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On the boundedness of integral transformations with rapidly oscillatory kernels

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The aim of this note is to prove the L^2 boundedness of some integral transformations with rapidly oscillatory kernels. We generalize a part of results in the previous paper [3] of this second author to the case of operators with multiple symbols. Techniques are the same as used in [3]. If our phase functions are homogeneous of degree one, our operators coincide with a special class of Fourier integral operators with amplitude functions of class $S_{0,0}^0$ in the notation of Hörmander [5]. Our result seems new even in this case. (See also Eskin [2].)

As an application, we shall elucidate the role of the canonical mapping associated with the phase function appeared in our previous paper [3].

We shall discuss only in the case that the dimension of the space is larger than one. But minor changes of discussions will prove our results in the one dimensional case.

§1. Assumptions.

In the present paper we assume the following assumptions.

(A-0) $S_j(x, \xi)$, $j=1, 2$, are real infinitely differentiable functions of (x, ξ) in $R^n \times R^n$, $n \geqq 2$.

(A-I) There exists a positive constant C such that we have

$$\left|\det\left(\frac{\partial^2 S_j(x, \xi)}{\partial x_k \partial \xi_l}\right)\right| \geqq C$$

for any (x, ξ) in $R^n \times R^n$.

(A-II) For any multi-indices α, β, $|\alpha|+|\beta| \geqq 2$, there exists a constant $C>0$ such that $|\partial_x^\alpha \partial_\xi^\beta S_j(x, \xi)| \leqq C$.

(A-III) The function $a(x, \xi, y, \eta)$ is an infinitely differentiable function of (x, ξ, y, η) in $R^n \times R^n \times R^n \times R^n$ which together with its derivatives of all orders are uniformly bounded.

§ 2. Main theorem.

The following two linear mappings A and B are defined at least for arbitrary infinitely differentiable function $f(x)$ with compact support.

$$Af(x)=\int a(x, \xi, y, \eta)e^{i\nu(S_1(y,\eta)-S_2(x,\xi)-\xi\cdot y)}f(\eta)d\eta dyd\xi\,, \tag{1}$$

$$Bf(x)=\int a(x, \xi, y, \eta)e^{i\nu(S_1(y,\xi)-S_2(y,\eta)+\xi\cdot x)}f(\eta)d\eta dyd\xi\,. \tag{2}$$

Here ν is a real parameter greater than 1. These operators are Fourier integral operators if phase functions are homogeneous of degree one in the variables (η, y, ξ) (cf. [2], [5]). But we do not assume this homogeneity of phase functions. Our main result is

THEOREM 1. *If assumptions* (A-0), (A-I), (A-II) *and* (A-III) *hold, then there exists a positive constant C such that we have*

$$\|Af\|\leqq C\nu^{-3n/2}\|f\| \tag{3}$$

and

$$\|Bf\|\leqq C\nu^{-3n/2}\|f\|\,, \tag{4}$$

for any $f\in C_0^{\infty}(R^n)$ and $\nu\geqq 1$.

Here and hereafter we shall denote by C positive constants which will take various different values in various occasions but will be independent of ν and f.

REMARK. We can so choose the constant C in the theorem as $C/\|a\|_{C^{3n+1}}$ is independent of a. Here $\|a\|_{C^l}$ is the supremum of absolute values of all derivatives of a of order not greater than l.

§ 3. Proof of the main theorem.

First we shall look for a global behaviour of the functions $S_j(x, \xi)$, $j=1, 2$.

PROPOSITION 1. *Under the assumptions* (A-0) *and* (A-II), *the condition* (A-I) *is equivalent to the following condition* (A-I$'$);

(A-I$'$) *There exists a positive constant C such that*

$$|\mathrm{grad}_x(S_j(x, \xi)-S_j(x, \eta))|\geqq C|\xi-\eta|\,,$$

$$|\mathrm{grad}_\xi(S_j(x, \xi)-S_j(y, \xi))|\geqq C|x-y|\,, \qquad j=1, 2\,,$$

for any x, y in R^n and ξ, η in R^n.

PROOF. Since it is obvious that (A-I$'$) implies (A-I), we have only to prove that (A-I$'$) follows from (A-I). We shall introduce new variables $\zeta_k=\partial_{x_k}S_1(x, \xi)$.

The Jacobian matrix of the correspondence $\xi \to \zeta$ is $J=(\partial_{\xi_k}\partial_{x_l}S_1(x, \xi))_{k,l}$. The condition (A-I) asserts that $|\det J| \geqq C>0$. Since R^n is connected and simply connected, the global implicit function theorem can be applied to our case and the correspondence $\xi \to \zeta$ is a global diffeomorphism of R^n. See, for example [6]. Thus ξ becomes a function of x and ζ, which we shall denote by $\xi(x, \zeta)$. The Jacobian matrix $(\partial_{\zeta_k}\xi_l(x, \zeta))_{k,l}$ is the inverse of $(\partial_{\xi_k}\partial_{x_l}S_1(x, \xi))_{k,l}$. This implies that the (k, l) element of the matrix $(\partial_{\zeta_k}\xi_l(x, \zeta))_{k,l}$ is uniformly bounded. Setting $\zeta'_k=\partial_{x_k}S_1(x, \eta)$, we have

$$
\begin{aligned}
|\xi_l-\eta_l| &= |\xi_l(x, \zeta)-\xi_l(x, \zeta')| \\
&= |\sum_k(\zeta-\zeta')_k\int_0^1\partial_{\zeta_k}\xi_l(x, t\zeta+(1-t)\zeta')dt| \\
&\leqq C|\zeta-\zeta'| .
\end{aligned}
$$

This turns out to be $|\mathrm{grad}_x(S_1(x, \xi)-S_1(x, \eta))| \geqq C^{-1}|\xi-\eta|$. Similar discussions prove that $|\mathrm{grad}_\xi(S_1(x, \xi)-S_1(y, \xi))| \geqq C|x-y|$. The same arguments are valid for $S_2(x, \xi)$.

Now we shall prove our Theorem 1. Let $\Lambda=\{0=g_0, g_1, g_2, \cdots\}$ be the set of unit lattice points in R^n and $\{\varphi_j(x)\}_{j=0}^\infty$ be a partition of unity subordinate to the covering of open cubes of side 2 centered at each of these lattice points. We may assume that $\varphi_j(x)=\varphi(x-g_j)$, $\varphi(x) \geqq 0$.

We shall first prove the inequality (3). We shall decompose the operator A into infinite sum of operators which are almost orthogonal to each other. (See Calderòn and Vaillancourt [1].) Let $p=(s_1, \sigma_1, s_2, \sigma_2) \in \Lambda^4$ and

$$A=\sum_{p\in\Lambda^4} A_p , \tag{5}$$

where

$$A_p f(x)=\int a_p(x, \xi, y, \eta)e^{i\nu(S_1(y,\eta)-S_2(x,\xi)-\xi\cdot y)}f(\eta)d\eta dy d\xi \tag{6}$$

and

$$a_p(x, \xi, y, \eta)=a(x, \xi, y, \eta)\varphi(x-s_1)\varphi(\xi-\sigma_1)\varphi(y-s_2)\varphi(\eta-\sigma_2) \quad \text{for } p=(s_1, \sigma_1, s_2, \sigma_2) . \tag{7}$$

The adjoint $A_{p'}^*$ of $A_{p'}$, $p'=(s'_1, \sigma'_1, s'_2, \sigma'_2)$, is

$$A_{p'}^* g(\eta')=\int \overline{a_{p'}(x', \xi', y', \eta')}e^{-i\nu(S_1(y',\eta')-S_2(x',\xi')-\xi' y')}g(x')dx'd\xi'dy' . \tag{8}$$

From these follows that

$$A_p A_{p'}^* g(x)=\int k_{pp'}(x, x')g(x')dx' , \tag{9}$$

where

$$k_{pp'}(x, x') = \int a_p(x, \xi, y, \eta)\overline{a_{p'}(x', \xi', y', \eta)}e^{i\nu\phi(x,\xi,y,\eta,y',\xi',x')}d\xi' dy' d\eta dy d\xi\,, \tag{10}$$

and

$$\phi(x, \xi, y, \eta, y', \xi', x') = S_1(y, \eta) - S_2(x, \xi) - \xi\cdot y - S_1(y', \eta) + S_2(x', \xi') + \xi' y'\,. \tag{11}$$

Function $k_{pp'}(x, x')$ is continuous in $(x, x')\in R^n\times R^n$ and vanishes if $|\sigma_2-\sigma_2'| \geqq 2\sqrt{n}$. We shall denote by grad ϕ the vector of $5n$ components:

$$\partial_\xi\phi = -\partial_\xi S_2(x, \xi) - y\,, \qquad \partial_{\xi'}\phi = \partial_{\xi'}S_2(x', \xi') + y'\,,$$

$$\partial_y\phi = \partial_y S_1(y, \eta) - \xi\,, \qquad \partial_{y'}\phi = -\partial_{y'}S_1(y', \eta) + \xi'\,,$$

$$\partial_\eta\phi = \partial_\eta S_1(y, \eta) - \partial_\eta S_1(y', \eta)\,.$$

If all of these vanish then $y=y'$, $\xi=\xi'$ and $x=x'$. This point is contained in the support of the integrand of (10) only if $|s_1-s_1'|\leqq 2\sqrt{n}$, $|\sigma_1-\sigma_1'|\leqq 2\sqrt{n}$ and $|s_2-s_2'|\leqq 2\sqrt{n}$. A differential operator of the first order

$$L_1 = |\mathrm{grad}\,\phi|^{-2}(\mathrm{grad}\,\phi\cdot\nabla)$$

$$= |\mathrm{grad}\,\phi|^{-2}(\partial_\xi\phi\cdot\partial_\xi + \partial_y\phi\cdot\partial_y + \partial_\eta\phi\cdot\partial_\eta + \partial_{y'}\phi\cdot\partial_{y'} + \partial_{\xi'}\phi\cdot\partial_{\xi'})$$

is defined for $x\neq x'$. Since $(L_1 - i\nu)e^{i\nu\phi}=0$, the expression

$$k_{pp'}(x, x') = (i\nu)^{-l}\int a_p(x, \xi, y, \eta)\overline{a_{p'}(\eta, y', \xi', x')}L_1^l(e^{i\nu\phi})d\xi' dy' d\eta dy d\xi$$

is valid for $x\neq x'$ and $l=0, 1, 2, \cdots$. An integration by part give

$$k_{pp'}(x, x') = (i\nu)^{-l}\int L_1^{*l}(a_p(x, \xi, y, \eta)\overline{a_{p'}(x', \xi', y', \eta)})e^{i\nu\phi}d\xi' dy' d\eta dy d\xi \tag{12}$$

for $x\neq x'$ and $l=0, 1, 2, 3, \cdots$. Here L_1^* is the formal adjoint of L_1, i.e.,

$$L_1^* = -L_1 - \mathrm{div}\,(|\mathrm{grad}\,\phi|^{-2}\,\mathrm{grad}\,\phi)\,. \tag{13}$$

Now we need the following lemma.

LEMMA 1. *For any positive integer l there exists a constant $C_l>0$ such that*

$$L_1^{*l}(a_p(x, \xi, y, \eta)\overline{a_{p'}(x', \xi', y', \eta)}) \leqq C_l|\mathrm{grad}\,\phi|^{-l}\,. \tag{14}$$

PROOF. Since $L_1^* = -|\mathrm{grad}\,\phi|^{-2}(\mathrm{grad}\,\phi\cdot\nabla) + \mathrm{div}\,(|\mathrm{grad}\,\phi|^{-2}\,\mathrm{grad}\,\phi)$, $L_1^{*l}(a_p(x, \xi, y, \eta)\overline{a_{p'}(x', \xi', y', \eta)})$ is a sum of terms of the form $(|\mathrm{grad}\,\phi|^{-k})$ $\times$(products of derivatives of ϕ)(products of derivatives of $a_p a_{p'}$) which is homogeneous of degree $-l$ in ϕ. This proves lemma.

In obtaining an upper bound of $k_{pp'}(x, x')$, we distinguish two cases.

Case 1.

$$|s_1-s_1'|\leqq 3\sqrt{n}, \qquad |s_2-s_2'|\leqq 3\sqrt{n} \qquad \text{and}$$
$$|\sigma_1-\sigma_1'|\leqq 3\sqrt{n}.$$

In this case $|x-x'|$, $|\xi-\xi'|$ and $|y-y'|\leqq 6\sqrt{n}$ in the support of the integrand of (12). We can show that there exists a constant $C>0$ such that

$$|\text{grad}\,\phi|\geqq C(|y-y'|+|\xi-\xi'|+|x-x'|). \tag{15}$$

In fact, (A-I′) implies that

$$|\text{grad}\,\phi|\geqq|\partial_\eta S_1(y,\eta)-\partial_\eta S_1(y',\eta)|\geqq C|y-y'|. \tag{16}$$

$$\begin{aligned}|\text{grad}\,\phi|&\geqq(|\partial_y S_1(y,\eta)-\xi|^2+|-\partial_{y'}S_1(y',\eta)+\xi'|^2)^{1/2}\\&\geqq\frac{1}{2}(|\partial_y S_1(y,\eta)-\partial_{y'}S_1(y',\eta)+\xi'-\xi|)\\&\geqq\frac{1}{2}(|\xi-\xi'|-|\partial_y S_1(y,\eta)-\partial_{y'}S_1(y',\eta)|)\\&\geqq\frac{1}{2}(|\xi-\xi'|-C|y-y'|).\end{aligned} \tag{17}$$

The last inequality is a consequence of (A-II). Similarly,

$$\begin{aligned}|\text{grad}\,\phi|&\geqq\frac{1}{2}(|-\partial_\xi S_2(x,\xi)-y|+|\partial_{\xi'}S_2(x',\xi')+y'|)\\&\geqq\frac{1}{2}(|-\partial_\xi S_2(x,\xi)+\partial_\xi S_2(x',\xi)|\\&\qquad-|\partial_\xi S_2(x',\xi)-\partial_{\xi'}S_2(x',\xi')|-|y-y'|).\end{aligned}$$

Hence, by virtue of (A-I),

$$|\text{grad}\,\phi|\geqq\frac{1}{2}|x-x'|-C|\xi-\xi'|-C|y-y'|. \tag{18}$$

These three inequalities (16), (17) and (18) yield inequality (15).

Combining (12), (14) and (15), we have

$$\begin{aligned}|k_{pp'}(x,x')|&\leqq C\nu^{-l}\varphi(x-s_1)\varphi(x'-s_1')\\&\quad\times\int_{R^{5n}}(|y-y'|+|\xi-\xi'|+|x-x'|)^{-l}\chi(y-s_2)\chi(y'-s_2')\\&\qquad\times\chi(\xi-\sigma_1)\chi(\xi'-\sigma_1')\chi(\eta-\sigma_2)\chi(\eta-\sigma_2')dyd\xi d\eta dy'd\xi'\end{aligned}$$

for $l=0, 1, 2, 3, \cdots$, where χ is the characteristic function of the ball of radius $3\sqrt{n}$ centered at the origin. We choose $l>2n$, then

$$(19)\qquad |k_{pp'}(x, x')| \leqq C\nu^{-l}\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)\varphi(x-s_1)\varphi(x'-s_1')\int_{R^{2n}}\chi(\xi-\sigma_1)\chi(y-s_2)dyd\xi$$

$$\times\int_{R^{2n}}(|y-y'|+|\xi-\xi'|+|x-x'|)^{-l}d\xi'dy'$$

$$\leqq C\nu^{-l}\chi\Big(\frac{\sigma_2-\sigma_2'}{2}\Big)\varphi(x-s_1)\varphi(x'-s_2')|x-x'|^{2n-l}.$$

Case 2. At least one of $|s_1-s_1'|$, $|s_2-s_2'|$, $|\sigma_1-\sigma_1'|$ is greater than $3\sqrt{n}$. We can find a constant C such that $|\text{grad}\,\phi| \geqq C|\text{grad}\,\phi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|$ on the support of $a_p\bar{a}_{p'}$, where $\sigma_2^* = \frac{1}{2}(\sigma_2+\sigma_2')$. Hence for any integer $l \geqq 0$,

$$(20)\qquad |k_{pp'}(x, x')| \leqq C\nu^{-l}|\text{grad}\,\phi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-l}$$

$$\times\varphi(x-s_1)\varphi(x'-s_1')\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big).$$

We shall denote by $\chi^{\#}$ the characteristic function for the set

$$|s_1'-s_1| \leqq 3\sqrt{n}, \qquad |s_2-s_2'| \leqq 3\sqrt{n}, \qquad |\sigma_1-\sigma_1'| \leqq 3\sqrt{n}.$$

Using this notation, we can unite estimates (19) and (20) into

$$(21)\qquad |k_{pp'}(x, x')| \leqq C_{ll'}\varphi(x-s_1)\varphi(x'-s_1')\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)(\nu^{-l}|x-x'|^{2n-l}\chi^{\#}$$

$$+(1-\chi^{\#})\nu^{-l'}|\text{grad}\,\phi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-l'})$$

for $x \neq x'$, $l > 2n$ and $l' = 0, 1, 2, 3, \cdots$.

We shall look for an upper bound of $\int|k_{pp'}(x, x')|dx'$. We divide this integral into two parts: $I_1 = \int_{|x-x'|<\rho}|k_{pp'}(x, x')|dx'$, $I_2 = \int_{|x-x'|>\rho}|k_{pp'}(x, x')|dx'$, where ρ is any number $1 > \rho > 0$. Making use of (21) for $l \geqq 2n+1$, $l' \geqq 3n+1$, we have

$$|I_1| \leqq C\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)\{\nu^{-(2n+1)}\rho^{n-1}\chi^{\#}$$

$$+\nu^{-l'}\rho^n|\text{grad}\,\phi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-l'}(1-\chi^{\#})\}.$$

Next we use (21) for $l \geqq 3n+1$, $l' \geqq 3n+1$, and we obtain

$$|I_2| \leqq C\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)(\nu^{-3n-1}\rho^{-1}\chi^{\#}$$

$$+\nu^{-l'}|\text{grad}\,\phi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-l'}(1-\chi^{\#})).$$

Consequently we obtain

$$(22)\qquad \int|k_{pp'}(x, x')|dx' \leqq C\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)\nu^{-3n}$$

$$\times(\chi^{\#}+(1-\chi^{\#})|\text{grad}\,\phi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-l'}),$$

if we choose $\rho=\nu^{-1}$. We have the same upper bound for integral $\int|k_{pp'}(x, x')|dx$. This implies that

(23) $$\|A_pA_{p'}^*\|\leqq C\nu^{-3n}h_1(p, p')^2,$$

where

(24) $$h_1(p, p')=\Big[\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)\times(\chi^\#+(1-\chi^\#)|\mathrm{grad}\,\phi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-l'})\Big]^{1/2}.$$

As a consequence of symmetry of our assumptions (A-0), (A-I), (A-II) and (A-III), we obtain the same upper bound for $\|A_p^*A_{p'}\|$. Calderòn and Vaillancourt lemma (see [1]) proves inequality (3) if we can prove that the kernel $h_1(p, p')$ defines a continuous linear mapping in $l^2(\Lambda^4)$. In order to prove this, we shall look for an upper bound of $\sum_{p'}h_1(p, p')$.

(25) $$\sum_{p'}h_1(p, p')\leqq\sum_{p'}\chi^\#\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)+\sum_{p'}(1-\chi^\#)\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)|\mathrm{grad}\,\phi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-l'/2}.$$

This first sum of the right side is smaller than a constant. The second is

(26) $$\begin{aligned}&\sum_{p'}(1-\chi^\#)\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)|\mathrm{grad}\,\phi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-(1/2)l'}\\ &\leqq C\int_{R^{4n}}(1-\chi^\#)\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)\\ &\quad\times(|\mathrm{grad}\,\phi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|+1)^{-(1/2)l'}ds_1'd\sigma_1'ds_2'd\sigma_2'\\ &\leqq C\int_{R^{4n}}(1-\chi^\#)\chi\Big(\frac{1}{2}(\sigma_2-\sigma_2')\Big)\{1+|\partial_{\sigma_1}S_2(s_1, \sigma_1)+s_2|+|\partial_{s_2}S_1(s_2, \sigma_2)-\sigma_1|\\ &\quad+|\sigma_1'-\partial_{s_2'}S_1(s_2', \sigma_2)|+|\partial_{\sigma_2}S_1(s_2, \sigma_2)-\partial_{\sigma_2}S_1(s_2', \sigma_2)|\\ &\quad+|\partial_{\sigma_1'}S_2(s_1', \sigma_1')+s_2'|\}^{-l'/2}ds_1'd\sigma_1'ds_2'd\sigma_2'.\end{aligned}$$

We introduce new variables $t=\partial_{\sigma_1}S_2(s_1, \sigma_1)$, $t'=\partial_{\sigma_1'}S_2(s_1', \sigma_1')$. Correspondences $s_1\leftrightarrow t$, $s_1'\leftrightarrow t'$ are diffeomorphisms because of (A-I′). We have $\Big|\det\Big(\frac{\partial t}{\partial s_1}\Big)\Big|\geqq C$ and $\Big|\det\Big(\frac{\partial t'}{\partial s_1}\Big)\Big|\geqq C$, with some constant $C>0$. We choose $\Big(2n-\frac{1}{2}l'\Big)<-n$, then the right side is estimated as

$$C\int(1+|t'+s_2'|+|s_2-s_2'|+|\sigma_1'-\partial_{s_2'}S_1(s_2', \sigma_2)|)^{-(1/2)l'}dt'ds_2'd\sigma_1'$$

(27) $$\leqq C\int(1+|s_2-s_2'|^{(2n-(1/2)l')}ds_2'$$

$$\leqq C.$$

As a consequence we obtained $\sum_{p'} h_1(p, p') \leqq C$ independent of p. Similarly, $\sum_{p} h_1(p, p') \leqq C$ holds. These two imply that the kernel $h_1(p, p')$ defines a bounded linear mapping in $l^2(\Lambda^4)$. Our inequality (3) has been proved.

Inequality (4) is proved similarly. We shall present the proof briefly. The operator B turns out to be

$$B = \sum_{p \in \Lambda^4} B_p \tag{28}$$

and

$$B_p f(x) = \int a_p(x, \xi, y, \eta) e^{i\nu(S_1(y,\xi) - S_2(y,\eta) + \xi \cdot x)} f(\eta) d\eta dy d\xi .$$

The adjoint of $B_{p'}$ is

$$B_{p'}^* g(\eta) = \int \overline{a_{p'}(x', \xi', y', \eta)} e^{-i\nu(S_1(y',\xi') - S_2(y',\eta) + \xi' \cdot x')} g(x') dx' d\xi' dy' .$$

Putting $B_p B_{p'}^* g(x) = \int b_{pp'}(x, x') g(x') dx'$, we have

$$b_{pp'}(x, x') = \int a_p(x, \xi, y, \eta) \overline{a_{p'}(x', \xi', y', \eta)} e^{i\nu\phi(x,\xi,y,\eta,y',\xi',x')} d\xi' dy' d\eta dy d\xi , \tag{29}$$

where

$$\begin{aligned} \phi &= \phi(x, \xi, y, \eta, y', \xi', x') \\ &= S_1(y, \xi) - S_2(y, \eta) + \xi \cdot x - S_1(y', \xi') + S_2(y', \eta) - \xi' x' . \end{aligned}$$

Function $b_{pp'}(x, x')$ is continuous and $b_{pp'}(x, x') \neq 0$ only if $|\sigma_2 - \sigma_2'| \leqq 2\sqrt{n}$. The gradient of the function ϕ is

$$\begin{aligned} &\partial_\xi \phi = \partial_\xi S_1(y, \xi) + x , \qquad \partial_{\xi'} \phi = -\partial_{\xi'} S_1(y', \xi') - x' , \\ &\partial_y \phi = \partial_y S_1(y, \xi) - \partial_y S_2(y, \eta) , \\ &\partial_{y'} \phi = -\partial_{y'} S_1(y', \xi') + \partial_{y'} S_2(y', \eta) , \\ &\partial_\eta \phi = -\partial_\eta S_2(y, \eta) + \partial_\eta S_2(y', \eta) . \end{aligned}$$

If all these vanish, then $x = x'$, $\xi = \xi'$, $y = y'$. We introduce differential operator $M_1 = (|\text{grad}\, \phi|^{-2})(\text{grad}\, \phi \cdot \nabla)$. The function $b_{pp'}(x, x')$ turns out to be

$$b_{pp'}(x, x') = (i\nu)^{-l} \int M_1^{*l} (a_p(x, \xi, y, \eta) \overline{a_{p'}(x', \xi', y', \eta)}) e^{i\nu\phi} d\xi' dy' d\eta dy d\xi \tag{30}$$

for $x \neq x'$ and $l = 0, 1, 2, 3, \cdots$. Lemma 1 where L is replaced by M_1 still holds. In case $|s_1 - s_1'| \leqq 3\sqrt{n}$, $|\sigma_1 - \sigma_1'| \leqq 3\sqrt{n}$, $|s_2 - s_2'| \leqq 3\sqrt{n}$, an inequality $|\text{grad}\, \phi| \geqq C(|y - y'| + |\xi - \xi'| + |x - x'|)$ holds. Consequently we have an estimate

$$|b_{pp'}(x, x')| \leqq C\nu^{-l} |x - x'|^{2n-l} \varphi(x - s_1) \varphi(x' - s_1') \chi(\sigma_2 - \sigma_2') . \tag{31}$$

In the other case, σ_2^* being $\frac{1}{2}(\sigma_2 + \sigma_2')$,

$$|\text{grad}\,\psi(x, \xi, y, \eta, y', \xi', x')| \geqq C|\text{grad}\,\psi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|$$

in the support of the integrand of (30). Therefore we obtain

$$(32) \qquad |b_{pp'}(x, x')| \leqq C\chi(\sigma_2-\sigma_2')\varphi(x-s_1)\varphi(x'-s_1')$$
$$\times(\chi^\# |x-x'|^{2n-l}\nu^{-l}+(1-\chi^\#)\nu^{-l'}|\text{grad}\,\psi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-l'})$$

for $l=2n+1, 2n+2, \cdots$ and $l'=0, 1, 2, 3, \cdots$. Consequently, just as in the previous discussions, we have

$$(33) \qquad \|B_pB_{p'}^*\| \leqq C\nu^{-3n}h_2(p, p')^2\,,$$
$$h_2(p, p')=\chi(\sigma_2-\sigma_2')(\chi^\# +(1-\chi^\#)|\text{grad}\,\psi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-l'})^{1/2}\,.$$

Just as before we obtain $\sum_{p'}\chi^\#\chi(\sigma_2-\sigma_2')\leqq C$ and

$$(34) \qquad \sum_{p'}(1-\chi^\#)\chi(\sigma_2-\sigma_2')|\text{grad}\,\psi(s_1, \sigma_1, s_2, \sigma_2^*, s_2', \sigma_1', s_1')|^{-l'}$$
$$\leqq C\int \chi(\sigma_2-\sigma_2')(1+|\partial_{\sigma_1}S_1(\sigma_2, \sigma_1)+s_1|+|\partial_{\sigma'}S_1(s_2', \sigma_1')+s_1'|$$
$$+|\partial_{\sigma_2}S_2(s_2, \sigma_2^*)-\partial_{\sigma_2}S_2(s_2', \sigma_2^*)|+|\partial_{s_2}S_1(s_2, \sigma_1)-\partial_{s_2}S_2(s_2, \sigma_2)|$$
$$+|-\partial_{s_2'}S_1(s_1', \sigma_1')+\partial_{s_2'}S_2(s_2', \sigma_2^*)|)^{-l'}ds_1'd\sigma_1'ds_2'd\sigma_2'\,.$$

The change of variables $t'=-\partial_{s_2'}S_1(s_2', \sigma_1')$ proves that the right side of (34) is majorized by

$$C\int(1+|s_1'+\partial_{\sigma_1'}S_1(s_2', \sigma_1')|+|s_2-s_2'|+|t'+\partial_{s_2'}S_2(s_2', \sigma_2^*)|)^{-l'}ds_1'dt'ds_2'$$
$$\leqq C\,.$$

These inequalities mean that $\sum_{p'}h_2(p, p')\leqq C$. Similarly, $\sum_{p}h_2(p, p')\leqq C$ holds. Consequently the kernel $h_2(p, p')$ defines a linear mapping bounded in $l^2(\Lambda^4)$.

Next we put $B_p^*B_{p'}f(\eta)=\int b_{pp'}^*(\eta, \eta')f(\eta')d\eta'$. Then

$$b_{pp'}^*(\eta, \eta')=\int \overline{a_p(x, \xi, y, \eta)}a_{p'}(x, \xi', y', \eta')e^{i\nu\psi_1}dxd\xi'dy'd\xi dy$$

and

$$\psi_1=\psi_1(\eta, y, \xi, x, \xi', y', \eta')$$
$$=S_1(y', \xi')-S_2(y', \eta')+\xi'\cdot x-S_1(y, \xi)+S_2(y, \eta)-\xi\cdot x\,.$$

The gradient of ψ_1 is

$$\partial_y\psi_1=\partial_yS_2(y, \eta)-\partial_yS_1(y, \xi), \quad \partial_{y'}\psi_1=\partial_{y'}S_1(y', \xi')-\partial_{y'}S_2(y', \eta')\,,$$
$$\partial_\xi\psi_1=-x-\partial_\xi S_1(y, \xi), \quad \partial_{\xi'}\psi_1=\partial_{\xi'}S_1(y', \xi')+x\,,$$
$$\partial_x\psi_1=\xi'-\xi\,.$$

If all these vanish, then $\xi'=\xi$, $y=y'$ and $\eta=\eta'$. Introducing a differential operator $M_2=|\operatorname{grad}\phi_1|^{-2}(\operatorname{grad}\phi_1\cdot\nabla)$, we obtain expression for $\eta\neq\eta'$ and $l=0, 1, 2, 3, \cdots$

$$b^*_{pp'}(\eta, \eta')=(i\nu)^{-l}\int M_1^{*}\overline{(a_p(x, \xi, y, \eta)}a_{p'}(x, \xi', y', \eta')e^{i\nu\phi_1}dxd\xi'dy'dyd\xi .$$

From this follows an estimate

$$(35)\qquad |b^*_{pp'}(\eta, \eta')|\leqq C_{ll'}\chi\Big(\frac{1}{2}(s_1-s_1')\Big)\varphi(\eta-\sigma_2)\varphi(\eta'-\sigma_2')$$
$$\times(\nu^{-l}|\eta-\eta'|^{2n-l}\chi^*+(1-\chi^*)\nu^{-l'}|\operatorname{grad}\phi_1(\sigma_2, s_2, \sigma_1, s_1^*, \sigma_1', s_2', \sigma_2')|^{-l'})$$

for $l\geqq 2n+1, \cdots$, and $l'=0, 1, 2, \cdots$, where χ^* is the characteristic function of the set $|\xi'-\xi|\leqq 3\sqrt{n}$, $|y-y'|\leqq 3\sqrt{n}$ and $|\eta-\eta'|\leqq 3\sqrt{n}$.

Therefore we obtain $\|B_p^*B_{p'}\|\leqq C\nu^{-3n}h_3(p, p')^2$,

$$(36)\qquad h_3(p, p')=\chi\Big(\frac{1}{2}(s_1-s_1')\Big)(\chi^*+(1-\chi^*)|\operatorname{grad}\phi(\sigma_2, s_2, \sigma_1, s_1^*, \sigma_1', s_2', \sigma_1')|^{-l'/2}) .$$

Here $s_1^*=\frac{1}{2}(s_1+s_1')$. As before

$$\sum_{p'}(1-\chi^*)\chi\Big(\frac{1}{2}(s_1-s_1')\Big)|\operatorname{grad}\phi_1(\sigma_2, s_2, \sigma_1, s_1^*, \sigma_1', s_2', \sigma_2)|^{-(1/2)l'}$$
$$\leqq C\int\chi\Big(\frac{1}{2}(s_1-s_1')\Big)(1+|\partial_{s_2}S_2(s_2, \sigma_2)-\partial_{s_2}S_1(s_2, \sigma_1)|$$
$$+|s_1^*+\partial_{\sigma_1}S_1(s_2, \sigma_1)|+|\sigma_1-\sigma_1'|+|\partial_{\sigma_1'}S_1(s_2', \sigma_1')+s_1^*|$$
$$+|\partial_{s_2'}S_1(s_2', \sigma_1')-\partial_{s_2'}S_2(s_2', \sigma_2)|)^{-(1/2)l'}ds_1'd\sigma_1'ds_2'd\sigma_2' .$$

The change of variables $u=\partial_{\sigma_1'}S_1(s_2', \sigma_1')$ and $v=\partial_{s_2'}(s_2', \sigma_2)$ proves that the right side of (37) is not greater than

$$C\int\chi\Big(\frac{1}{2}(s_1-s_1')\Big)ds_1'\int(1+|v-\partial_{s_2'}S_1(s_2', \sigma_1')|+|\sigma_1-\sigma_1'|+|u+s_1^*|)^{-(1/2)l'}d\sigma_1'd\sigma_2'du\leqq C ,$$

if l' is large enough.

This implies that $\sum_{p'}h_3(p, p')\leqq C$. Similarly we have $\sum_{p}h_3(p, p')\leqq C$. Therefore $h_3(p, p')$ is the kernel of a bounded linear mapping in $l^2(\Lambda^4)$. This and Calderòn-Vaillancourt lemma prove inequality (4).

§ 4. Applications.

Let $S_1(x, \xi)$ and $S_2(x, \xi)$ be as in § 1 and $e_j(x, \xi)$, $j=1, 2$, be functions in $C^\infty(R^n\times R^n)$ uniformly bounded with their derivatives of all orders. We proved in [3] that operators E_j, $j=1, 2$, defined by

$$E_j f(x)=(2\pi)^{-n}\int e_j(x,\xi)e^{i\nu(S_j(x,\xi)-y\xi)}f(y)dyd\xi$$

are bounded in $L^2(R^n)$. More precisely, we proved that there exists a constant $C>0$ such that $\|E_j f\|\leqq C\nu^{-n}\|f\|$, $j=1, 2$.

Let Φ_j be the canonical mapping defined by the generating function $S_j(y, \xi)$, $j=1, 2$. Φ_j are diffeomorphism by (A-I). In connection with these mappings we shall use notations: $(z(y, \xi), \eta(y, \xi))=\Phi_1^{-1}(y, \xi)$, and $(x(y, \xi), \zeta(y, \xi))=\Phi_2(y, \xi)$, i.e., $z=\partial_\eta S_1(y, \eta)$, $\xi=\partial_y S_1(y, \eta)$, $y=\partial_\xi S_2(x, \xi)$ and $\zeta=\partial_x S_2(x, \xi)$.

THEOREM 2. *Assume that* $e_2(x, \xi)e_1(y, \eta)=0$ *for any* (x, ξ, y, η) *such that* $|x-x(y, \xi)|^2+|\eta-\eta(y, \xi)|^2\leqq d^2$. *Then for any integer* $l\geqq 0$ *there exists a constant* $C_l\geqq 0$ *such that*

$$\|E_2\circ E_1\|\leqq C_l(\nu d)^{-l}\nu^{-2n}.$$

PROOF. Let $f\in C_0^\infty(R^n)$. Then

$$E_2\circ E_1 f(x)=(2\pi)^{-2n}\int e_2(x, \xi)e_1(y, \eta)e^{i\nu(S_2(x,\xi)-\xi\cdot y+S_1(y,\eta))}\hat{f}_\nu(\eta)dyd\eta d\xi, \tag{38}$$

where $\hat{f}_\nu(\eta)=\int e^{-i\nu\eta z}f(z)dz$. We put $\phi=\phi(x, \xi, y, \eta)=S_2(x, \xi)-\xi\cdot y+S_1(y, \eta)$. Then $\partial_\xi\phi=\partial_\xi S_2(x, \xi)-y$, $\partial_y\phi=-\xi+\partial_y S_1(y, \eta)$. If all these vanish, then $x=x(y, \xi)$, $\eta=(y, \xi)$ hold. Thus on the support of $e_2(x, \xi)e_1(y, \eta)$ there holds inequality

$$\begin{aligned}|\partial_\xi\phi| &\geqq |\partial_\xi S_2(x, \xi)-y|\\ &\geqq -|\partial_\xi S_2(x(y, \xi), \xi)-y|+|\partial_\xi S_2(x, \xi)-\partial_\xi S_2(x(y, \xi), \xi)|\\ &\geqq |\partial_\xi S_2(x, \xi)-\partial_\xi S_2(x(y, \xi), \xi)|\geqq C|x-x(y, \xi)|.\end{aligned}$$

Similarly, we have $|\partial_y\phi|\geqq C|\eta-\eta(x, \xi)|$. Consequently we proved $|\partial_\xi\phi|+|\partial_y\phi|\geqq Cd$ on the support of the integrand of (38). We define differential operator of the first order

$$P_1=\frac{d}{|\partial_\xi\phi|^2+|\partial_y\phi|^2}(\partial_\xi\phi\cdot\partial_\xi+\partial_y\phi\cdot\partial_y).$$

We have expression

$$E_2E_1 f(x)=(2\pi)^{-2n}(i\nu d)^{-l}\int P_1^{*l}(e_2(x, \xi)e_1(y, \eta))e^{i\nu\phi}\hat{f}_\nu(\eta)dyd\eta d\xi,$$

where P_1^* is the formal adjoint of P_1. Since we can easily show that $P_1^{*l}(e_2(x, \xi)e_1(y, \eta))$ satisfies condition (A-III), we obtain estimate

$$\|E_2E_1 f\|\leqq C(\nu d)^{-l}\nu^{-2n}\|f\|.$$

The graph of the canonical map Φ_1 is parametrized by independent variables (y, η) of the generating function $S_1(y, \eta)$. The canonical map Φ_2^{-1} sends this point to $(x(y, \eta), \xi(y, \eta))$.

THEOREM 3. *Assume that* $|\xi(y, \eta)-\xi|\geqq d$ *for any* (ξ, y, η) *on the support of*

$\overline{e_2(y, \xi)}e_1(y, \eta)$. *Then we have estimate*

$$\|E_2^* E_1 f\| \leqq C(\nu d)^{-l}\|f\|,$$

where l *is any positive integer and* C_l *is positive constant independent of* d *and* ν *and* f.

PROOF. By definition

$$E_2^* E_1 f(x) = (2\pi)^{-2n}\int e_2(y, \xi)e_1(y, \eta)e^{i\nu\phi(x,\xi,y,\eta)}\hat{f}_\nu(\eta)d\eta dy d\xi \tag{39}$$

where $\phi(x, \xi, y, \eta) = S_1(y, \eta) - S_2(y, \xi) + x\cdot\xi$. Since we can prove that $|\partial_y\phi| \geqq |\partial_y S_2(y, \xi(y, \eta)) - \partial_y S_2(y, \xi)| \geqq Cd$ on the support of the integrand of (39), the same argument as in the proof of Theorem 2 proves this theorem.

Theorems 2 and 3 will be used in [4].

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