J. Math. Soc. Japan
Vol. 27, No. 4, 1975

Difference approximation of Cauchy problems for quasi-dissipative operators and generation of nonlinear semigroups

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(Received April 8, 1975)

In this paper we consider the evolution equation

$$\text{(DE)} \qquad (d/dt)u(t) \in Au(t), \qquad t \in (0, T)$$

for an "ω-quasi-dissipative" operator A in a Banach space X, from the viewpoint of difference approximation. We introduce a notion of "DS-limit solution" of (DE) and discuss the construction of the DS-limit solutions. We also give a generation theorem of nonlinear semigroups through difference approximation.

Recently several authors have treated the evolution equation (DE) from the view-point of difference approximation. The result of Crandall and Liggett [**7**] is the first fundamental one in this direction. Kenmochi and Oharu [**9**] extended the result in [**7**] to the case where the difference scheme for (DE) permits errors. Takahashi [**16**], [**17**] formulated a more general approximate difference scheme and determined the conditions under which the solution of the difference scheme converges.

In this paper we introduce a notion of ω-quasi-dissipative operator as a generalization of ω-dissipative operator. We consider the approximate difference scheme for the Cauchy problem for (DE) under the same formulation as in [**17**]. Our first purpose is to give a convergence theorem for difference approximation and to improve the result in [**17**]. At the same time, it is shown that the limit function of solutions of difference approximation is uniquely determinded by the initial condition and is independent of the choice of difference scheme. Hence we shall call the limit function a DS-limit solution of the Cauchy problem.

Recently Bénilan [**2**] has introduced the notion of "integral solution" and "bonne solution" and investigated properties of bonne solutions. Our second purpose is to investigate basic properties of DS-limit solutions and to study the relationship between those solutions.

The results mentioned above can be considered from the view point of the theory of nonlinear semigroups. Our third purpose is to discuss the generation of a nonlinear semigroup associated with a given operator A.

Our fourth purpose is to give a sufficient condition which assures generation of nonlinear semigroups through the difference approximation. Many authors have already treated the generation of semigroups. Our results imply the results of Crandall and Liggett [7] and Martin [12]. Also we give a simple application of our results to the continuous perturbation of m-dissipative operators.

This paper consists of five sections. In § 1, we introduce basic notions and give some fundamental facts concerning these notions. In § 2, we deal with the convergence of the difference approximation and introduce the notion of DS-limit solutions. In § 3, some basic properties of DS-limit solutions are studied. § 4 treats the generation of semigroups. In § 5, we give a sufficient condition for the generation of semigroups and its applications.

The author would like to express his hearty thanks to Prof. I. Miyadera and Mr. T. Takahashi for their advices.

§ 1. Preliminaries.

In this section, we list some notation and basic notions along with their fundamental properties.

Let X be a real Banach space with norm $\|\cdot\|$. By an operator A in X we mean a multi-valued operator with domain $D(A)$ and range $R(A)$ in X, where $D(A)$ is the set $\{x \in X;\ Ax \neq \emptyset\}$ and $R(A) = \bigcup_{x \in D(A)} Ax$. We identify the operator A with its graph, so that we write $[x, y] \in A$ if $y \in Ax$. For each $x \in D(A)$, we write

$$|||Ax||| = \inf\{\|y\|\ ;\ y \in Ax\}$$

and define $A^0 x = \{y\ ;\ y \in Ax,\ \|y\| = |||Ax|||\}$. Also we define $|Ax|$, $x \in X$, by

$$\begin{aligned} |Ax| &= \inf\{\sup |||Ax_n|||\ ;\ x_n \in D(A) \text{ and } \lim_{n\to\infty} x_n = x\} \quad \text{for} \quad x \in \overline{D(A)} \\ &= \infty \quad \text{for} \quad x \notin \overline{D(A)} \end{aligned}$$

and set $D_a(A) = \{x \in X\ ;\ |Ax| < \infty\}$. Clearly, we have $D(A) \subset D_a(A) \subset \overline{D(A)}$. We see easily that the functional $x \mapsto |Ax|$ is lower semi-continuous on X. (See [11] or [18].)

We define the sum $A+B$ and the scalar multiple αA of operators A, B in X as in [9] and use the symbol I for the identity operator in X.

We denote by $\langle x, f \rangle$ the natural pairing between $x \in X$ and $f \in X^*$, where

X^* is the dual space of X. Let F be the duality map from X into X^*, i.e.,

$$F(x)=\{f\in X^*;\ \|f\|^2=\|x\|^2=\langle x, f\rangle\} \qquad \text{for} \quad x\in X.$$

We define the functionals $\langle , \rangle_s$ and $\langle , \rangle_i$ on $X\times X$, by

$$\langle y, x\rangle_s=\sup\{\langle y, f\rangle;\ f\in F(x)\} \tag{1.1}$$

and

$$\langle y, x\rangle_i=\inf\{\langle y, f\rangle;\ f\in F(x)\} \tag{1.2}$$

for each $x, y\in X$. Clearly $\langle y, x\rangle_s=-\langle -y, x\rangle_i=-\langle y, -x\rangle_i$ for $x, y\in X$. It has been shown in [2] that

$$\langle y, x\rangle_s=\tau(x, y)\cdot\|x\| \qquad \text{for} \quad x, y\in X, \tag{1.3}$$

where

$$\tau(x, y)=\inf_{t>0} t^{-1}(\|x+ty\|-\|x\|)=\lim_{t\to+0} t^{-1}(\|x+ty\|-\|x\|)$$

for $x, y\in X$. This shows that the value of $\langle , \rangle_s$ at $[x, y]\in X\times X$ is not changed even if x and y are regarded as elements in X^{**}, the bidual space of X. Therefore we use the same notation (1.1) and (1.2), for the corresponding functionals from $X^{**}\times X^{**}$ into R. We see easily that the fnnctional $\langle , \rangle_s$ is upper semi-continuous with respect to the strong topology of $X\times X$. We refer to [2] and [7] for other properties of the functionals.

Let ω be a real number. An operator A in X is said to be *ω-dissipative* if for any $[x_i, y_i]\in A$ $(i=1, 2)$,

$$\langle y_1-y_2, x_1-x_2\rangle_i\leqq\omega\|x_1-x_2\|^2.$$

An operator A in X is said to be *strictly ω-dissipative* if for any $[x_i, y_i]\in A$ $(i=1, 2)$,

$$\langle y_1-y_2, x_1-x_2\rangle_s\leqq\omega\|x_1-x_2\|^2.$$

(Strictly) 0-dissipative operator is simply called (strictly) dissipative. Apparently A is ω-dissipative if and only if $A-\omega I$ is dissipative. We refer to [2] and [7] for the properties of ω-dissipative operators.

Following Takahashi [16], we introduce the following notion.

DEFINITION 1.1. An operator A in X is said to be *ω-quasi-dissipative* if for any $[x_i, y_i]\in A$ $(i=1, 2)$,

$$\langle y_1, x_1-x_2\rangle_i+\langle y_2, x_2-x_1\rangle_i\leqq\omega\|x_1-x_2\|^2. \tag{1.4}$$

0-quasi-dissipative operator is simply called *quasi-dissipative*.

Apparently an ω-dissipative operator is ω-quasi-dissipative; those notions are equivalent when the duality map is single-valued but not so in general. (See Example 1.1 at the end of this section.)

Let $S\subset X$ and A be an ω (-quasi)-dissipative operator in X. Then we say that A is *maximal* ω *(-quasi)-dissipative on* S if any ω (-quasi)-dissipative extension of A coincides with A on S.

If A is a dissipative operator such that $R(I-\lambda A)=X$ for all $\lambda>0$, then we say that A is *m-dissipative.* It is well known that if A is a dissipative operator such that $R(I-\lambda_0 A)=X$ for some $\lambda_0>0$, then A is m-dissipative. We refer to [8] for other properties of m-dissipative operators.

An ω-quasi-dissipative operator A in X can be regarded as an ω-quasi-dissipative operator in X^{**}. Therefore, we can associate with A an operator $\mathcal{A}$ in X^{**} such that $\mathcal{A}$ is an extension of A, $D(\mathcal{A})\subset\overline{D(A)}$ and $\mathcal{A}$ is maximal ω-quasi-dissipative on $\overline{D(A)}$ in X^{**}. We call such $\mathcal{A}$ a *maximal* (**) *extension* of A. (See [**17**].)

Let $X_0\subset X$. A one parameter family $\{T(t)\,;\, t\geqq 0\}$ of operators from X_0 into itself is called a *semigroup of type* ω *on* X_0 if it has the following properties:

(i) for $x, y\in X_0$ and $t\geqq 0$, $\|T(t)x-T(t)y\|\leqq e^{\omega t}\|x-y\|$;

(ii) $T(0)x=x$ for $x\in X_0$ and $T(t+s)=T(t)T(s)$ for $t, s\geqq 0$;

(iii) for each $x\in X_0$, $T(t)x$ is strongly continuous in $t\geqq 0$.

In the following, we prepare some estimates which will play a central role in later argument.

LEMMA 1.1. *Let A be an operator in X and ω a real number. Then the following three conditions are equivalent:*

(i) *A is ω-quasi-dissipative;*

(ii) *for any $[x_i, y_i]\in A$ $(i=1, 2)$ and λ, $\mu>0$,*

$$(\lambda+\mu-\lambda\mu\omega)\|x_1-x_2\|\leqq\lambda\|x_1-x_2-\mu y_1\|+\mu\|x_2-x_1-\lambda y_2\|\,;$$

(iii) *for any $[x_i, y_i]\in A$ $(i=1, 2)$ and $\lambda>0$,*

$$(2-\lambda\omega)\|x_1-x_2\|\leqq\|x_1-x_2-\lambda y_1\|+\|x_2-x_1-\lambda y_2\|\,.$$

Furthermore, in these cases we have

(iv) *for any $[x, y]\in A$, $u\in D(A)$ and $\lambda>0$,*

$$(1-\lambda\omega)\|x-u\|\leqq\|x-u-\lambda y\|+\lambda\|Au\|\,.$$

PROOF. Suppose that A is ω-quasi-dissipative. Let $[x_i, y_i]\in A$ $(i=1, 2)$ and λ, $\mu>0$. By definition, there exist $f\in F(x_1-x_2)$ and $g\in F(x_2-x_1)$ such that

$$\langle y_1, f\rangle+\langle y_2, g\rangle\leqq\omega\|x_1-x_2\|^2\,.$$

Therefore, we have

$$
\begin{aligned}
(\lambda+\mu)\|x_1-x_2\|^2 &= \lambda\langle x_1-x_2, f\rangle+\mu\langle x_2-x_1, g\rangle \\
&\leqq \lambda\langle x_1-x_2-\mu y_1, f\rangle+\mu\langle x_2-x_1-\lambda y_2, g\rangle+\lambda\mu\omega\|x_1-x_2\|^2 \\
&\leqq (\lambda\|x_1-x_2-\mu y_1\|+\mu\|x_2-x_1-\lambda y_2\|+\lambda\mu\omega\|x_1-x_2\|)\|x_1-x_2\| .
\end{aligned}
$$

Hence we have (ii). Apparently (ii) implies (iii). Suppose that (iii) is satisfied. Then we have

$$
\begin{aligned}
&t^{-1}(\|x_1-x_2\|-\|x_1-x_2-ty_1\|) \\
&\qquad +t^{-1}(\|x_2-x_1\|-\|x_2-x_1-ty_2\|)\leqq \omega\|x_1-x_2\| ,
\end{aligned}
$$

for $[x_i, y_i]\in A$ $(i=1, 2)$ and $t>0$. Letting $t\rightarrow+0$, we have

$$-\tau(x_1-x_2, -y_1)-\tau(x_2-x_1, -y_2)\leqq \omega\|x_1-x_2\|$$

or

$$\langle y_1, x_1-x_2\rangle_i+\langle y_2, x_2-x_1\rangle_i\leqq \omega\|x_1-x_2\|^2$$

by (1.3). Hence A is ω-quasi-dissipative.

Let $[x, y]\in A$, $u\in D(A)$ and $\lambda>0$. Then we have immediately by (iii)

$$(2-\lambda\omega)\|x-u\|\leqq\|x-u-\lambda y\|+\|u-x\|+\lambda\|v\|$$

or

$$(1-\lambda\omega)\|x-u\|\leqq\|x-u-\lambda y\|+\lambda\|v\| \qquad \text{for} \quad v\in Au .$$

Since v is arbitrary in Au, we have (iv). Q. E. D.

The following example is due to Miyadera.

EXAMPLE 1.1. Let $X=R^2$ with the maximum norm. Let $x_1=(1, 1)$ and $x_2=(0, 0)$. We set $D(A)=\{x_1, x_2\}$, $Ax_1=\{(\alpha, \beta); \alpha\leqq 0 \text{ or } \beta\leqq 0\}$ and $Ax_2=\{(\alpha, \beta); \alpha\geqq 0 \text{ or } \beta\geqq 0\}$. Then A is quasi-dissipative in X but A is not ω-dissipative in X for any real ω. In addition, $R(I-\lambda A)\supset D(A)$ for any $\lambda>0$. (See also Remark 5.2.)

REMARK 1.1. The inequality (ii) is suggested by Takahashi [**16**]. Crandall and Evans [**6**] also proves the inequality (ii) for the case A is dissipative. The assertion (iii)$\Rightarrow$(i) was pointed out by Prof. I. Miyadera.

§ 2. Convergence of differences approximation.

In this section we treat the convergence of difference approximation of the Cauchy problem for the evolution equation (DE) and introduce a notion of DS-limit solution of the Cauchy problem. At the same time, the uniqueness of DS-limit solutions will be established. We also give some fundamental properties of DS-limit solutions, which are immediately derived from the convergence theorem.

Let ω be a real number and A be an ω-quasi-dissipative operator in X.

Let $T>0$ be fixed. We consider the following Cauchy problem, formulated for A on a finite interval $[0, T]$:

$$\text{(CP)} \qquad \begin{cases} (d/dt)u(t) \in Au(t), & t \in (0, T), \\ u(0) = x_0, \end{cases}$$

where $x_0 \in X$ is given. We shall denote by $(CP; x_0)$ the Cauchy problem (CP) with the initial condition $u(0) = x_0$.

Let $u_n(t)$ be a sequence of X-valued simple functions on $[0, T]$ defined by

$$\text{(2.1)} \qquad u_n(t) = \begin{cases} x_0^n & \text{for } t = 0, \\ x_i^n & \text{for } t \in (t_{i-1}^n, t_i^n] \cap (0, T], \quad i = 1, 2, \cdots, N_n, \end{cases}$$

and $n \geqq 1$, where $\{t_i^n\}$ represents the partition $\Delta_n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n-1}^n < T \leqq t_{N_n}^n\}$ of the interval $[0, T]$ satisfying the condition:

$$\text{(2.2)} \qquad |\Delta_n| = \max_{1 \leqq i \leqq N_n} (t_i^n - t_{i-1}^n) \to 0 \quad \text{as} \quad n \to \infty.$$

DEFINITION 2.1. Let $x_0 \in X$. We say that the sequence $u_n(t)$ is a (backward) *DS-approximate solution* of $(CP; x_0)$ if it satisfies

$$\text{(2.3)} \qquad \begin{cases} \dfrac{x_i^n - x_{i-1}^n}{t_i^n - t_{i-1}^n} - \varepsilon_i^n \in Ax_i^n, & i = 1, 2, \cdots, N_n; \ n \geqq 1 \\ x_0^n \longrightarrow x_0 & \text{as} \quad n \to \infty, \end{cases}$$

and also

$$\text{(2.4)} \qquad \varepsilon_n = \sum_{i=1}^{N_n} \|\varepsilon_i^n\| (t_i^n - t_{i-1}^n) \to 0 \quad \text{as} \quad n \to \infty.$$

We call ε_n as the error bounds of the DS-approximate solution $u_n(t)$.

The following is the main result of this paper and will be proved below in this section.

THEOREM 2.1. *Let $x_0 \in \overline{D(A)}$ and $u_n(t)$ be a DS-approximate solution of $(CP; x_0)$ on $[0, T]$. Then there exists a continuous function $u(t)$ on $[0, T]$ satisfying the following*:

$$\text{(i)} \qquad u(t) = \lim_{n \to \infty} u_n(t) \quad \text{for} \quad t \in [0, T]$$

and the convergence is uniform on $[0, T]$;

$$\text{(ii)} \qquad u(t) \in \overline{D(A)} \quad \text{for} \quad t \in [0, T] \text{ and } u(0) = x_0;$$

(iii) *for any DS-approximate solution $\hat{u}_n(t)$ of $(CP; x_0)$ on $[0, T]$,*

$$u(t) = \lim_{n \to \infty} \hat{u}_n(t) \quad \text{for} \quad t \in [0, T].$$

REMARK 2.1. Takahashi showed in [16] and [17] the convergence (i) under an additional condition:

$$\text{(S)} \qquad \|u_n(t)-u_n(s)\| \leqq \rho(|t-s|)+\delta_n\,, \qquad \text{for} \quad t, s\in[0, T] \text{ and } n\geqq 1\,,$$

where $\rho(r)$ is a bounded, nondecreasing function on $[0, T]$ such that $\rho(r)\to 0$ as $r\to +0$ and δ_n is a sequence of nonnegative numbers, converging to 0 as $n\to\infty$. Therefore, our result is an extension of his result, although the condition (S) is necessary for our convergence (i) to hold. See Remark after Theorem I in [17].

REMARK 2.2. After the preparation of this manuscript, Prof. M. Crandall informed me that Crandall and Evans [6] proved Theorem 2.1 by an entirely different method, which is interesting in itself. They treat a more general evolution equation

$$(d/dt)u(t)\in Au(t)+f(t)\,, \qquad \text{for} \quad t\in(0, T)\,,$$

where $f\in L^1(0, T; X)$ is given. Our method is also applicable to this case if A is ω-dissipative.

By virtue of Theorem 2.1, we define the following.

DEFINITION 2.2. Let $u(t)$ be a continuous function on $[0, T]$ and $x_0\in\overline{D(A)}$. We say that $u(t)$ is a (backward) *DS-limit solution* of $(CP; x_0)$ on $[0, T]$ if there exists a (backward) DS-approximate solution $u_n(t)$ of $(CP; x_0)$ on $[0, T]$ such that $u_n(t)$ converges to $u(t)$, uniformly on $[0, T]$.

The proof of Theorem 2.1 is based on the following lemma. We set $\omega_0=\max(\omega, 0)$ in this section.

LEMMA 2.1. *Let $u_n(t)$ and $\hat{u}_n(t)$ be two DS-approximate solutions of the Cauchy problem* (CP) *on $[0, T]$. Then we have*

$$\begin{aligned}(2.5) \qquad \|x_i^n-\hat{x}_j^m\| \leqq \exp(2\omega_0(t_i^n+\hat{t}_j^m))\cdot[\|x_0^n-u\|+\|\hat{x}_0^m-u\| \\ +\{(t_i^n-\hat{t}_j^m)^2+|\Delta_n|t_i^n+|\hat{\Delta}_m|\hat{t}_j^m\}^{1/2}\cdot|||Au|||+\varepsilon_n+\hat{\varepsilon}_m]\,,\end{aligned}$$

for $u\in D(A)$, $0\leqq i\leqq N_n$ and $0\leqq j\leqq \hat{N}_m$ with $|\Delta_n|\omega_0$, $|\hat{\Delta}_m|\omega_0\leqq 1/2$. Here the notations with the symbol "$\hat{}$" correspond to the solution $\hat{u}_n(t)$.

We prove Lemma 2.1 by the method of Crandall and Liggett [7], modified by Rasmussen [15]. (See also Yosida [21].)

We start with

LEMMA 2.2. *Let λ and μ be positive numbers such that $\lambda\omega_0$, $\mu\omega_0<1$. Then we have the following:*

$$(2.6) \qquad \lambda(1-\mu\omega_0)+\mu(1-\lambda\omega_0)\leqq\lambda+\mu-\lambda\mu\omega_0\,,$$

$$(2.7) \qquad (\lambda+\mu)(\lambda(1-\mu\omega_0)^2+\mu(1-\lambda\omega_0)^2)\leqq(\lambda+\mu-\lambda\mu\omega_0)^2\,.$$

PROOF. The inequality (2.6) is evident. To prove (2.7), we note that

$\lambda+\mu-\lambda\mu\omega_0>0$, and $1\geqq 1-\lambda\omega_0$, $1-\mu\omega_0>0$. Therefore we have

$$\begin{aligned}&\{\lambda(1-\mu\omega_0)^2+\mu(1-\lambda\omega_0)^2\}/(\lambda+\mu-\lambda\mu\omega_0)\\ &\quad\leqq\{\lambda(1-\mu\omega_0)+\mu(1-\lambda\omega_0)\}/(\lambda+\mu-\lambda\mu\omega_0)\\ &\quad=1-\lambda\mu\omega_0/(\lambda+\mu-\lambda\mu\omega_0).\end{aligned}$$

Since $\lambda\mu\omega_0/(\lambda+\mu-\lambda\mu\omega_0)\geqq\lambda\mu\omega_0/(\lambda+\mu)$, we have

$$\begin{aligned}&\{\lambda(1-\mu\omega_0)^2+\mu(1-\lambda\omega_0)^2\}/(\lambda+\mu-\lambda\mu\omega_0)\\ &\quad\leqq 1-\lambda\mu\omega_0/(\lambda+\mu)=(\lambda+\mu-\lambda\mu\omega_0)/(\lambda+\mu).\end{aligned}$$

Hence we have (2.7). Q. E. D.

PROOF OF LEMMA 2.1. For simplicity, we omit the indices n and m, so that we write $x_i=x_i^n$, $\hat{x}_j=\hat{x}_j^m$, $N=N_n$, $\hat{N}=\hat{N}_m$, etc. Also we set $h_i=t_i-t_{i-1}$ and $\hat{h}_j=\hat{t}_j-\hat{t}_{j-1}$ for $1\leqq i\leqq N$ and $1\leqq j\leqq\hat{N}$. Furthermore, we define $a_{i,j}$ and $\gamma_{i,j}$ by

$$a_{i,j}=\|x_i-\hat{x}_j\| \quad\text{and}\quad \gamma_{i,j}=\prod_{k=1}^{i}(1-\omega_0h_k)\cdot\prod_{k=1}^{j}(1-\omega_0\hat{h}_k)$$

for $0\leqq i\leqq N$ and $0\leqq j\leqq\hat{N}$, where $\prod_{k=1}^{0}(1-\omega_0h_k)=1$ and $\prod_{k=1}^{0}(1-\omega_0\hat{h}_k)=1$.

Let $u\in D(A)$ and assume $|\Delta|\omega_0$, $|\hat{\Delta}|\omega_0\leqq 1/2$. We then show that

$$\begin{aligned}\gamma_{i,j}a_{i,j}\leqq&\|x_0-u\|+\|\hat{x}_0-u\|+\{(t_i-\hat{t}_j)^2+|\Delta|t_i+|\hat{\Delta}|\hat{t}_j\}^{1/2}\cdot|||Au|||\\ &+\sum_{k=1}^{i}\|\varepsilon_k\|h_k+\sum_{k=1}^{j}\|\hat{\varepsilon}_k\|\hat{h}_k,\end{aligned}\tag{2.8}$$

for $0\leqq i\leqq N$ and $0\leqq j\leqq\hat{N}$. Apparently (2.8) implies (2.5), since we have

$$\sum_{k=1}^{i}\|\varepsilon_k\|h_k\leqq\varepsilon,\qquad\sum_{k=1}^{j}\|\hat{\varepsilon}_k\|\hat{h}_k\leqq\hat{\varepsilon}$$

and

$$\gamma_{i,j}\leqq\exp(2\omega_0(t_i+\hat{t}_j))$$

for $0\leqq i\leqq N$ and $0\leqq j\leqq\hat{N}$. Here we have used the following inequality:

$$(1-t)^{-1}\leqq\exp(2t)\qquad\text{for}\quad t\in[0,1/2].\tag{2.9}$$

We prove (2.8) by induction on (i,j). We first show $a_{i,0}$ satisfies (2.8) for $0\leqq i\leqq N$. In fact, by (2.3) and (iv) of Lemma 1.1, we have

$$\begin{aligned}(1-\omega_0h_k)\|x_k-u\|&\leqq(1-\omega h_k)\|x_k-u\|\\ &\leqq\|x_k-u-(x_k-x_{k-1}-h_k\varepsilon_k)\|+h_k|||Au|||\\ &\leqq\|x_{k-1}-u\|+h_k\|\varepsilon_k\|+h_k|||Au|||\end{aligned}$$

or

$$\|x_k-u\|\leqq(1-\omega_0 h_k)^{-1}(\|x_{k-1}-u\|+h_k\|\varepsilon_k\|+h_k|||Au|||),$$

for $1\leqq k\leqq N$. Therefore, we have inductively

$$\|x_i-u\|\leqq\prod_{k=1}^{i}(1-\omega_0 h_k)^{-1}(\|x_0-u\|+t_i|||Au|||+\sum_{k=1}^{i}\|\varepsilon_k\|h_k)$$

and hence we have

$$\begin{aligned}a_{i,0}&\leqq\|x_i-u\|+\|u-\hat{x}_0\|\\&\leqq\prod_{k=1}^{i}(1-\omega_0 h_k)^{-1}(\|\hat{x}_0-u\|+\|x_0-u\|+t_i|||Au|||+\sum_{k=1}^{i}\|\varepsilon_k\|h_k)\end{aligned}$$

for $0\leqq i\leqq N$. Hence $a_{i,0}$ satisfies (2.8) if $0\leqq i\leqq N$. Similarly, we see that $a_{0,j}$ satisfies (2.8) if $0\leqq j\leqq\hat{N}$.

Now let i and j be integers such that $1\leqq i\leqq N$ and $1\leqq j\leqq\hat{N}$ and assume that $a_{i-1,j}$ and $a_{i,j-1}$ satisfy (2.8). We then show that $a_{i,j}$ satisfies (2.8). In fact, by (2.3) and (ii) of Lemma 1.1, we have

$$\begin{aligned}(h_i+\hat{h}_j-\omega_0 h_i\hat{h}_j)a_{i,j}&\leqq(h_i+\hat{h}_j-\omega h_i\hat{h}_j)a_{i,j}\\&\leqq\hat{h}_j\|x_i-\hat{x}_j-(x_i-x_{i-1}-h_i\varepsilon_i)\|+h_i\|\hat{x}_j-x_i-(\hat{x}_j-\hat{x}_{j-1}-\hat{h}_j\hat{\varepsilon}_j)\|\\&\leqq\hat{h}_j a_{i-1,j}+h_i a_{i,j-1}+(\|\varepsilon_i\|+\|\hat{\varepsilon}_j\|)\cdot h_i\hat{h}_j.\end{aligned}$$

Hence by the hypothesis of induction, we have

$$\begin{aligned}&(h_i+\hat{h}_j-\omega_0 h_i\hat{h}_j)\gamma_{i,j}a_{i,j}\\&\quad\leqq\hat{h}_j(1-\omega_0 h_i)\gamma_{i-1,j}a_{i-1,j}+h_i(1-\omega_0\hat{h}_j)\gamma_{i,j-1}a_{i,j-1}\\&\qquad+\gamma_{i,j}(\|\varepsilon_i\|+\|\hat{\varepsilon}_j\|)h_i\hat{h}_j\\&\quad\leqq(\hat{h}_j(1-\omega_0 h_i)+h_i(1-\omega_0\hat{h}_j))(\|x_0-u\|+\|\hat{x}_0-u\|)\\&\qquad+[\hat{h}_j(1-\omega_0 h_i)\{(t_{i-1}-\hat{t}_j)^2+|\varDelta|t_{i-1}+|\hat{\varDelta}|\hat{t}_j\}^{1/2}\\&\qquad+h_i(1-\omega_0\hat{h}_j)\{(t_i-\hat{t}_{j-1})^2+|\varDelta|t_i+|\hat{\varDelta}|\hat{t}_{j-1}\}^{1/2}]\cdot|||Au|||\\&\qquad+[\hat{h}_j(1-\omega_0 h_i)(\sum_{k=1}^{i-1}\|\varepsilon_k\|h_k+\sum_{k=1}^{j}\|\hat{\varepsilon}_k\|\hat{h}_k)\\&\qquad+h_i(1-\omega_0\hat{h}_j)(\sum_{k=1}^{i}\|\varepsilon_k\|h_k+\sum_{k=1}^{j-1}\|\hat{\varepsilon}_k\|\hat{h}_k)\\&\qquad+\gamma_{i,j}(\|\varepsilon_i\|+\|\hat{\varepsilon}_j\|)h_i\hat{h}_j].\end{aligned}\tag{2.10}$$

We denote by I_q the q-th term $(q=1, 2, 3)$ of the right side of the inequality (2.10). Then by (2.6) in Lemma 2.2, we have

$$I_1\leqq(h_i+\hat{h}_j-\omega_0 h_i\hat{h}_j)(\|x_0-u\|+\|\hat{x}_0-u\|).\tag{2.11}$$

On I_2, using Cauchy-Schwarz's inequality, we have

$$I_2 \leqq \{\hat{h}_j(1-\omega_0 h_i)^2+h_i(1-\omega_0\hat{h}_j)^2\}^{1/2}\cdot\{\hat{h}_j((t_{i-1}-\hat{t}_j)^2+|\Delta|t_{i-1}+|\hat{\Delta}|\hat{t}_j)$$

$$+h_i((t_i-\hat{t}_{j-1})^2+|\Delta|t_i+|\hat{\Delta}|\hat{t}_{j-1})\}^{1/2}\cdot|||Au|||$$

$$\leqq \{\hat{h}_j(1-\omega_0 h_i)^2+h_i(1-\omega_0\hat{h}_j)^2\}^{1/2}\{\hat{h}_j+h_i\}^{1/2}$$

$$\cdot\{(t_i-\hat{t}_j)^2+|\Delta|t_i+|\hat{\Delta}|\hat{t}_j\}^{1/2}\cdot|||Au|||,$$

where we used the following fact:

$$\hat{h}_j((t_{i-1}-\hat{t}_j)^2+|\Delta|t_{i-1}+|\hat{\Delta}|\hat{t}_j)+h_i((t_i-\hat{t}_{j-1})^2+|\Delta|t_i+|\hat{\Delta}|\hat{t}_{j-1})$$

$$=\hat{h}_j((t_i-\hat{t}_j)^2-2h_i(t_i-\hat{t}_j)+h_i^2+|\Delta|t_{i-1}+|\hat{\Delta}|\hat{t}_j)$$

$$+h_i((t_i-\hat{t}_j)^2+2\hat{h}_j(t_i-\hat{t}_j)+|\Delta|t_i+\hat{h}_j^2+|\hat{\Delta}|\hat{t}_{j-1})$$

$$\leqq(h_i+\hat{h}_j)((t_i-\hat{t}_j)^2+|\Delta|t_i+|\hat{\Delta}|\hat{t}_j);$$

note that $h_i \leqq |\Delta|$ and $\hat{h}_j \leqq |\hat{\Delta}|$. Therefore, by (2.7) in Lemma 2.2, we have

$$I_2 \leqq (h_i+\hat{h}_j-\omega_0 h_i\hat{h}_j)\{(t_i-\hat{t}_j)^2+|\Delta|t_i+|\hat{\Delta}|\hat{t}_j\}^{1/2}\cdot|||Au|||. \tag{2.12}$$

To estimate I_3, we note that $\gamma_{i,j} \leqq (1-\omega_0 h_i)$, $(1-\omega_0\hat{h}_j)$. Then we have

$$I_3 \leqq (\hat{h}_j(1-\omega_0 h_i)+h_i(1-\omega_0\hat{h}_j))(\sum_{k=1}^{i}\|\varepsilon_k\|h_k+\sum_{k=1}^{j}\|\hat{\varepsilon}_k\|\hat{h}_k) \tag{2.13}$$

$$\leqq (h_i+\hat{h}_j-\omega_0 h_j\hat{h}_j)(\sum_{k=1}^{i}\|\varepsilon_k\|h_k+\sum_{k=1}^{j}\|\hat{\varepsilon}_k\|\hat{h}_k),$$

where we used (2.6) again.

Combining (2.10)-(2.13), we see that $a_{i,j}$ satisfies (2.8). Hence we have completed the induction and (2.8) holds. Q. E. D.

REMARK 2.3. Let A be an ω-dissipative operator in X such that $R(I-\lambda A) \supset D(A)$ for $0 \leqq \lambda \leqq \lambda_0$. Then the estimate (2.5) gives

$$\|(I-\lambda A)^{-m}x-(I-\mu A)^{-n}x\|$$

$$\leqq \exp(2\omega_0(m\lambda+n\mu))\{(m\lambda-n\mu)^2+m\lambda^2+n\mu^2\}^{1/2}\cdot|||Ax|||$$

for $x \in D(A)$ and λ, $\mu>0$ such that $\lambda\omega_0$, $\mu\omega_0 \leqq 1/2$. This estimate is similar to but different from that of Crandall and Liggett [7]. Also, the inequality (2.5) (or (2.8)) itself is sharper than that of Rasmussen [15].

PROOF OF THEOREM 2.1. Let $u_n(t)$ be a DS-approximate solution of $(CP; x_0)$ with $x_0 \in \overline{D(A)}$. Let $\{u_p\} \subset D(A)$ be a sequence such that $u_p \to x_0$ as $p \to \infty$. Then by Lemma 2.1, we have

$$\|x_i^n-x_j^m\| \leqq \exp(2\omega_0(t_i^n+t_j^m))\cdot[\|x_0^n-u_p\|+\|x_0^m-u_p\| \tag{2.14}$$

$$+\{(t_i^n-t_j^m)^2+|\Delta_n|t_i^n+|\Delta_m|t_j^m\}^{1/2}\cdot|||Au_p|||+\varepsilon_n+\varepsilon_m]$$

for $0 \leqq i \leqq N_n$, $0 \leqq j \leqq N_m$, $p \geqq 1$ and n, $m \geqq 1$ such that $\omega_0|\Delta_m|$, $\omega_0|\Delta_n| \leqq 1/2$. Letting $n, m \to \infty$ with t_i^n, $t_j^m \to t$ in (2.14), we have

$$\limsup_{t_i^n, t_j^m \to t} \|x_i^n - x_j^m\| \leqq 2 \exp(4\omega_0 t)\|x_0 - u_p\| \qquad \text{for } p \geqq 1.$$

Since $u_p \to x_0$ as $p \to \infty$, we see that there exists

$$\begin{aligned} u(t) &= \lim x_i^n \qquad \text{as } t_i^n \to t,\ n \to \infty, \\ &= \lim_{n \to \infty} u_n(t) \end{aligned}$$

for $t \in [0, T]$. Apparently (2.14) shows that the convergence is uniform for $t \in [0, T]$. Furthermore, letting $t_i^n \to t$, $t_j^m \to s$, $n, m \to \infty$ in (2.14), we have

$$\|u(t) - u(s)\| \leqq \exp(2\omega_0(t+s)) \cdot (2\|x_0 - u_p\| + |t-s| \cdot |||Au_p|||) \tag{2.15}$$

for $t, s \in [0, T]$ and $p \geqq 1$. This shows that $u(t)$ is continuous on $[0, T]$. Thus, the assertion (i) has been proved.

The property (ii) is evident. For the proof of (iii), let $\hat{u}_m(t)$ be a DS-approximate solution of $(CP;\hat{x}_0)$ with $\hat{x}_0 \in \overline{D(A)}$. Then the assertion (i) implies that there exists

$$\hat{u}(t) = \lim_{m \to \infty} \hat{u}_m(t) \qquad \text{for } t \in [0, T].$$

Then by (2.5) in Lemma 2.1, we have

$$\|u(t) - \hat{u}(t)\| \leqq \exp(4\omega_0 t) \cdot (\|x_0 - u_p\| + \|\hat{x}_0 - u_p\|)$$

for $t \in [0, T]$ and $p \geqq 1$. Letting $p \to \infty$, we have

$$\|u(t) - \hat{u}(t)\| \leqq \exp(4\omega_0 t) \cdot \|x_0 - \hat{x}_0\| \qquad \text{for } t \in [0, T]. \tag{2.16}$$

In particular, when $\hat{x}_0 = x_0$, we have (iii). Q.E.D.

REMARK 2.4. Let $u(t)$ be a DS-limit solution of $(CP;\ x_0)$ on $[0, T]$ and let $\mathcal{A}$ be a maximal (**) extension of A. Then the argument in the proof of Lemma 2.1 shows that the inequality (2.5) remains true even if A is replaced by $\mathcal{A}$. Therefore, similarly to (2.15), we have

$$\|u(t) - u(s)\| \leqq \exp(2\omega_0(t+s)) \cdot (2\|x_0 - u_p\| + |t-s| \cdot |||\mathcal{A}u_p|||)$$

for $t, s \in [0, T]$, where $\{u_p\} \subset D(\mathcal{A})$ is a sequence such that $u_p \to x_0$ as $p \to \infty$. Suppose that $x_0 \in D_a(\mathcal{A})$. Then we can take $\{u_p\}$ so that $\|u_p\| \leqq |\mathcal{A}x_0| + 1/p$. Letting $p \to \infty$, we have

$$\|u(t) - u(s)\| \leqq \exp(2\omega_0(t+s)) \cdot |t-s| \cdot |\mathcal{A}x_0| \tag{2.17}$$

for $t, s \in [0, T]$. That is, $u(t)$ is Lipschitz continuous on $[0, T]$ if $x_0 \in D_a(\mathcal{A})$.

In the remainder of this section, we give some fundamental properties of DS-limit solutions.

PROPOSITION 2.1. *The DS-limit solution is uniquely determined by the initial condition.*

PROOF. It is a direct consequence of (iii) of Theorem 2.1. Q. E. D.

PROPOSITION 2.2. *Let $u(t)$ be a DS-limit solution of (CP) on $[0, T]$. Let $T_0 \in (0, T)$. Then we have*

(i) *the function $v(t)=u(t)$ defined on $[0, T_0]$ is a DS-limit solution of $(CP; u(0))$ on $[0, T_0]$;*

(ii) *the function $v(t)=u(t+T_0)$ defined on $[0, T-T_0]$ is a DS-limit solution of $(CP; u(T_0))$ on $[0, T-T_0]$.*

PROOF. Let $u_n(t)$ be a DS-approximate solution of $(CP; u(0))$ on $[0, T]$. Then apparently, $v_n(t)=u_n(t)$ defined on $[0, T_0]$ is a DS-approximate solution of $(CP; u(0))$ on $[0, T_0]$. Since $u_n(t)$ converges to $u(t)$ as $n\to\infty$, (i) is evident. Furthermore, $v_n(t)=u_n(t+T_0)$ defined on $[0, T-T_0]$ is a DS-approximate solution of $(CP; u(T_0))$ on $[0, T-T_0]$, since $v_n(t)=u_n(T_0)$ converges to $u(T_0)$ as $n\to\infty$. Hence (ii) is also evident. Q. E. D.

PROPOSITION 2.3. *Let $u^{(l)}(t)$ be a sequence of DS-limit solutions of $(CP; x_l)$ on $[0, T]$. Suppose that $\{x_l\}\subset\overline{D(A)}$ converges to x_0. Then there exists a DS-limit solution $u(t)$ of $(CP; x_0)$ on $[0, T]$ such that $u^{(l)}(t)$ converges to $u(t)$, uniformly on $[0, T]$ as $l\to\infty$.*

PROOF. By (2.16), we have

$$\|u^{(l)}(t)-u^{(m)}(t)\|\leqq \exp(4\omega_0 t)\|x_l-x_m\|$$

for $t\in[0, T]$ and $l, m\geqq 1$. Therefore $u^{(l)}(t)$ converges to a continuous function $u(t)$ on $[0, T]$ as $l\to\infty$, uniformly for $t\in[0, T]$. Let $u_n^{(l)}(t)$ be a DS-approximate solution of $(CP; x_l)$ and let $\varepsilon_n^{(l)}$ be the error bound of $u_n^{(l)}(t)$. For each l, $u_n^{(l)}(t)$ converges to $u^{(l)}(t)$ as $n\to\infty$, uniformly on $[0, T]$ and $\varepsilon_n^{(l)}\to 0$ as $n\to\infty$. Hence there exists a subsequence $\{n(l)\}$ of $\{n\}$ such that $u_{n(l)}^{(l)}(t)$ converges to $u(t)$ as $l\to\infty$ and $u_{n(l)}^{(l)}(t)$ is a DS-approximate solution. Therefore $u(t)$ becomes a DS-limit solution of $(CP; x_0)$. Q. E. D.

§ 3. Abstract Cauchy problems.

In this section, we investigate some basic properties of DS-limit solutions of the Cauchy problem for (DE).

Let ω be a real number and let A be an ω-quasi-dissipative operator in X. Let $T>0$ be fixed.

Following Brezis and Pazy [**5**], we define the strong solution.

DEFINITION 3.1. Let $x_0\in X$. An X-valued function $u(t)$ on $[0, T]$ is said to be a *strong solution* of $(CP; x_0)$ on $[0, T]$ if it satisfies

(i) $u(0)=x_0$;

(ii) $u(t)$ is Lipschitz continuous on $[0, T]$;

(iii) $u(t)$ is differentiable for a.a. $t\in(0, T)$ and

$$(d/dt)u(t)\in Au(t), \quad \text{for a.a. } t\in(0, T).$$

The following theorem is essentially due to Kenmochi and Oharu [**9**] and we omit the proof.

THEOREM 3.1. *A strong solution of* (CP) *on* $[0, T]$ *is a DS-limit solution of* (CP) *on* $[0, T]$.

By Proposition 2.1 and Theorem 3.1, we have the following.

PROPOSITION 3.1. *There exists at most one strong solution of the Cauchy problem* $(CP; x_0)$ *with* $x_0\in X$.

REMARK 3.1. We can prove Proposition 3.1 directly, using the Lemma 3.1 given later. See Proposition 2.2 in [**16**], where the case $\omega=0$ is treated.

As Bénilan [**2**], we define the following.

DEFINITION 3.2. Let $x_0\in X$. An X-valued continuous function $u(t)$ on $[0, T]$ is said to be an *integral solution of type* ω of $(CP; x_0)$ on $[0, T]$ if it satisfies the followings:

(i) $u(0)=x_0$;

(ii) for every $s, t\in[0, T]$ such that $s\leqq t$ and $[x, y]\in A$,

$$e^{-2\omega t}\|u(t)-x\|^2-e^{-2\omega s}\|u(s)-x\|^2 \leqq 2\int_s^t e^{-2\omega\tau}\langle y, u(\tau)-x\rangle_s d\tau . \tag{3.1}$$

Then we have

THEOREM 3.2. *Let* $x_0\in\overline{D(A)}$. *A DS-limit solution of* $(CP; x_0)$ *is the unique integral solution of* $(CP; x_0)$.

We first prove the following proposition, which gives a characterization of the integral solutions of type ω.

PROPOSITION 3.2. *Let* $u(t)$ *be a continuous function on* $[0, T]$ *with* $u(0)=x_0\in X$. *Then* $u(t)$ *is an integral solution of type* ω *of* $(CP; x_0)$ *if and only if it satisfies*

$$\|u(t)-x\|^2-\|u(s)-x\|^2\leqq 2\omega\int_s^t\|u(\tau)-x\|^2 d\tau +2\int_s^t\langle y, u(\tau)-x\rangle_s d\tau , \tag{3.2}$$

for $s, t\in[0, T]$ *such that* $s\leqq t$ *and* $[x, y]\in A$.

PROOF. We may assume $\omega\neq 0$. Let $[x, y]\in A$ be fixed. Suppose that $u(t)$ is an integral solution of type ω of $(CP; x_0)$. We first assume $\omega>0$. Then by

(3.1), we have

$$e^{-2\omega(t-s)}\|u(t)-x\|^2-\|u(s)-x\|^2\leqq 2\int_s^t e^{-2\omega(\tau-s)}\langle y,\ u(\tau)-x\rangle_s d\tau \tag{3.3}$$

for $s\leqq t$. Integrating (3.3) with respect to s from s to t, we have

$$\begin{aligned}\int_s^t e^{-2\omega(t-\sigma)}d\sigma\|u(t)-x\|^2-\int_s^t\|u(\sigma)-x\|^2 d\sigma\\ \leqq 2\int_s^t\int_\sigma^t e^{-2\omega(\tau-\sigma)}\langle y,\ u(\tau)-x\rangle_s d\sigma d\tau\\ =2\int_s^t\int_s^\tau e^{-2\omega(\tau-\sigma)}d\sigma\langle y,\ u(\tau)-x\rangle_s d\tau\end{aligned}$$

for $s\leqq t$. Hence we have

$$\begin{aligned}(2\omega)^{-1}(1-e^{-2\omega(t-s)})\|u(t)-x\|^2-\int_s^t\|u(\sigma)-x\|^2 d\sigma\\ \leqq 2\int_s^t(2\omega)^{-1}(1-e^{-2\omega(\tau-s)})\langle y,\ u(\tau)-x\rangle_s d\tau\end{aligned}$$

or

$$\begin{aligned}&2\int_s^t e^{-2\omega(\tau-s)}\langle y,\ u(\tau)-x\rangle_s d\tau\\ &\leqq 2\omega\int_s^t\|u(\sigma)-x\|^2 d\sigma+2\int_s^t\langle y,\ u(\tau)-x\rangle_s d\tau \qquad (3.4)\\ &\quad+(e^{-2\omega(t-s)}-1)\|u(t)-x\|^2\end{aligned}$$

for $s\leqq t$. Combining (3.4) with (3.3), we have (3.2). Next, we assume $\omega<0$. Then by (3.1), we have

$$\begin{aligned}&\|u(t)-x\|^2-e^{2\omega(t-s)}\|u(s)-x\|^2 \qquad (3.5)\\ &\leqq 2\omega\int_s^t e^{2\omega(t-\tau)}\langle y,\ u(\tau)-x\rangle_s d\tau\end{aligned}$$

for $s\leqq t$. Integrating (3.5) with respect to t from s to t, we have

$$\begin{aligned}\int_s^t\|u(\sigma)-x\|^2 d\sigma-\int_s^t e^{2\omega(\sigma-s)}d\sigma\|u(s)-x\|^2\\ \leqq 2\int_s^t\int_s^\sigma e^{2\omega(\sigma-\tau)}\langle y,\ u(\tau)-x\rangle_s d\tau d\sigma\\ =2\int_s^t\int_\tau^t e^{2\omega(\sigma-\tau)}d\sigma\langle y,\ u(\tau)-x\rangle_s d\tau\end{aligned}$$

for $s\leqq t$. Hence, in the same way as above, we have

$$\begin{aligned}&2\int_s^t e^{2\omega(t-\tau)}\langle y,\ u(\tau)-x\rangle_s d\tau \qquad (3.6)\\ &\leqq 2\omega\int_s^t\|u(\sigma)-x\|^2 d\sigma+2\int_s^t\langle y,\ u(\tau)-x\rangle_s d\tau\\ &\quad+(1-e^{2\omega(t-s)})\|u(s)-x\|^2\end{aligned}$$

for $s \leqq t$. Combining this with (3.5), we have (3.2).

Conversely, suppose that $u(t)$ satisfies (3.2) for $s \leqq t$. Then we have

$$
\begin{aligned}
(3.7)\qquad & e^{-2\omega t}\|u(t)-x\|^2 - e^{-2\omega t}\|u(s)-x\|^2 \\
& \leqq 2\omega e^{-2\omega t}\int_s^t \|u(\tau)-x\|^2 d\tau + 2e^{-2\omega t}\int_s^t \langle y, u(\tau)-x\rangle_s d\tau
\end{aligned}
$$

or

$$
\begin{aligned}
(3.8)\qquad & (d/dt)\Big(e^{-2\omega t}\int_s^t \|u(\tau)-x\|^2 d\tau\Big) - e^{-2\omega t}\|u(s)-x\|^2 \\
& \leqq 2e^{-2\omega t}\int_s^t \langle y, u(\tau)-x\rangle_s d\tau
\end{aligned}
$$

for $s \leqq t$. Integrating (3.8) with respect to t from s to t, we have

$$
\begin{aligned}
(3.9)\qquad & e^{-2\omega t}\int_s^t \|u(\tau)-x\|^2 d\tau - \int_s^t e^{-2\omega\sigma} d\sigma \|u(s)-x\|^2 \\
& \leqq 2\int_s^t \int_s^\sigma e^{-2\omega\sigma}\langle y, u(\tau)-x\rangle_s d\tau d\sigma \\
& = 2\int_s^t \int_\tau^t e^{-2\omega\sigma} d\sigma \langle y, u(\tau)-x\rangle_s d\tau
\end{aligned}
$$

for $s \leqq t$. Combining (3.9) with (3.7), we have (3.1). Q. E. D.

PROOF OF THEOREM 3.2. We follow the argument of Bénilan [**2**]. (See also Kenmochi and Oharu [**9**] and Takahashi [**17**].) Let $u(t)$ be a DS-limit solution of $(CP; x_0)$ on $[0, T]$. Let $u_n(t)$ be a DS-approximate solution of $(CP; x_0)$ on $[0, T]$, defined by (2.1). We set $h_k^n = t_k^n - t_{k-1}^n$ for $k = 1, 2, \cdots, N_n$. We first show that $u(t)$ is an integral solution of type ω of $(CP; x_0)$. Let $[x, y] \in A$. Since A is ω-quasi-dissipative, we have by (2.3),

$$\langle (h_k^n)^{-1}(x_k^n - x_{k-1}^n) - \varepsilon_k^n, x_k^n - x\rangle_i + \langle y, x - x_k^n\rangle_i \leqq \omega\|x_k^n - x\|^2$$

for $k = 1, 2, \cdots, N_n$. Noting that

$$\langle (h_k^n)^{-1}(x_k^n - x_{k-1}^n), x_k^n - x\rangle_i \geqq (2h_k^n)^{-1}(\|x_k^n - x\|^2 - \|x_{k-1}^n - x\|^2),$$

we have

$$\|x_k^n - x\|^2 - \|x_{k-1}^n - x\|^2 \leqq 2h_k^n(\omega\|x_k^n - x\|^2 + \langle y, x_k^n - x\rangle_s + \|\varepsilon_k^n\|\cdot\|x_k^n - x\|)$$

for $k = 1, 2, \cdots, N_n$. Adding these inequalities for $k = j+1, j+2, \cdots, i$, $(i > j)$, we have

$$
\begin{aligned}
& \|x_i^n - x\|^2 - \|x_j^n - x\|^2 \\
& \leqq 2\sum_{k=j+1}^{i} h_k^n(\omega\|x_k^n - x\|^2 + \langle y, x_k^n - x\rangle_s + \|\varepsilon_k^n\|\cdot\|x_k^n - x\|)
\end{aligned}
$$

for $0 \leqq j < i \leqq N_n$. Letting $t_i^n \to t$, $t_j^n \to s$ and $n \to \infty$, we have

$$\|u(t)-x\|^2-\|u(s)-x\|^2\leqq 2\int_s^t(\omega\|u(\tau)-x\|^2+\langle y, u(\tau)-x\rangle_s)d\tau$$

for $s, t\in[0, T]$ such that $s\leqq t$. Hence, by Proposition 3.2, $u(t)$ is an integral solution of type ω of $(CP; x_0)$.

Let $v(t)$ be an integral solution of type ω of (CP). Let $\alpha, \beta\in[0, T]$ such that $\alpha\leqq\beta$. Then by (2.3) and (3.2), we have

$$\begin{aligned}&\|v(\beta)-x_k^n\|^2-\|v(\alpha)-x_k^n\|^2\\ &\quad\leqq 2\omega\int_\alpha^\beta\|v(\sigma)-x_k^n\|^2d\sigma+2\int_\alpha^\beta\langle(h_k^n)^{-1}(x_k^n-x_{k-1}^n)-\varepsilon_k^n, v(\sigma)-x_k^n\rangle_s d\sigma\end{aligned}$$

for $k=1, 2, \cdots, N_n$. Noting that

$$\langle(h_k^n)^{-1}(x_k^n-x_{k-1}^n), v(\sigma)-x_k^n\rangle_s\leqq(2h_k^n)^{-1}(\|v(\sigma)-x_{k-1}^n\|^2-\|v(\sigma)-x_k^n\|^2),$$

we have

$$\begin{aligned}&h_k^n(\|v(\beta)-x_k^n\|^2-\|v(\alpha)-x_k^n\|^2)\\ &\quad\leqq 2\omega h_k^n\int_\alpha^\beta\|v(\sigma)-x_k^n\|^2d\sigma+\int_\alpha^\beta(\|v(\sigma)-x_{k-1}^n\|^2-\|v(\sigma)-x_k^n\|^2)d\sigma\\ &\qquad+2\|\varepsilon_k^n\|\cdot h_k^n\int_\alpha^\beta\|v(\sigma)-x_k^n\|d\sigma\end{aligned}$$

for $k=1, 2, \cdots, N_n$. Adding these inequalities for $k=j+1, j+2, \cdots, i$, $(i>j)$, we have

$$\begin{aligned}&\sum_{k=j+1}^i h_k^n(\|v(\beta)-x_k^n\|^2-\|v(\alpha)-x_k^n\|^2)\\ &\quad\leqq 2\omega\sum_{k=j+1}^i h_k^n\int_\alpha^\beta\|v(\sigma)-x_k^n\|^2d\sigma\\ &\qquad+\int_\alpha^\beta(\|v(\sigma)-x_j^n\|^2-\|v(\sigma)-x_i^n\|^2)d\sigma+2\sum_{k=j+1}^i\|\varepsilon_k^n\|h_k^n\int_\alpha^\beta\|v(\sigma)-x_k^n\|d\sigma\end{aligned}$$

for $0\leqq j<i\leqq N_n$. Letting $t_i^n\to t$, $t_j^n\to s$ and $n\to\infty$, we have

$$\begin{aligned}\text{(3.10)}\qquad&\int_s^t(\|v(\beta)-u(\tau)\|^2-\|v(\alpha)-u(\tau)\|^2)d\tau\\ &\quad\leqq 2\omega\int_s^t\int_\alpha^\beta\|v(\sigma)-u(\tau)\|^2d\sigma d\tau\\ &\qquad+\int_\alpha^\beta(\|v(\sigma)-u(s)\|^2-\|v(\sigma)-u(t)\|^2)d\sigma\end{aligned}$$

for $s, t\in[0, T]$ such that $s\leqq t$. Hence, by the following Lemma 3.1, we have

$$\text{(3.11)}\qquad e^{-\omega t}\|u(t)-v(t)\|\leqq e^{-\omega s}\|u(s)-v(s)\|$$

for $s, t\in[0, T]$ such that $s\leqq t$. In particular, we see that $u(t)$ is the unique integral solution of type ω of $(CP; x_0)$ on $[0, T]$. Q. E. D.

LEMMA 3.1. *Let $u(t)$ and $v(t)$ be X-valued continuous functions on* $[0, T]$ *satisfying the inequality* (3.10) *for every s, t, α and $\beta\in[0, T]$ such that $s\leqq t$ and $\alpha\leqq\beta$. Then they satisfy the inequality* (3.11) *for every $s, t\in[0, T]$ such that $s\leqq t$.*

Lemma 3.1 is essentially due to Bénilan [**2**]. For the proof, see the proof of Theorem 4.4 in [**9**]. We have immediately

COROLLARY 3.1. *Let $u(t)$ and $\hat{u}(t)$ be two DS-limit solutions of* (CP) *on* $[0, T]$. *Then we have*

$$e^{-\omega t}\|u(t)-\hat{u}(t)\|\leqq e^{-\omega s}\|u(s)-\hat{u}(s)\| \tag{3.12}$$

for $s, t\in[0, T]$ such that $s\leqq t$.

REMARK 3.2. Let $u(t)$ be a DS-limit solution of (CP) on $[0, T]$ and let $\mathcal{A}$ be a maximal (**) extension of A. Then the proof of Theorem 3.2 shows that $u(t)$ satisfies the inequalities (3.1) and (3.2) for $[x, y]\in\mathcal{A}$ and $s, t\in[0, T]$ such that $s\leqq t$.

REMARK 3.3. The proof of Theorem 3.2 shows also that DS-limit solutions are "bonne solutions" in the sense of Bénilan [**2**].

Let $u(t)$ be a X-valued strongly continuous function on $[0, T]$. Then we define

$$D^{+}u(t)=\bigcap_{\varepsilon>0}\overline{\mathrm{conv}}^{\sigma(X^{**},X^{*})}\{h^{-1}(u(t+h)-u(t))\,;\ 0<h<\varepsilon\}\,,$$

for $t\in[0, T)$, where the symbol $\overline{\mathrm{conv}}^{\sigma(X^{**},X^{*})}$ denotes the closed convex hull in the weak* topology of X^{**}. It is clear that if $\liminf_{h\to+0} h^{-1}\|u(t+h)-u(t)\|<+\infty$, then $D^{+}u(t)\neq\emptyset$.

The following is proved in [**16**] for the case $\omega=0$. Thus we omit the proof. (See Proposition 2.5 in [**16**].)

PROPOSITION 3.3. *Let $\mathcal{A}$ be a maximal* (**) *extension of A. Let $u(t)$ be an X-valued continuous function on $[0, T]$ satisfying the inequalities* (3.1) *or* (3.2) *for every $[x, y]\in\mathcal{A}$ and $s, t\in[0, T]$ such that $s\leqq t$. Suppose that $u(t)\in\overline{D(A)}$ for $t\in[0, T]$. Let $t_0\in[0, T)$. Then the following properties are equivalent:*

(i) $u(t_0)\in D(\mathcal{A})$;

(ii) $\limsup_{h\to+0} h^{-1}\|u(t_0+h)-u(t_0)\|<+\infty$;

(iii) $D^{+}u(t_0)\neq\emptyset$.

In these cases, we have also $\lim_{h\to+0} h^{-1}\|u(t_0+h)-u(t_0)\|=|||\mathcal{A}u(t_0)|||$ *and* $D^{+}u(t_0)\subset\mathcal{A}^{0}u(t_0)$.

On the Lipschitz continuity of DS-limit solutions, we have the following.

THEOREM 3.3. *Let $\mathcal{A}$ be a maximal* (**) *extension of A. Let $u(t)$ be a DS-limit solution of $(CP; x_0)$ on $[0, T]$ with $x_0\in\overline{D(A)}$. Then $u(t)$ is Lipschitz continuous on $[0, T]$ if and only if $x_0\in D_a(\mathcal{A})$. In this case we have also $u(t)\in$*

$D(\mathcal{A})$ for $t\in[0, T)$, $u(T)\in D_a(\mathcal{A})$ and

$$\|u(t+h)-u(t)\|\leqq|\mathcal{A}x_0|e^{\omega t}\int_0^h e^{\omega(h-\tau)}d\tau \tag{3.13}$$

for t, $t+h\in[0, T]$ with $h\geqq 0$.

PROOF. By Remark 3.2 and Proposition 3.3, if $u(t)$ is Lipschitz continuous, then $u(t)\in D(\mathcal{A})$ for $t\in[0, T)$ and hence $u(0)=x_0\in D_a(\mathcal{A})$. Furthermore, we have by Corollary 3.1

$$e^{-\omega t}\|u(t+h)-u(t)\|\leqq\|u(h)-u(0)\| \tag{3.14}$$

for $h\geqq 0$ and $t\in[0, T]$ such that $t+h\in[0, T]$. Hence we have by Proposition 3.3,

$$e^{-\omega t}|||\mathcal{A}u(t)|||\leqq|||\mathcal{A}x_0||| \qquad \text{for} \quad t\in[0, T).$$

Hence $u(T)\in D_a(\mathcal{A})$.

Conversely, suppose that $x_0\in D_a(\mathcal{A})$. Then there exists a sequence $[x_n, y_n]\in\mathcal{A}$ such that $x_n\to x_0$ as $n\to\infty$ with $\|y_n\|\leqq|\mathcal{A}x_0|+(1/n)$. Therefore we have by Remark 3.2,

$$\begin{aligned} e^{-2\omega t}\|u(t)-x_n\|^2-\|u(0)-x_n\|^2 &\leqq 2\int_0^t e^{-2\omega\tau}\langle y_n, u(\tau)-x_n\rangle_s d\tau \\ &\leqq 2(|\mathcal{A}x_0|+(1/n))\int_0^t e^{-2\omega\tau}\|u(\tau)-x_n\|d\tau \end{aligned}$$

for $t\in[0, T]$. Letting $n\to\infty$, we have

$$e^{-2\omega t}\|u(t)-x_0\|^2\leqq|\mathcal{A}x_0|\int_0^t e^{-2\omega\tau}\|u(\tau)-x_0\|d\tau$$

for $t\in[0, T]$. Therefore, by a standard argument (see Lemma A.5 in Brezis [**4**]), we have

$$e^{-\omega t}\|u(t)-x_0\|\leqq|\mathcal{A}x_0|\int_0^t e^{-\omega\tau}d\tau \qquad \text{for} \quad t\in[0, T].$$

Combining this inequality with (3.14), we have (3.13). Hence $u(t)$ is Lipschitz continuous on $[0, T]$. (See also Remark 2.4.) Q. E. D.

§ 4. Nonlinear semigroups.

In this section we review the results obtained so far from the view point of the theory of nonlinear semigroups.

Let A be an ω-quasi-dissipative operator in X. We denote by $(\mathrm{CP})_\infty$ the Cauchy problem formulated for A on $[0, \infty)$. We say that an X-valued continuous function on $[0, \infty)$ is a (backward) DS-limit solution of $(\mathrm{CP})_\infty$ if $u(t)$ restricted on any finite interval $[0, T]$ is a DS-limit solution of (CP) on $[0, T]$.

DEFINITION 4.1. Let A be an ω-quasi-dissipative operator in X. Then we say that A has *property* $(\mathcal{D})$ if for any $x\in\overline{D(A)}$ there exists a DS-limit solution of $(\mathrm{CP})_\infty$ with the initial value x.

THEOREM 4.1. *Let A be an ω-quasi-dissipative operator in X. Let D be a subset of $\overline{D(A)}$ such that $\bar{D}=\overline{D(A)}$. If for each $x\in D$ and $T>0$, there exists a DS-approximate solution of $(CP; x)$ on $[0, T]$, then A has property $(\mathcal{D})$.*

PEOOF. Let $x\in\overline{D(A)}$ and $T>0$. Let $\{x_l\}\subset D$ be a sequence such that $x_l\to x$ as $l\to\infty$. By assumption and Theorem 2.1, there exists a DS-limit solution $u^{(l)}(t)$ of $(CP; x_l)$ on $[0, T]$ for each l. Hence by Proposition 2.3, there exists a DS-limit solution of $(CP; x)$ on $[0, T]$.

Let $x\in\overline{D(A)}$ and T_m be a sequence of positive numbers such that $T_m\uparrow\infty$ as $m\to\infty$. Then for each m, there exists a DS-limit solution $u^{(m)}(t)$ of $(CP; x)$ on $[0, T_m]$. Then we have by Proposition 2.1 and 2.2,

$$u^{(m)}(t)=u^{(n)}(t) \qquad \text{for} \quad t\in[0, T_m] \text{ and } m\leqq n\,.$$

Therefore we define a continuous function $u(t)$ on $[0, \infty)$ by

$$u(t)=u^{(m)}(t) \qquad \text{for} \quad t\in[0, T_m] \text{ and } m=1, 2, \cdots.$$

By Proposition 2.2, $u(t)$ is a DS-limit solution of $(\mathrm{CP})_\infty$ with the initial value x.

Q. E. D.

THEOREM 4.2. *Let A be an ω-quasi-dissipative operator in X, having property $(\mathcal{D})$. Then there exists a unique semigroup $\{T(t); t\geqq 0\}$ of type ω on $\overline{D(A)}$ such that for each $x\in\overline{D(A)}$, $u(t)=T(t)x$ is the unique DS-limit solution of* $(\mathrm{CP})_\infty$ *with the initial value x.*

PROOF. For each $x\in\overline{D(A)}$, let $u(t, x)$ be a DS-limit solution of $(\mathrm{CP})_\infty$ with the initial value x. Then, by Proposition 2.1, we can define a family of operators, $T(t)$ for $t>0$, from $\overline{D(A)}$ into itself by setting $T(t)x=u(t, x)$ for $x\in\overline{D(A)}$ and $t>0$. Then we have by Corollary 3.1,

$$\|T(t)x-T(t)y\|\leqq e^{\omega t}\|x-y\| \qquad \text{for} \quad x, y\in\overline{D(A)} \text{ and } t>0.$$

To prove the semigroup property of $\{T(t); t\geqq 0\}$, let $x\in\overline{D(A)}$ and $t, s\geqq 0$. Then by Proposition 2.2, the function $u(t)=T(t+s)x=u(t+s, x)$ is a DS-limit solution of $(\mathrm{CP})_\infty$ with the initial value $u(0)=u(s, x)$. Therefore, by Proposition 2.1, we have $u(t)=u(t, u(s, x))=T(t)T(s)x$. Hence $T(t+s)=T(t)T(s)$. Q. E. D.

By Theorem 3.3, we have immediately the following.

PROPOSITION 4.1. *Let A be an ω-quasi-dissipative operator in X, having property $(\mathcal{D})$ and let $\mathcal{A}$ be a maximal (**) extension of A. Then, for the semigroup $\{T(t); t>0\}$ obtained in Theorem 4.2, we have the following:*

(i) *Put*

$$\hat{D}=\{x\in\overline{D(A)};\ T(t)x \text{ is Lipschitz continuous on any bounded interval of } [0, \infty)\}\,.$$

Then $\hat{D}=D(\mathcal{A})=D_a(\mathcal{A})$ *and hence, each* $T(t)$ *maps* $D(\mathcal{A})$ *into itself.*

(ii) *If* $D^+T(t_0)x\neq\emptyset$ *for* $x\in\overline{D(A)}$ *and* $t_0\geqq 0$, *then* $T(t_0)x\in D(\mathcal{A})$ *and* $D^+T(t_0)x\subset\mathcal{A}^0T(t_0)x$.

§ 5. Existence of DS-approximate solutions.

In this section, we give a sufficient condition for an ω-quasi-dissipative operator to have property $(\mathcal{D})$.

Let A be an ω-quasi-dissipative operator in X, where ω is a real number. We consider the following condition on A:

$$(R_t)\qquad \liminf_{\delta\to+0}\delta^{-1}d(R(I-\delta A),x)=0 \qquad \text{for any}\quad x\in\overline{D(A)},$$

where $d(C,x)=\inf\{\|x-y\|\,;\,y\in C\}$ for $x\in X$ and $C\subset X$.

Then we have

THEOREM 5.1. *Let* A *be an* ω*-quasi-dissipative operator in* X, *satisfying condition* (R_t). *Then* A *has property* $(\mathcal{D})$.

Hence A satisfying condition (R_t) generates a semigroup of type ω on $\overline{D(A)}$, in the sense of Theorem 4.2.

REMARK 5.1. Yorke announces in [**20**] that he has obtained a similar result.

For the proof of Theorem 5.1, we start with

LEMMA 5.1. *Let* A *be an* ω*-quasi-dissipative operator in* X, *satisfying condition* (R_t). *Let* $x_0\in\overline{D(A)}$ *and* $\varepsilon>0$. *Then there exist a sequence* $[x_k,y_k]\in A$, $k=1,2,\cdots$ *and a sequence* $\{t_k\}_{k=1}^{\infty}$ *of positive numbers such that they satisfy the following:*

(i) $0=t_0<t_1<\cdots<t_k<\cdots$ *and* $t_k\to\infty$ *as* $k\to\infty$;

(ii) $t_k-t_{k-1}\leqq\varepsilon$ *for* $k=1,2,\cdots$;

(iii) $\|x_k-x_{k-1}-(t_k-t_{k-1})y_k\|\leqq(t_k-t_{k-1})\varepsilon$ *for* $k=1,2,\cdots$.

PROOF. We set $\omega_0=\max(\omega,0)$. We may assume ε to be so small that $2\omega_0\varepsilon<1$. Then for each $x\in\overline{D(A)}$, we define $\delta(x)$ as the supremum of δ with the following properties:

$$(5.1)\qquad 0\leqq\delta\leqq\varepsilon;$$

and

(5.2) there exists $[x_\delta,y_\delta]\in A$ such that

$$\|x_\delta-x-\delta y_\delta\|\leqq\delta\varepsilon.$$

Then by condition (R_t), $\delta(x)$ is positive for any $x\in\overline{D(A)}$. Therefore we can choose inductively $[x_k,y_k]\in A$ and $h_k>0$ for $k=1,2,\cdots$, so that they satisfy the following:

$$(5.3)\qquad (1/2)\delta(x_{k-1})\leqq h_k\leqq\varepsilon \qquad \text{for}\quad k=1,2,\cdots;$$

(5.4) $$\|x_k-x_{k-1}-h_k y_k\| \leqq \varepsilon h_k \quad \text{for} \quad k=1, 2, \cdots.$$

We define $t_k=\sum_{i=1}^{k} h_i$ for $k=1, 2, \cdots$. In order to complete the proof, we want to show that $t_k\to\infty$ as $k\to\infty$. For this purpose, we establish the following estimate:

(5.5) $$a_{i,j} \leqq \prod_{p=k+1}^{i}(1-\omega_0 h_p)^{-1} \prod_{p=k+1}^{j}(1-\omega_0 h_p)^{-1} \cdot\{(t_i-t_j)|||Ax_k|||+\varepsilon(t_i-t_k)+\varepsilon(t_j-t_k)\}$$

for every $i\geqq j\geqq k\geqq 1$, where we set $a_{i,j}=\|x_i-x_j\|$.

To prove the estimate (5.5), let $k\geqq 1$ be fixed and define

$$\gamma_{i,j}=\prod_{p=k+1}^{i}(1-\omega_0 h_p)\cdot \prod_{p=k+1}^{j}(1-\omega_0 h_p) \quad \text{for} \quad i\geqq j\geqq k.$$

Also we recall the proof of Lemma 2.1. We have by (iv) of Lemma 1.1 and (5.4),

$$\begin{aligned}(1-\omega_0 h_i)\|x_i-x_k\| &\leqq \|x_i-x_k-h_i y_i\|+h_i|||Ax_k||| \\ &\leqq \|x_{i-1}-x_k\|+h_i\varepsilon+h_i|||Ax_k|||\end{aligned}$$

or

$$a_{i,k}\leqq(1-\omega_0 h_i)^{-1}(a_{i-1,k}+h_i\varepsilon+h_i|||Ax_k|||)$$

for $i\geqq k$. Therefore, inductively, we have

$$\gamma_{i,k}a_{i,k}\leqq(t_i-t_k)|||Ax_k|||+\varepsilon(t_i-t_k)$$

for $i\geqq k$. Hence $a_{i,k}$ satisfies (5.5) for $i\geqq k$. Furthermore, $a_{i,j}$ apparently satisfies (5.5) if $i=j$.

Now let us assume that $a_{i-1,j}$ and $a_{i,j-1}$ satisfy (5.5) with $i>j>k$. Then by (ii) of Lemma 1.1 and (5.4), we have

$$\begin{aligned}(h_i+h_j-\omega_0 h_i h_j)a_{i,j} &\leqq h_j\|x_i-x_j-h_i y_i\|+h_i\|x_j-x_i-h_j y_j\| \\ &\leqq h_j a_{i-1,j}+h_i a_{i,j-1}+2\varepsilon h_i h_j.\end{aligned}$$

Hence we have by the assumptions,

(5.6) $$\begin{aligned}&(h_i+h_j-\omega_0 h_i h_j)\gamma_{i,j}a_{i,j} \\ &\quad\leqq[h_j(1-\omega_0 h_i)(t_{i-1}-t_j)+h_i(1-\omega_0 h_j)(t_i-t_{j-1})]\cdot|||Ax_k||| \\ &\qquad+[h_j(1-\omega_0 h_i)(\varepsilon(t_{i-1}-t_k)+\varepsilon(t_j-t_k)) \\ &\qquad+h_i(1-\omega_0 h_j)(\varepsilon(t_i-t_k)+\varepsilon(t_{j-1}-t_k))+2\varepsilon\gamma_{i,j}h_i h_j].\end{aligned}$$

We denote by I_q $(q=1, 2)$ the q-th term of the right side of the inequality

(5.6). Then we have

$$(5.7)\qquad I_1=[(h_j(1-\omega_0 h_i)+h_i(1-\omega_0 h_j))(t_i-t_j)+h_i h_j\omega_0(h_i-h_j)]\cdot|||Ax_k|||$$
$$=[(h_i+h_j-\omega_0 h_i h_j)(t_i-t_j)+h_i h_j\omega_0(h_i-h_j)-h_i h_j\omega_0(t_i-t_j)]\cdot|||Ax_k|||$$
$$\leqq(h_i+h_j-\omega_0 h_i h_j)(t_i-t_j)\cdot|||Ax_k|||\,,$$

where we used the fact that $t_i-t_j=h_i-h_j+(t_{i-1}-t_{j-1})\geqq h_i-h_j$. Since $\gamma_{i,j}\leqq(1-\omega_0 h_i)$, $(1-\omega_0 h_j)$, we have

$$\begin{aligned}I_2&\leqq h_j(1-\omega_0 h_i)(\varepsilon(t_{i-1}-t_k)+\varepsilon(t_j-t_k))\\&\quad+h_i(1-\omega_0 h_j)(\varepsilon(t_i-t_k)+\varepsilon(t_{j-1}-t_k))\\&\quad+h_i h_j(\varepsilon(1-\omega_0 h_i)+\varepsilon(1-\omega_0 h_j))\\&=(h_j(1-\omega_0 h_i)+h_i(1-\omega_0 h_j))(\varepsilon(t_i-t_k)+\varepsilon(t_j-t_k))\,.\end{aligned}$$

Therefore, we have by (2.6) in Lemma 2.2,

$$(5.8)\qquad I_2\leqq(h_i+h_j-\omega_0 h_i h_j)(\varepsilon(t_i-t_k)+\varepsilon(t_j-t_k))\,.$$

Combining (5.6)-(5.8), we see that $a_{i,j}$ satisfies (5.5). Hence by induction, $a_{i,j}$ satisfies (5.5) for all $i\geqq j\geqq k$.

For showing $t_i\to\infty$ as $i\to\infty$, we suppose that $t_i\to s_0<+\infty$ as $i\to\infty$. Then we have by (5.5),

$$\limsup_{i,j\to\infty}\|x_i-x_j\|\leqq 2\exp(4\omega_0(s_0-t_k))\cdot\varepsilon(s_0-t_k)$$

for $k\geqq 1$, where we used the inequality (2.9). Since $t_i\to s_0$ as $i\to\infty$, we see that $\{x_i\}_{i=1}^{\infty}\subset D(A)$ is a Cauchy sequence in X. Hence there exists $u_0\in\overline{D(A)}$ such that $x_i\to u_0$ as $i\to\infty$. Then by condition (R_t), there exist a positive number δ and $[u_\delta, v_\delta]\in A$ such that $\delta\leqq\varepsilon$ and

$$(5.9)\qquad \|u_\delta-u_0-\delta v_\delta\|\leqq\delta\varepsilon/2\,.$$

Since h_i (and hence $\delta(x_i)$) converges to 0 as $i\to\infty$, there is an i_0 such that $\delta(x_i)<\delta$ for all $i\geqq i_0$. Then by definition of $\delta(x_i)$, we have

$$\|u_\delta-x_i-\delta v_\delta\|>\varepsilon\delta\qquad\text{for}\quad i\geqq i_0\,.$$

Letting $i\to\infty$, we have

$$\|u_\delta-u_0-\delta v_\delta\|\geqq\varepsilon\delta\,,$$

which is contrary to (5.9). Hence $t_i\to\infty$ as $i\to\infty$. Q. E. D.

PROOF OF THEOREM 5.1. Let $x_0\in\overline{D(A)}$ and $T>0$. Let $\varepsilon_n\downarrow 0$ as $n\to\infty$. Then by Lemma 5.1, there exist $[x_k^n, y_k^n]\in A$ and $t_k^n>0$ for $k=1, 2, \cdots, N_n$

such that

(i) $0=t_0^n<t_1^n<\cdots<t_{N_n-1}^n<T\leqq t_{N_n}^n,\ n\geqq 1$;

(ii) $\max_{1\leqq i\leqq N_n}(t_i^n-t_{i-1}^n)\leqq\varepsilon_n,\ n\geqq 1$;

(iii) $\|x_i^n-x_{i-1}^n-(t_i^n-t_{i-1}^n)y_i^n\|\leqq\varepsilon_n(t_i^n-t_{i-1}^n),\ i=1, 2, \cdots, N_n,\ n\geqq 1$,

where $x_0^n=x_0$. We define $u_n(t)$ by

$$u_n(t)=\begin{cases} x_0 & \text{for } t=0, \\ x_i^n & \text{for } t\in(t_{i-1}^n, t_i^n]\cap(0, T],\ i=1, 2, \cdots, N_n. \end{cases}$$

Then $u_n(t)$ is a DS-approximate solution of $(CP; x_0)$ on $[0, T]$. Hence, by Theorem 4.1, A has property $(\mathscr{D})$. Q. E. D.

REMARK 5.2. Let A be an ω-quasi-dissipative operator satisfying the following condition (see Crandall and Liggett [**7**]):

(R_0) $$R(I-\lambda A)\supset\overline{D(A)} \quad \text{for} \quad 0<\lambda\leqq\lambda_0.$$

Then apparently A satisfies condition (R_t).

REMARK 5.3. Let A be an ω-quasi-dissipative operator. Suppose that $D(A)$ is closed and A is continuous on $D(A)$. Then condition (R_t) is equivalent to the following condition:

(C) $$\liminf_{\delta\to+0}\delta^{-1}d(D(A), x+\delta Ax)=0 \quad \text{for any} \quad x\in D(A).$$

See Martin [**12**], where A is assumed to be *strictly* ω-dissipative. See also Takahashi [**17**].

REMARK 5.4. The following example due to Martin [**12**] shows that condition (R_t) does not imply condition (R_0) in general. Let $X=R^2$ with the Euclidean norm. Let $D(A)=\{(x, y); x^2+y^2=1\}$ and let $A(x, y)=(y, -x)$ for each $(x, y)\in D(A)$. Then A is continuous and dissipative and satisfies condition (R_t) or (C). But $R(I-\lambda A)$ does not intersect $D(A)$ for any $\lambda>0$.

As an application of Theorem 5.1, we give the following.

THEOREM 5.2. *Let A be dissipative operator in X. Then the following* (i) *and* (ii) *are equivalent:*

(i) *A is closed and $A-z$ satisfies condition (R_t) for any $z\in X$;*

(ii) *A is m-dissipative.*

PROOF. Apparently (ii) implies (i). Suppose that A satisfies (i). We first show that A is maximal dissipative on $\overline{D(A)}$. For this purpose, let $x\in\overline{D(A)}$ and $z\in X$ satisfy that $A\cup[x, z]$ is dissipative in X. Then by the assumption, there exist $[x_n, y_n]\in A$ and $\delta_n\downarrow 0$ such that $\lim_{n\to\infty}\delta_n^{-1}\|x_n-x-\delta_n(y_n-z)\|=0$. Since $A\cup[x, z]$ is dissipative, we have

$$\begin{aligned}\delta_n^2\|y_n-z\|^2&\leqq\delta_n^2\|y_n-z\|^2+2\delta_n\langle y_n-z, x-x_n\rangle_s\\ &\leqq\|\delta_n(y_n-z)-(x_n-x)\|^2\end{aligned}$$

or

$$\|y_n-z\| \leqq \delta_n^{-1}\|x_n-x-\delta_n(y_n-z)\| .$$

Therefore $y_n \to z$ as $n\to\infty$ and hence $x_n \to x$ as $n\to\infty$. Since A is closed, we have $[x, z]\in A$. Hence A is maximal dissipative on $\overline{D(A)}$.

Let $z\in X$ and $B=A-I+z$. Then B satisfies condition (R_t). In fact, let $x\in\overline{D(B)}=\overline{D(A)}$. Then by the assumption, there exist $[x_n, y_n]\in A$ and $\delta_n\downarrow 0$ such that $\lim_{n\to\infty}\delta_n^{-1}\|x_n-x-\delta_n(y_n-x+z)\|=0$. Take $h_n=\delta_n/(1-\delta_n)$ for sufficiently large n. Then we have

$$\begin{aligned} h_n^{-1}\|x_n-x-h_n(y_n-x_n+z)\| &= h_n^{-1}\|(1+h_n)(x_n-x)-h_n(y_n-x+z)\| \\ &= \delta_n^{-1}\|x_n-x-\delta_n(y_n-x+z)\| . \end{aligned}$$

Therefore we see that B satisfies condition (R_t). Hence, by Theorem 5.1, B generates a semigroup $\{T(t)\,;\,t\geqq 0\}$ on $\overline{D(A)}$ such that $\|T(t)x-T(t)y\|\leqq e^{-t}\|x-y\|$ for $x, y\in\overline{D(A)}$ and

(5.10) $$\begin{aligned} &\|T(t)x-u\|^2-\|x-u\|^2 \\ &\leqq -2\int_0^t\|T(\tau)x-u\|^2 d\tau+2\int_0^t\langle v-u+z, T(\tau)x-u\rangle_s d\tau \end{aligned}$$

for $x\in\overline{D(A)}$, $t>0$ and $[u, v]\in A$, since B is (-1)-dissipative. Since each $T(t)$ is a strict contraction on $\overline{D(A)}$, there is a unique $x_t\in\overline{D(A)}$ such that $T(t)x_t=x_t$. Since $T(s)x_t=T(s)T(t)x_t=T(t)T(s)x_t$ for any $s>0$, we see that $x_t=x_0$ for all $t>0$. That is, there exists a unique $x_0\in\overline{D(A)}$ such that $T(t)x_0=x_0$ for all $t>0$. Let $x=x_0$ in (5.10). Then we have

$$\begin{aligned} 0 &\leqq -\|x_0-u\|^2+\langle v-u+z, x_0-u\rangle_s \\ &\leqq \langle v-x_0+z, x_0-u\rangle_s \end{aligned}$$

for any $[u, v]\in A$. Since A is maximal dissipative on $\overline{D(A)}$, we have $x_0\in D(A)$ and $x_0-z\in Ax_0$. That is $z\in R(I-A)$. Since $z\in X$ is arbitrary, we have $R(I-A)=X$. Q. E. D.

THEOREM 3.3. *Let A be an m-dissipative operator in X. Let B be a continuous operator in X such that $\overline{D(A)}\subset D(B)$ and $A+B$ is dissipative. Then $A+B$ is m-dissipative.*

PROOF. We define the operator J_δ on $\overline{D(A)}$ for $\delta>0$ by setting

$$J_\delta x=(I-\delta A)^{-1}(x+\delta Bx) \qquad \text{for} \quad x\in\overline{D(A)} .$$

Then we see that $J_\delta x\to x$ as $\delta\to +0$ for each $x\in\overline{D(A)}$. In fact, we have

$$\|J_\delta x-x\| \leqq \|(I-\delta A)^{-1}(x+\delta Bx)-(I-\delta A)^{-1}x\|+\|(I-\delta A)^{-1}x-x\|$$

$$\leqq \delta\|Bx\|+\|(I-\delta A)^{-1}x-x\| \qquad \text{for} \quad x\in \overline{D(A)}\,.$$

Let $x\in \overline{D(A)}$ and $x_\delta=J_\delta x$. Then by definition of $J_\delta x$, there exists $y_\delta\in Ax_\delta$ such that $x_\delta-\delta y_\delta=x+\delta Bx$. For such $[x_\delta, y_\delta]$, we have

$$\lim_{\delta\to+0}\delta^{-1}\|x_\delta-x-\delta(y_\delta+Bx_\delta)\|=\lim_{\delta\to+0}\|Bx_\delta-Bx\|=0\,,$$

since B is continuous on $\overline{D(A)}$. Therefore, $A+B$ satisfies condition (R_t). Similarly we see that $A+B+z$ satisfies condition (R_t) for any $z\in X$. Furthermore, $A+B$ is apparently closed in X. Hence, by Theorem 5.2, we see that $A+B$ is m-dissipative. Q. E. D.

REMARK 5.5. When B is a continuous dissipative operator defined on X, B is strictly dissipative and hence $A+B$ is dissipative. Therefore, Theorem 3.3 is an extension of the results of Barbu [**1**] and Webb [**19**].

REMARK 5.6. Recently a similar result has been obtained by Pierre [**14**] by a quite different method.

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