\varphi,q}(xv)$$

is a class function on $L_J(q)$, and (5.6) combined with Lemma (5.4) imply that

$$(5.8)\qquad \theta=\sum(\zeta_{\varphi,q}, \tilde{\eta}^G)\eta,$$

where the sum is taken over $\eta\in 1_{B_J(q)}^{L_J(q)}$. Because $C_G(x)\leqq L_J(q)$, the map $v\to(x, v)$, for $x\in L_J(q)$, $v\in V_J(q)$, is a bijection of $V_J(q)$, and hence every element in the coset $xV_J(q)$ is conjugate to x. Therefore $\theta(x)=\zeta_{\varphi,q}(x)$, by (5.7), and (5.2) follows by substitution in (5.8).

We now take up the proof of (5.3). The method is suggested by the work of Scott [**17**]. Let $q\in\{q\}\cup\{1\}$ be fixed, and let $\mathfrak{O}$ be the valuation ring in $\bar{K}$, containing $\mathfrak{o}^*$, associated with a prime ideal in $\mathfrak{o}^*$ containing $X-q$. Then $\mathfrak{O}$ is a discrete valuation ring, and a principal ideal domain with quotient field $\bar{K}$. The homomorphism $f:\mathfrak{o}\to Q$ given by $X\to q$ extends to a homomorphism $f^*:\mathfrak{O}\to\bar{Q}$. For each irreducible character χ of $A^{\bar{K}}$, $\chi(a_w)\in\mathfrak{O}$ for all $w\in W$, and $\chi_{f^*}: a_{wf}\to f^*(\chi(a_w))$, $w\in W$, is an irreducible character of $A_f^{\bar{Q}}$, where A_f is the specialized algebra $A\otimes_{\mathfrak{o}}Q$, by [**1**], Proposition (2.2).

(5.9) LEMMA. *Let χ be an irreducible character of $A^{\bar{K}}$. Then there exists a primitive idempotent $e\in A^{\bar{K}}$ affording χ, such that $\chi(ea_w)\in\mathfrak{O}$, for all standard basis elements $\{a_w\}_{w\in W}$ of A.*

PROOF. Let V be an irreducible $A^{\bar{K}}$-module affording χ, and let V_0 be an A-submodule such that V_0 is a free $\mathfrak{O}$-module with the property that a $\mathfrak{O}$-basis for V_0 is a $\bar{K}$-basis of V. (The existence of V_0 is guaranteed because $\mathfrak{O}$ is a principal ideal domain with quotient field $\bar{K}$.) Let $\{v_1, \cdots, v_d\}$ be this $\mathfrak{O}$-basis of V_0. For all $a \in A$, the matrix of $a_L : v \to av$, $v \in V$, with respect to $\{v_1, \cdots, v_d\}$ is in $M_d(\mathfrak{O})$. The matrix representation of $A^{\bar{K}}$ with respect to this basis, faithfully represents the Wedderburn component corresponding to χ. Moreover, $\chi(a) = \sum \alpha_{ii}$, if $a_L = (\alpha_{ij})$, $\alpha_{ij} \in \mathfrak{O}$. Let e be the primitive idempotent in $A^{\bar{K}}$ corresponding to the matrix unit E_{11}. If $a \in A^{\bar{K}}$, $a_L = (\alpha_{ij})$, then the matrix corresponding to ea is

$$\begin{pmatrix} \alpha_{11} \cdots \alpha_{1n} \\ 0 \end{pmatrix}.$$

Therefore, for all $a \in A$, $\chi(ea) = \alpha_{11} \in \mathfrak{O}$ as required.

(5.10) LEMMA. *Let e be an arbitrary primitive idempotent in $A^{\bar{K}}$ affording χ. Then*

$$e = \frac{d_\chi(X)}{P(X)} \sum_{w \in W} \frac{\chi(ea_{w^{-1}})}{\mathrm{IND}(a_w)} a_w ,$$

where $d_\chi(X)$ is the generic degree associated with the irreducible character φ of W corresponding to χ, and $P(X) = \sum \mathrm{IND}\,(a_w)$ is the Poincare polynomial of A.

PROOF. We first introduce the function

$$\rho : A^{\bar{K}} \longrightarrow \bar{K}$$

given by

$$\rho = \sum_\chi d_\chi(X)\chi .$$

By specialization it follows that

$$\rho(a_w) = \begin{cases} 0 & w \neq 1 \\ P(X) & w = 1 \end{cases},$$

as $P(q) = |G(q) : B(q)|$ for all q, and $\chi(a_1) =$ degree χ is the multiplicity of $\zeta_{\varphi,q}$ in $1_{B(q)}^{G(q)}$. Another specialization argument yields

$$\rho(a_w a_{w'}) = \begin{cases} 0 & ww' \neq 1 \\ P(X)\,\mathrm{IND}\,(a_w) , & ww' = 1 \end{cases}.$$

Now let $e = \sum_{w \in W} \lambda_w a_w$; then

$$ea_{w_1^{-1}} = \sum \lambda_w a_w a_{w_1^{-1}} .$$

Applying ρ to both sides we obtain

$$d_\chi(X)\chi(ea_{w_1^{-1}}) = \lambda_{w_1} P(X)\,\mathrm{IND}\,(a_{w_1}) ,$$

and the Lemma is proved.

(5.11) LEMMA. *Let* $f^*: \mathfrak{O} \to \bar{Q}$ *extend* $f: u \mapsto q$. *Then* f^* *defines a homomorphism of* $\mathfrak{O}$*-algebras* $A \to A_f^{f^*(\mathfrak{O})}$, *where* $A_f = Q \otimes A$ *is the specialized algebra associated with* f (see [1], p. 252). *Let* e *be a primitive idempotent in* $A^{\bar{K}}$ *affording* χ, *and satisfying Lemmas* (5.9) *and* (5.10). *Then* $f^*(e)$ *is a primitive idempotent in* $A_f^{\bar{Q}}$ *affording* χ_{f^*}.

PROOF. We have $\chi(e)=1$. As $P(X)\,\mathrm{IND}\,(a_w)$ is a unit in $\mathfrak{O}$ for all $w \in W$, the idempotent e in Lemma (5.10) belongs to A, $f^*(e)$ is defined, and is an idempotent in A_f such that $\chi_{f^*}(f^*(e))=1$. Finally, $f^*(e)$ is primitive, because $eA^{\bar{K}}e = \bar{K}e$ implies $eA^{\mathfrak{O}}e \subseteq \mathfrak{O}e$, and hence $f^*(e)A_f^{\bar{Q}} \cdot f^*(e) = \bar{Q} \cdot f^*(e)$.

PROOF OF (5.3). Let J be as in the hypothesis of Theorem (5.1), and let A_J be the subring of A generated by $\{a_w\}_{w \in W_J}$. Then A_J is a generic ring of the system $\mathcal{S}_J = \{P_J(q)\}$ of type (W_J, J). Let $e \in A_J^{\bar{K}}$ correspond to the character of $A_J^{\bar{K}}$ associated with $\psi \in \mathcal{E}(W_J)$, and let χ be the irreducible character of $A^{\bar{K}}$ associated with $\zeta_{\varphi,q}$. Then, for a given q, we may assume e is as in Lemma (5.11). Then $\chi(e)=m$, a non-negative integer, which is independent of q, because all primitive idempotents affording a given irreducible character of $A_J^{\bar{K}}$ are conjugate in $A_J^{\bar{K}}$. Apply f^*, and obtain $\chi_{f^*}(f^*(e))=m$. It follows that $m=(\zeta_{\varphi,q}, \tilde{\eta}_{\psi,q}^{G})$ if $q \neq 1$, and that $m=(\varphi, \psi^w)$ if $q=1$, by Lemma (5.11), because, for example, if $q \neq 1$, $f^*(e) \in A_{Jf}^{\bar{Q}}$ corresponds to a primitive idempotent $\tilde{e}$ in $H(P_J(q), B(q))$ affording $\tilde{\eta}_{\psi,q}$, and $m=\chi(e)=\chi_{f^*}(f^*(e))=\zeta_{\varphi,q}(\tilde{e})=(\zeta_{\varphi,q}, \tilde{\eta}_{\psi,q}^{G(q)})$. As m is independent of q, the proof of Theorem (5.1) is completed.

As in the case of the First Reduction Theorem, there is a sharp form of Theorem (5.1), for elements $t \in T(q)$ of *parabolic centralizer* in the sense of Definition (4.7).

(5.12) THEOREM. *Let* $G(q) \in \mathcal{S}$, *as in Theorem* (5.1). *Let* $t \in T(q)$ *have a parabolic centralizer* $L_J(q)$, *for some* $J \subseteq R$. *Then there exist rational integers* $\{a_{\varphi,\psi}^J\}$ *depending on* $J \subseteq R$, $\varphi \in \mathcal{E}(W)$ *and* $\psi \in \mathcal{E}(W_J)$, *and independent of* q, *such that for each* p*-element* $u \in C_{G(q)}(t)$,

$$\zeta_{\varphi,q}(tu) = \sum_{\psi \in \mathcal{E}(W_J)} a_{\varphi,\psi}^J \eta_{\psi,q}(u)\,.$$

Moreover, the decomposition numbers are given by $a_{\varphi,\psi}^J = (\varphi, \psi^W)$, *for* $\varphi \in \mathcal{E}(W)$, $\psi \in \mathcal{E}(W_J)$.

PROOF. As $C_G(tu) \leq C_G(t) = L_J(q)$, Theorem (5.1) can be applied, and gives

$$\zeta_{\varphi,q}(tu) = \sum_{\psi \in \mathcal{E}(W_J)} a_{\varphi,\psi}^J \eta_{\psi,q}(tu)\,,$$

with $a_{\varphi,\psi}^J = (\varphi, \psi^W)$. As $C_G(t) = L_J(q)$, $\eta_{\psi,q}(tu) = (\eta_{\psi,q}(t)/\eta_{\psi,q}(1))(\eta_{\psi,q}(u))$. Finally, $Z(L_J(q)) \leq B_J(q)$, and $\eta_{\psi,q} \in 1_{B_J(q)}^{L_J(q)}$. It follows that $\eta_{\psi,q}(t) = \eta_{\psi,q}(1)$, and Theorem (5.12) is proved.

6. Generic character values on elements of $T(q)$.

We continue to work with a system $\mathcal{S}$ of finite groups $\{G(q)\}$ with split (B, N)-pairs, such that $\{q\}$ contains almost all primes, so that the generic degrees $d_\varphi(X) \in Q[X]$ are available.

(6.1) THEOREM. *For each pair $\{J, \varphi\}$, with $J \subseteq R$ and $\varphi \in \mathcal{E}(W)$, there exists a polynomial $v_{J,\varphi}(X) \in Q[X]$ such that if $t \in T(q)$ has the centralizer $C_{G(q)}(t) = L_J(q)$, then*

$$\zeta_{\varphi,q}(t) = v_{J,\varphi}(q) .$$

The polynomial $v_{J,\varphi}(X)$ is given by

$$v_{J,\varphi}(X) = \sum_{\phi \in \mathcal{E}(W_J)} (\varphi, \phi^W) d_\phi(X) ,$$

where $d_\phi(X)$ is the generic degree associated with $\phi \in \mathcal{E}(W_J)$.

PROOF. By Theorem (5.12), taking $u = 1$, we have

$$\zeta_{\varphi,q}(t) = \sum_{\phi \in \mathcal{E}(W_J)} (\varphi, \phi^W) \eta_{\phi,q}(1) ,$$

and $\eta_{\phi,q}(1) = d_\phi(q)$, for all $\phi \in \mathcal{E}(W_J)$. This completes the proof.

An element $t \in T(q)$ is called *regular* if $C_{G(q)}(t) = T(q)$. The first Corollary of Theorem (6.1) has been proved independently by Kilmoyer, Seitz, and the author [**9**].

(6.2) COROLLARY. *Let $t \in T(q)$ be a regular element, such that $C_{G(q)}(t) = T(q)$. Then for $\varphi \in \mathcal{E}(W)$,*

$$\zeta_{\varphi,q}(t) = \varphi(1) .$$

PROOF. In this case, $J = \emptyset$, $W_J = \{1\}$, and

$$\zeta_{\varphi,q}(t) = v_{\emptyset,\varphi}(q) = (\varphi, 1_{\{1\}}^W) 1 = \varphi(1) ,$$

because $1_{\{1\}}^W$ is the regular character of W, and the multiplicity of φ in the regular character is equal to its degree.

REMARK. Another statement of Corollary (6.2) is that, for a regular element $t \in T(q)$,

$$\zeta_{\varphi,q}(t) = (\zeta_{\varphi,q}, 1_{B(q)}^{G(q)}) . \tag{6.3}$$

It is always interesting to set $q = 1$ in various polynomials in q associated with groups of Lie type. For example, if $d_\varphi(X)$ is a generic degree, $d_\varphi(1) = \varphi(1)$, the degree of the corresponding character of W.

(6.4) COROLLARY. *Let $J \subseteq R$, and $\varphi \in \mathcal{E}(W)$. Then*

$$v_{J,\varphi}(1) = v_{\emptyset,\varphi}(q) ,$$

the value of the character $\zeta_{\varphi,q}$ on a regular element of $T(q)$.

PROOF. From Theorem (6.1), we have

$$\begin{aligned} v_{J,\varphi}(1) &= \sum_{\psi\in\mathcal{E}(W_J)}(\varphi, \psi^W)d_\psi(1) \\ &= \sum_{\psi\in\mathcal{E}(W_J)}(\varphi, \psi^W)\psi(1) \\ &= (\varphi, (\sum_{\psi\in\mathcal{E}(W_J)}\psi(1)\psi)^W) \\ &= (\varphi, \rho_J^W) \\ &= (\varphi, \rho_R) = \varphi(1) = v_{\emptyset,\varphi}(q), \end{aligned}$$

where ρ_J and ρ_R are the regular characters of W_J and W, respectively.

The next result shows the connection between the character values $\zeta_{\varphi,q}(t)$ on elements $t\in T(q)$ having parabolic centralizers, and the class intersections $\mathfrak{C}\cap B(q)wB(q)$, where $\mathfrak{C}$ is the conjugacy class in $G(q)$ containing t. In [**10**] information about the class intersections $\mathfrak{C}\cap B(q)wB(q)$ was used to obtain information about the character values $\zeta_{\varphi,q}(t)$, for p'-elements t which are not necessarily conjugate to elements of the standard torus $T(q)$.

For the discussion to follow, we adopt the set-up introduced immediately before the proof of Theorem (5.1). Then A denotes the generic ring of $\mathcal{S}$, with basis $\{a_w\}_{w\in W}$ over $\mathfrak{o}=Q[X]$. We have $K=Q(X)$, and $\bar{K}$ a finite extension of K which is a splitting field for A. For each q, $\mathfrak{O}$ denotes a valuation ring in $\bar{K}$ associated with a prime ideal in $\mathfrak{o}^*$ containing $X-q$, where $\mathfrak{o}^*$ is the integral closure of $\mathfrak{o}$ in $\bar{K}$. We denote by $f^*:\mathfrak{O}\to\bar{Q}$ an extension of $f: X\mapsto q$.

There is an isomorphism between the specialized algebra A_f^Q and the Hecke algebra $H(G(q), B(q))$, with the basis element $a_{wf}=1\otimes a_w$ of A_f corresponding to $\tilde{a}_{wf}=|B(q)|^{-1}\sum_{x\in B(q)wB(q)}x$, for each $w\in W$. The correspondence between irreducible characters φ of W, irreducible characters χ_φ of $A^{\bar{K}}$, and components $\zeta_{\varphi,q}\in 1_{B(q)}^{G(q)}$ is such that

$$\zeta_{\varphi,q}(\tilde{a}_{wf})=\chi_{\varphi,f^*}(a_{w,f}),$$

where χ_{φ,f^*} is the character of $A_f^{\bar{Q}}$ defined by $\chi_{\varphi,f^*}(a_{w,f})=f^*(\chi_\varphi(a_w))$ ([**1**], § 2).

Suppose for the moment that (W, R) is indecomposable. By Theorem (2.8) of [**1**], corrected in [**8**], every irreducible character χ_φ of A is rational, in the sense that $\chi_\varphi(a_w)\in\mathfrak{o}$, with the exception of two characters of degree $512=2^9$, in case (W, R) is of type E_7, and four characters of degree $4096=2^{12}$, in case (W, R) is of type E_8.

(6.5) THEOREM. *For each subset $J\subseteq R$ and element $w\in W$, there exists a polynomial $h_{J,w}(X)\in Q[X]$, such that if $\mathfrak{C}$ is a conjugacy class in $G(q)\in\mathcal{S}$, containing an element $t\in T(q)$ with parabolic centralizer $C_{G(q)}(t)=L_J(q)$, then*

$$|\mathfrak{C}\cap B(q)wB(q)|=h_{J,w}(q)\,.$$

The polynomial $h_{J,w}(X)$ *is given by*

$$h_{J,w}(X)=\frac{\mathrm{IND}\,(a_{w_R})}{\mathrm{IND}\,(a_{w_J})\cdot P_J(X)}\sum_{\phi\in\mathcal{E}(W_J)}d_\phi(X)\omega_\phi(a_w)\,,$$

where

$$\omega_\phi=\sum_{\varphi\in\mathcal{E}(W)}(\varphi,\phi^W)\chi_\varphi\,,$$

w_R *and* w_J *are the elements of maximal length in* W *and* W_J, *respectively, and*

$$P_J(X)=\sum_{w\in W_J}\mathrm{IND}\,(a_w)$$

is the Poincare polynomial of W_J.

PROOF. We shall first derive the formula for $|\mathfrak{C}\cap BwB|$, and then prove that it is given by a polynomial. Let ζ be an irreducible character of $G(q)$. Then, for $w\in W$,

$$\zeta(\sum_{x\in B(q)wB(q)}x)=|B(q)wB(q)\cap\mathfrak{C}|\zeta(t)+\sum_{\mathfrak{C}'\neq\mathfrak{C}}|B(q)wB(q)\cap\mathfrak{C}'|\zeta(t')$$

with $t\in\mathfrak{C}$, $t'\in\mathfrak{C}'$. If the left side is different from zero, then $\zeta(\tilde{a}_{wf})\neq 0$, and hence $\zeta=\zeta_{\varphi,q}\in 1_{B(q)}^{G(q)}$, because the only characters of $G(q)$ having a non-zero restriction to the Hecke algebra $H(G(q), B(q))$ are the components of the permutation character $1_{B(q)}^{G(q)}$. Multiply both sides of the equation by $\zeta(t^{-1})$, and sum over $\zeta\in\mathcal{E}(G(q))$. By the second orthogonality relation ([**11**], (2.14)),

$$|C_{G(q)}(t)|\,|B(q)wB(q)\cap\mathfrak{C}|=\sum_{\varphi\in\mathcal{E}(W)}\zeta_{\varphi,q}(t^{-1})\zeta_{\varphi,q}(|B(q)|\tilde{a}_{wf})\,,$$

because of the remark above and the fact that $\tilde{a}_{wf}=|B(q)|^{-1}\sum_{x\in B(q)wB(q)}x$. Combining this formula with Theorem (6.1), and using the hypothesis that $C_{G(q)}(t)=C_{G(q)}(t^{-1})=L_J(q)$, we obtain

$$|B(q)wB(q)\cap\mathfrak{C}|=|B(q)|\,|L_J(q)|^{-1}\sum_{\varphi\in\mathcal{E}(W)}v_{J,\varphi}(q)\chi_{\varphi,f^*}(a_{wf})\,.$$

We have $|B(q)|=|T(q)|\,|U(q)|=|T(q)|f(\mathrm{IND}\,(a_{w_R}))$ and $|L_J(q)|=|T(q)|\,|U_J(q)|P_J(q)=|T(q)|f(\mathrm{IND}\,(a_{w_J}))P_J(q)$. Consider

$$h_{J,w}=\frac{\mathrm{IND}\,(a_{w_R})}{\mathrm{IND}\,(a_{w_J})P_J(X)}\sum_{\phi\in\mathcal{E}(W_J)}d_\phi(X)\omega_\phi(a_w)\,,$$

with $\omega_\phi=\sum_{\varphi\in\mathcal{E}(W)}(\varphi,\phi^W)\chi_\varphi$. As $\mathrm{IND}\,(a_{w_J})\,|\,\mathrm{IND}\,(a_{w_R})$ in $\mathfrak{o}$, and $P_J(q)\equiv 1\pmod{q}$, it follows that $h_{J,w}(X)\in\mathfrak{O}$, for each q, and that $f^*(h_{J,w}(X))=|B(q)wB(q)\cap\mathfrak{C}|$, where $f^*:\mathfrak{O}\to\bar{Q}$ is an extension of $f: X\to q$.

It remains to prove that $h_{J,w}\in\mathfrak{o}$, and for this it is sufficient to prove that

$$\hat{h}_{J,w}=\sum_{\phi\in\mathcal{E}(W_J)}d_\phi(X)\omega_\phi(a_w)\in Q(X)\,,$$

because $h_{J,w}(X)$ is then a rational function taking integer values at infinitely many positive integers $\{q\}$, and hence belongs to $Q[X]$. What has to be shown is that if $\sigma\in \mathrm{Gal}_{\bar{K}/K}$, then $\sigma(\hat{h}_{J,w})\in K$. We adopt the point of view of the proof of (5.3), following Lemma (5.11). Let $\{\chi\}$ denote the irreducible characters of $A^{\bar{K}}$, and $\{\xi\}$ the irreducible characters of $A_J^{\bar{K}}$. For each ξ let e_ξ be a primitive idempotent in $A_J^{\bar{K}}$ affording ξ, chosen according to Lemma (5.11). Then from the proof of (5.3),

$$\hat{h}_{J,w}=\sum_{\xi} d_\xi(X)\Big(\sum_{\chi}\chi(e_\xi)\chi(a_w)\Big),$$

where $d_\xi(X)$ is the generic degree corresponding to the character ξ.

The following proof that $\hat{h}_{J,w}\in Q(X)$ is independent of the discussion of the rationality of the characters χ in Theorem (2.8) of [1]. For each automorphism σ, let χ^σ and ξ^σ denote the irreducible characters of $A^{\bar{K}}$ and $A_J^{\bar{K}}$, respectively, defined by

$$\chi^\sigma\Big(\sum \lambda_w a_w\Big)=\sum \lambda_w \sigma(\chi(a_w)), \qquad \lambda_w\in\bar{K},$$

and similarly for ξ^σ.

(6.6) LEMMA. *Let $\sigma\in\mathrm{Gal}_{\bar{K}/K}$. Then $d_{\xi^\sigma}=d_\xi$, for all irreducible characters ξ of W_J.*

PROOF. From the formula for a generic degree ([1], (2.4)) it is sufficient to prove that

$$\sum_{w\in W_J}\mathrm{IND}\,(a_w)^{-1}\xi(\hat{a}_w)\xi(a_w)=\sum_{w\in W_J}\mathrm{IND}\,(a_w)^{-1}\xi^\sigma(\hat{a}_w)\xi^\sigma(a_w).$$

As $\{q\}$ contains almost all primes, both expressions above belong to K, by Theorem (2.6) of [1]. Therefore the expression on the left is invariant under σ, and as σ is an automorphism leaving the elements of K fixed, the right side is the image of the left side under σ.

(6.7) LEMMA. *$\chi^\sigma(e_{\xi^\sigma})=\chi(e_\xi)$, for each pair of irreducible characters χ and ξ, and all $\sigma\in\mathrm{Gal}_{\bar{K}/K}$.*

PROOF. Let $e_\xi=\sum_{w\in W_J}\lambda_w a_w$. Then $\sigma\otimes 1$ is an automorphism of $A_J^{\bar{K}}$, and $(\sigma\otimes 1)(e_\xi)=\sum\sigma(\lambda_w)a_w$ is a primitive idempotent affording ξ^σ, as

$$\begin{aligned}\xi^\sigma\Big(\sum\sigma(\lambda_w)a_w\Big)&=\sum\sigma(\lambda_w)\sigma(\xi(a_w))\\&=\sigma\Big(\sum\lambda_w\xi(a_w)\Big)\\&=\sigma(\xi(e_\xi))\neq 0.\end{aligned}$$

We now have, as $\chi(e_\xi)\in K$,

$$\chi^\sigma(e_{\xi^\sigma})=\sum\sigma(\lambda_w)\chi^\sigma(a_w)=\sigma(\chi(e_\xi))=\chi(e_\xi),$$

and the Lemma is proved.

We can now complete the proof that $\hat{h}_{J,w} \in K$. Let $\sigma \in \mathrm{Gal}_{\bar{K}/K}$, and let

$$\omega_\xi = \sum_\chi \chi(e_\xi)\chi .$$

Then $\sigma(\omega_\xi) = \sum_\chi \chi(e_\xi)\chi^\sigma = \sum_\chi \chi^\sigma(e_{\xi^\sigma})\chi^\sigma = \omega_{\xi^\sigma}$, by Lemma (6.7). As the generic degrees $d_\xi(X)$ are constant over $\langle\sigma\rangle$ orbits $\{\xi, \xi^\sigma, \xi^{\sigma 2}, \cdots\}$ it follows that $\sum d_\xi(X)\omega_\xi$ is invariant under σ, and hence

$$\sum d_\xi(X)\omega_\xi(a_w) \in K$$

for each $w \in W$, as required.

The following elegant expression for the polynomials $h_{J,w}(X)$ in case $J=\emptyset$ is due to R. Kilmoyer. Note that $J=\emptyset$ is equivalent to the statement that t is a regular element of $T(q)$.

(6.8) COROLLARY (Kilmoyer). *The polynomial $h_{J,w}(X)$, for $J=\emptyset$, is given by*

$$h_{\emptyset,w}(X) = \mathrm{IND}\,(a_{w_R})\rho_A(a_w) ,$$

where $\rho_A = \sum_{\varphi \in \mathcal{E}(W)} \varphi(1)\chi_\varphi$ is the regular character of the generic algebra A.

PROOF. From the formula for $h_{J,w}$, in case $J=\emptyset$, in Theorem (6.5), we have

$$\begin{aligned} h_{\emptyset,w}(X) &= \mathrm{IND}\,(a_{w_R}) \sum_{\varphi\in\mathcal{E}(W)} v_{\emptyset,\varphi}(X)\chi_\varphi(a_w) \\ &= \mathrm{IND}\,(a_{w_R}) \sum_{\varphi\in\mathcal{E}(W)} \varphi(1)\chi_\varphi(a_w) \\ &= \mathrm{IND}\,(a_{w_R})\rho_A(a_w) , \end{aligned}$$

because $P_\emptyset(X)=1$, $\mathrm{IND}\,(a_{w_\emptyset})=1$, and $v_{\emptyset,\varphi}(X)=\varphi(1)$ by Corollary (6.2).

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