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A mathematical study of the circular Couétte flow

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§1. Introduction.

The aim of the present paper is to study some detailed properties of special solutions of the 3-dimensional stationary and nonstationary Navier-Stokes equations from several points of view. The solution to be considered in this paper is the classical Couétte flow between two rotating concentric cylinders.

There are two main reasons why we wish to study this flow. First it is important to study detailed properties of special solutions of the Navier-Stokes equations(1) in special cases in order to guide the mathematical analysis for the equation in the general formulation. Secondly, the Couétte flow itself has many properties which are quite interesting mathematically as well as physically. For example, as the celebrated experiment by G. I. Taylor in 1923 revealed and as was rigorously proved mathematically by Velte [6] in 1966, the Couétte flow is not necessarily the unique solution. And moreover, what is more interesting and challenging to mathematicians is the physical fact that in experiments the Couétte flow is actually observed in one case but not in another case, notwithstanding mathematically it is equally a solution for both cases. In the latter case flows different from the Couétte flow are observed. Explanations of this phenomenon have been tried by physicists from the standpoint of the stability theory (See, for example, C. C. Lin [3]). But at the present state of the mathematical study of the N-S equations where we do not know whether a unique and global in time regular solution of the 3-dimensional N-S equation exists or not, the stability theory is confronted with theoretical difficulties.

In this paper we shall treat some general problems related to the Couétte flows. First we shall study the problem whether there exists for any given T (>0) a regular solution in the interval $[0, T]$ of the nonstationary N-S equation for every initial data given closely to the Couétte flow. After establishing an affirmative answer to this question, we next prove the differentiability in the sense of Fréchet of the evolution operator of this initial value

(1) For simplicity, we call it an N-S equation.

problem which gives a mathematical foundation to the linear stability theory. Thirdly, we shall study the eigenvalue problem of the Fréchet derivative evaluated at the Couétte flow and show that the Couétte flow is unstable for infinitesimal perturbations under certain circumstances. Finally, we shall prove that the Couétte flow is an isolated solution under almost all circumstances.

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§2. Formulation of the problem and the results.

We consider the non-stationary and stationary N-S equations in a special domain G between two concentric cylinders of radii R_1 and R_2 $(>R_1)$. More precisely, G is defined by $G=\{x=(x_1, x_2, x_3)\in \boldsymbol{R}^3;\ R_1^2\leqq x_1^2+x_2^2\leqq R_2^2\}$. The two cylinders rotates with constant angular velocities Ω_1 and Ω_2; the inner with Ω_1 and the outer with Ω_2 counter clockwise.

In G the N-S equation is written for the non-stationary motion as

(NSE) $$\begin{cases} \dfrac{\partial v}{\partial t}=\Delta v-\nabla_v v-\operatorname{grad} q,\ t>0,\ x\in G \\ \operatorname{div} v(x)=0,\ x\in G \\ v(0, x)=a(x) \\ \text{and the boundary condition of adherence at the boundary} \\ \text{that the fluid on the boundary move with the boundary,} \end{cases}$$

and for the stationary motion as

(SE) $$\begin{cases} \Delta v-\nabla_v v-\operatorname{grad} q=0 \\ \operatorname{div} v=0 \\ \text{the boundary condition of adherence at the boundary.} \end{cases}$$

The boundary condition will be made explicit later. In these equations v is the velocity vector field of the fluid in question and q is a scalar function which is the pressure in the fluid. The unknowns are v and q. ∇ means the canonical affine connection in $\boldsymbol{R}^3$ and Δ is the Laplacian. In the sequel to treat the equations (NSE) and (SE) conveniently, we use two coordinate systems, the cartesian coordinate system (x_1, x_2, x_3) and the cylindrical coordinate system (r, ϕ, z). In these coordinate systems a vector field v is expressed; in the first, $v=(v_1, v_2, v_3)=\sum_{i=1}^{3} v_i \dfrac{\partial}{\partial x_i}$, and in the latter, $v=(v_r, v_\phi, v_z)= v_r\dfrac{\partial}{\partial r}+\dfrac{v_\phi}{r}\dfrac{\partial}{\partial \phi}+v_z\dfrac{\partial}{\partial z}$. The well-known Couétte flow is then expressed by

$w=\left(\alpha+\frac{\beta}{r^2}\right)\frac{\partial}{\partial\phi}$ and $q_0=\int^r\frac{1}{\rho}\left(\alpha\rho+\frac{\beta}{\rho}\right)d\rho$ where $\alpha=(R_2^2\Omega_2-R_1^2\Omega_1)/(R_2^2-R_1^2)$ and $\beta=R_1^2R_2^2(\Omega_1-\Omega_2)/(R_2^2-R_1^2)$. This is a solution of (SE) for all R_1, R_2, Ω_1, Ω_2. We use the letter w to denote the Couétte flow exclusively in this paper. It is the aim of this paper to study the properties of the Couétte flow and those of the solutions of the equations (NSE) and (SE) near the Couétte flow. To that end, we consider a portion G_h of G and treat the equations in G_h. G_h is defined by $G_h=\{x\in G\,;\,0\leqq x_3\leqq h\}$ and the union of the top and the bottom of G_h is denoted ∂G_h. In G_h we consider the following initial value problem and boundary value problem.

$$
\text{(IVP)}\qquad\begin{cases}\dfrac{\partial v}{\partial t}=\Delta v-\nabla_v v-\operatorname{grad} q\\[2mm] \operatorname{div} v=0\\[2mm] v(0,x)=a(x)\\[2mm] v(x_1,x_2,0)=v(x_1,x_2,h)\,,\quad \dfrac{\partial v}{\partial x_3}(x_1,x_2,0)=\dfrac{\partial v}{\partial x_3}(x_1,x_2,h)\\[2mm] v=\left(\alpha+\dfrac{\beta}{R_i^2}\right)\dfrac{\partial}{\partial\phi}\qquad\text{for}\quad r^2=x_1^2+x_2^2=R_i^2,\quad i=1,2\,,\end{cases}
$$

and

$$
\text{(BVP)}\qquad\begin{cases}\Delta v-\nabla_v v-\operatorname{grad} q=0\\[2mm] \operatorname{div} v=0\\[2mm] v(x_1,x_2,0)=v(x_1,x_2,h)\,,\quad \dfrac{\partial v}{\partial x_3}(x_1,x_2,0)=\dfrac{\partial v}{\partial x_3}(x_1,x_2,h)\\[2mm] v=\left(\alpha+\dfrac{\beta}{R_i^2}\right)\dfrac{\partial}{\partial\phi}\qquad\text{for}\quad r=R_i,\quad i=1,2\,.\end{cases}
$$

In order to treat the problem in a functional analysis setting, we introduce some function spaces and operators. $L^2\equiv L^2(G_h)$ is a Hilbert space of all $\boldsymbol{R}^3$-valued functions $v=(v_1(x),v_2(x),v_3(x))$ defined in G_h for which the norm $\|v\|=\left(\int_{G_h}\sum_{i=1}^{3}v_i^2(x)dx\right)^{\frac{1}{2}}$ is finite. $C_{0,\sigma}^\infty=C_{0,\sigma}(G_h)$ is a space of all $\boldsymbol{R}^3$-valued functions $\varphi=(\varphi_1(x),\varphi_2(x),\varphi_3(x))$ such that (i) every component $\varphi_j\in C^\infty(\bar{G}_h)$ (ii) $\varphi_j=0$ near the lateral Γ_h of G_h. $\Gamma_h=\{x\,;\,x_1^2+x_2^2=R_i^2,\ i=1,2,\ 0\leqq x_3\leqq h\}$. (iii) $\operatorname{div}\varphi=0$. (iv) $\varphi(x_1,x_2,0)=\varphi(x_1,x_2,h)$, $\frac{\partial\varphi}{\partial x_3}(x_1,x_2,0)=\frac{\partial\varphi}{\partial x_3}(x_1,x_2,h)$. $L_\sigma^2=L_\sigma^2(G_h)$ is the completion of $C_{0,\sigma}^\infty$ with respect to the norm of $L^2(G_h)$. By P we denote the orthogonal projection of L^2 onto L_σ^2. For $\varphi\in C_{0,\sigma}^\infty$ we define an operator A by $A\varphi=-P\Delta\varphi$. It is easy to verify that A is a strictly positive symmetric operator in the Hilbert space L_σ^2. The positivity follows from the Poincaré inequality. We take the Friedrichs extension of A which we denote also by the same letter A. Then A is a strictly positive self-adjoint operator with compact inverse A^{-1}. For real γ we denote A^{γ} the fractional power of A and by $\mathscr{D}(A^{\gamma})$ the domain of definition of A^{γ} endowed with its graph norm $\|\varphi\|_{\gamma}=\|A^{\gamma}\varphi\|$. Transforming the unknowns from (v, q) to (u, p) by the identities $v=u+w$, $q=p+q_0$, and making use of the above notations, the equations (IVP) and (BVP) are transformed (formally) into the following abstract evolution equations (EE) and operator equation (E) in L_{σ}^2, respectively.

$$\text{(EE)} \qquad \begin{cases} \dfrac{du}{dt}=-Au-P(\nabla_u u+\nabla_w u+\nabla_u w) \\ u(0)=a \end{cases}$$

and

$$\text{(E)} \qquad Au+P(\nabla_u u+\nabla_w u+\nabla_u w)=0\,.$$

In (EE) $u=u(t)$ is regarded as an L_{σ}^2-valued function defined on $\{t\geqq 0\}$. In order to investigate the integrability of the equation (EE), we introduce the following integral equation (IE),

$$\text{(IE)} \qquad u(t)=e^{-tA}a-\int_0^t e^{-(t-s)A}P(\nabla_{u(s)}u(s)+\nabla_w u(s)+\nabla_{u(s)}w)ds$$

where by e^{-tA} we denote the semi-group of operators generated by $-A$. If we can prove the existence of solution $u(t)$ of (IE) with a certain regularity property, it is easy to verify that it is a regular solution of (EE). Hence we shall be engaged exclusively in (IE).

Now we can write the main statements of our theorems which we are going to prove in this paper. Full statements of these theorems will be given in later sections.

THEOREM 1 (an existence theorem). (1) *For every $r>0$ there exists $T>0$ such that there exists uniquely a solution of* (IE) *on the interval* $[0, T]$ *for every $a\in\mathscr{D}(A^{1/2})$ with $\|A^{1/2}a\|<r$.*

(2) *For every $T>0$ there exists $r>0$ such that the statement in* (1) *holds.*

According to Theorem 1, the evolution operator, $S_t: a\mapsto u(t)$ $(0\leqq t\leqq T)$ is well defined in the r-neighbourhood of the origin for T and γ in (1) or (2). Then we have

THEOREM 2 (differentiability of the evolution operator). *The evolution operator $S_t: \mathscr{D}(A^{1/2})\to\mathscr{D}(A^{1/2})$ is Fréchet differentiable at $0\in\mathscr{D}(A^{1/2})$ for any $t>0$.*

THEOREM 3 (eigenvalue problem for the Fréchet derivative of the evolution operator). *For any $\omega=(\omega_1, \omega_2)\in S^1$ (the 1-sphere) such that $(R_2^2\omega_2-R_1^2\omega_1)(\omega_2-\omega_1)>0$ there exists $\rho_\omega>0$ such that if $\sqrt{\Omega_1^2+\Omega_2^2}>\rho_\omega$, then the Fréchet derivative of the evolution operator S_t at 0 has real positive eigenvalue greater than 1 for* every $t>0$.

THEOREM 4. *0 is an isolated solution of* (E) *in* $\mathcal{D}(A^{\gamma})$ *with* $\gamma>\frac{3}{4}$ *except for a countable set of values of* $\Omega_1^2+\Omega_2^2$.

§ 3. Existence theorems.

First we state two lemmas concerning the operators A and e^{-tA}. For the statement of Lemma 1, we introduce the operator B defined as follows. The domain of definition of B is $\mathcal{D}(B)=W_2^2(G_h)\cap H_2^1(G_h)$ where $W_2^m(G_h)$ is a L^2-Sobolev space of order m, and $H_2^1(G_h)=\Big\{u\in W_2^1(G_h)\,;\ u|_{\Gamma_h}=0,\ u(x_1, x_2, 0)=u(x_1, x_2, h),\ \frac{\partial u}{\partial x_3}(x_1, x_2, 0)=\frac{\partial u}{\partial x_3}(x_1, x_2, h)\Big\}$. And for $u\in\mathcal{D}(B)$, $Bu=-\Delta u$.

LEMMA 1. *For* $0<\gamma<1$, $\mathcal{D}(A^{\gamma})=\mathcal{D}(B^{\gamma})\cap L_{\sigma}^2$. *And therefore* $\mathcal{D}(A^{\gamma})\subset C(\overline{G}_h)$ *for* $\gamma>\frac{3}{4}$ *and* $\sup_{x\in\overline{G}_h}|u(x)|\leqq C_{\gamma}\|A^{\gamma}u\|$ *for* $u\in\mathcal{D}(A^{\gamma})$, *with some constant* $C_{\gamma}>0$.

Here, $C(\overline{G}_h)$ *is the space of all continuous* $\boldsymbol{R}^3$*-valued functions defined on* $\overline{G}_h$.

We can prove the lemma by the interpolation theory of Lions and a certain fact concerning $\mathcal{D}(A)$. For details, see H. Fujita and H. Morimoto [2] where analogous result is proved.

LEMMA 2. *For* $0<\gamma<e$, $\|A^{\gamma}e^{-tA}\|\leqq t^{-\gamma}$.

PROOF. The proof is easy if we use the spectral representation of A and so we omit the proof.

We note here that by virtue of Lemma 1, the nonlinear operator $P\nabla_u v$ is well-defined for every $u\in\mathcal{D}(A^{\gamma})$ if $\gamma>\frac{3}{4}$, and $v\in\mathcal{D}(A^{1/2})$ and we have $\|P\nabla_u v\|\leqq C_{\gamma}\|A^{\gamma}u\|\,\|A^{1/2}v\|$. Actually in the cartesian coordinate system, $\nabla_u v=\sum_{j=1}^{3}\Big(\sum_{i=1}^{3}u_i\frac{\partial v_j}{\partial x_i}\Big)\frac{\partial}{\partial x_j}\equiv(u\cdot\nabla)v$ and hence the above estimate is an immediate consequence.

Now we are ready to study the integral equation (IE). For that purpose we introduce a function space Ψ_T^{γ} for T and γ with $T>0$ and $\frac{3}{4}<\gamma<1$. Ψ_T^{γ} is a Banach space of all $\mathcal{D}(A^{1/2})$-valued continuous functions $u(t)$ defined on the interval $[0, T]$ such that (i) $u(t)\in\mathcal{D}(A^{\gamma})$ for $0<t\leqq T$, (ii) $u(t)\in C([0, T]\,;\ \mathcal{D}(A^{1/2}))\cap C((0, T]\,;\ \mathcal{D}(A^{\gamma}))$, and (iii) the norm

$$|||u|||\equiv\sup_{0\leqq t\leqq T}\|A^{1/2}u(t)\|+\sup_{0\leqq t\leqq T}\ \sup_{0<s\leqq t}s^{\gamma-1/2}\|A^{\gamma}u(s)\|$$

is finite.

We are going to obtain solutions of the integral equation (IE) in the class Ψ_T^{γ} by the iteration method. We use the following iteration scheme.

$$(3.1)\qquad \begin{cases} u_0(t)=0\,, \\ u_{n+1}(t)=e^{-tA}a-\displaystyle\int_0^t e^{-(t-s)A}P[(u_n(s)\cdot\nabla)u_n(s) \\ \qquad +(w\cdot\nabla)u_n(s)+(u_n(s)\cdot\nabla)w]ds \quad \text{for} \quad n=0,1,2,\cdots. \end{cases}$$

First we must verify that the iteration is possible in Ψ_T^γ. To that end we introduce functions $K_u(t)$, $M_u(t)$ for functions $u(t)$ in Ψ_T^γ. They are defined as follows.

$$K_u(t)=\sup_{0<s\leq t} s^{\gamma-1/2}\|A^\gamma u(s)\|\,, \qquad M_u(t)=\max_{0\leq s\leq t}\|A^{1/2}u(s)\|\,.$$

And in addition we define operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ by

$$\mathcal{A}(u)(t)=\int_0^t e^{-(t-s)A}P(u(s)\cdot\nabla)u(s)ds\,,$$

$$\mathcal{B}(u)(t)=\int_0^t e^{-(t-s)A}P(w\cdot\nabla)u(s)ds\,,$$

$$\mathcal{C}(u)(t)=\int_0^t e^{-(t-s)A}P(u(s)\cdot\nabla)wds\,.$$

Then we obtain the following estimates which justify the feasibility of the iteration.

LEMMA 3.

$$\begin{cases} \displaystyle\sup_{0<s\leq t} s^{\gamma-1/2}\|A^\gamma\mathcal{A}(u)(s)\|\leq t^{1-\gamma}c_\gamma B_1K_u(t)M_u(t) \\ \displaystyle\sup_{0<s\leq t} s^{\gamma-1/2}\|A^\gamma\mathcal{B}(u)(s)\|\leq t^{1/2}\frac{c_\gamma}{1-\gamma}\|A^\gamma w\|M_u(t) \\ \displaystyle\sup_{0<s\leq t} s^{\gamma-1/2}\|A^\gamma\mathcal{C}(u)(s)\|\leq t^{1-\gamma}c_\gamma B_1\|A^{1/2}w\|K_u(t)\,, \end{cases}$$

$$\begin{cases} \displaystyle\sup_{0<s\leq t}\|A^{1/2}\mathcal{A}(u)(s)\|\leq t^{1-\gamma}c_\gamma B_1K_u(t)M_u(t) \\ \displaystyle\sup_{0<s\leq t}\|A^{1/2}\mathcal{B}(u)(s)\|\leq t^{1/2}2c_\gamma\|A^\gamma w\|M_u(t) \\ \displaystyle\sup_{0<s\leq t}\|A^{1/2}\mathcal{C}(u)(s)\|\leq t^{1-\gamma}c_\gamma B_1\|A^{1/2}w\|K_u(t)\,. \end{cases}$$

Here $B_1=B\left(1-\gamma,\dfrac{3}{2}-\gamma\right)$ *where* $B(\cdot,\cdot)$ *is the beta function.*

PROOF. We prove the first estimate only. The others are proved similarly.

$$\begin{aligned}\int_0^s\|A^\gamma e^{-(s-\sigma)A}P(u(\sigma)\cdot\nabla)u(\sigma)\|d\sigma &\leq\int_0^s(s-\sigma)^{-\gamma}c_\gamma\|A^\gamma u(\sigma)\|\|A^{1/2}u(\sigma)\|d\sigma \\ &\leq c_\gamma\int_0^s(s-\sigma)^{-\gamma}\sigma^{1/2-\gamma}K_u(s)M_u(s)d\sigma\end{aligned}$$

$$=c_\gamma K_u(s)M_u(s)\int_0^1(1-\rho)^{-\gamma}\rho^{1/2-\gamma}s^{3/2-2\gamma}d\rho$$

$$=s^{3/2-2\gamma}c_\gamma B\Big(1-\gamma,\ \frac{3}{2}-\gamma\Big)K_u(s)M_u(s)\,.$$

Hence we have

$$\sup_{0<s\leqq t} s^{\gamma-1/2}\|A^\gamma\mathcal{A}(u)(s)\|\leqq c_\gamma B_1\sup_{0<s\leqq t}s^{1-\gamma}K_u(s)M_u(s)\leqq c_\gamma B_1 t^{1-\gamma}K_u(t)M_u(t)\,.$$

In the last inequality we used the fact that $K_u(t)$ and $M_u(t)$ are increasing functions and the assumption that $\frac{3}{4}<\gamma<1$. Q. E. D.

By Lemma 3, we have

$$\sup_{0<s\leqq t} s^{\gamma-1/2}\|A^\gamma[\mathcal{A}(u(s))+\mathcal{B}(u)(s)+\mathcal{C}(u)(s)]\|$$
$$\leqq t^{1-\gamma}[c_1K_u(t)M_u(t)+c_2K_u(t)+c_3t^{\gamma-1/2}M_u(t)]$$

and

$$\sup_{0<s\leqq t}\|A^{1/2}[\mathcal{A}(u)(s)+\mathcal{B}(u)(s)+\mathcal{C}(u)(s)]\|$$
$$\leqq t^{1-\gamma}[c_1K_u(t)M_u(t)+c_2K_u(t)+c_3t^{\gamma-1/2}M_u(t)]\,.$$

Therefore, defining $N_u(t)=\max\{K_u(t),\ M_u(t)\}$ and $\mathcal{M}(u)=\mathcal{A}(u)+\mathcal{B}(u)+\mathcal{C}(u)$, we have

$$N_{\mathcal{M}(u)}(t)\leqq t^{1-\gamma}[c_1N^2_{\mathcal{M}(u)}(t)+(c_2+c_3t^{\gamma-1/2})N_{\mathcal{M}(u)}(t)] \tag{3.2}$$

where we used c_1, c_2, c_3 to denote positive constants depending only on γ and the Couétte flow w.

We now return to the iteration scheme (3.1). By the estimate (3.2) we have the following reccurence inequality, writing $N_{\mathcal{M}(u_n)}(t)$ simply as $N_n(t)$

$$N_{n+1}(t)\leqq\|A^{1/2}a\|+t^{1-\gamma}[c_1N^2_n(t)+(c_2+c_3t^{\gamma-1/2})N_n(t)]\,. \tag{3.3}$$

By a simple consideration we have for every u_n

$$N_n(t)\leqq\chi(t) \tag{3.4}$$

if $c_2t^{1-\gamma}+c_3t^{1/2}<1$ and $\Delta(t)>0$ where we define

$$\Delta(t)=(c_2t^{1-\gamma}+c_3t^{1/2}-1)^2-4c_1t^{1-\gamma}\|A^{1/2}a\| \tag{3.5}$$

and

$$\chi(t)=[1-(c_2t^{1-\gamma}+c_3t^{1/2})-\Delta(t)^{1/2}]/2c_1t^{1-\gamma}\,. \tag{3.6}$$

We can now study the convergence of the iteration. Setting $v_n(t)=u_{n+1}(t)-u_n(t)$, we have $v_0(t)=e^{-tA}a$ and

$$v_n(t)=-\int_0^t e^{-(t-s)A}P[(v_{n-1}(s)\cdot\nabla)u_n(s)+(u_{n-1}(s)\cdot\nabla)v_{n-1}(s)$$
$$+(w\cdot\nabla)v_{n-1}(s)+(v_{n-1}(s)\cdot\nabla)w]ds\,.$$

In order to estimate $v_n(t)$ we define

$$D_n(t)=\sup_{0<s\leqq t} s^{\gamma-1/2}\|A^\gamma v_n(s)\|\,,\qquad E_n(t)=\sup_{0<s\leqq t}\|A^{1/2}v_n(s)\|\,,$$

$$F_n(t)=\max\{D_n(t),\,E_n(t)\},\ K_n(t)=K_{u_n}(t),\ M_n(t)=M_{u_n}(t),\ N_n(t)=N_{u_n}(t)\,.$$

Then we have

$$F_n(t)\leqq t^{1-\gamma}(d_1\chi(t)+d_2\|A^\gamma w\|t^{\gamma-1/2}+d_3\|A^{1/2}w\|)F_{n-1}(t)\equiv\rho(t,w)F_{n-1}(t)$$

and

$$F_0(t)\leqq\|A^{1/2}a\|$$

where d_1, d_2, d_3 are positive constants depending only on γ. If we note that for every fixed w, $\chi(t)$ tends to 0 as t tends to 0, we immediately see that there exists a positive T such that $\rho(T,w)<1$. For such T, $\sum_{n=0}^{\infty}F_n(t)$ converges uniformly in $t\in[0,T]$. Hence we see, noting that A is a closed operator and has a continuous inverse, that $u(t)=\lim_{n\to\infty}u_n(t)$ exists in L_σ^2 and $\mathcal{D}(A^{1/2})$ for every $t\in[0,T]$ and in $\mathcal{D}(A^\gamma)$ for every $t\in(0,T]$ and that the former convergence is uniform on $[0,T]$ and the latter locally uniform in $(0,T]$. The fact that the limit function belongs to Ψ_T^γ is evident.

Now let us suppose that arbitrary positive r is given. We consider a problem whether there exists a positive T such that the integral equation (IE) has a solution in Ψ_T^γ for every initial data $a\in\mathcal{D}(A^{1/2})$ with $\|A^{1/2}a\|\leqq r$. From the discussions above we see that it suffices that T satisfies the following three inequalities.

$$c_2T^{1-\gamma}+c_3T^{1/2}<1\,,\qquad \Delta(T)>0\qquad\text{and}\qquad\rho(T,w)<1$$

where $\|A^{1/2}a\|$ is replaced by r in the expression of $\Delta(T)$ and $\rho(T,w)$. The fact that there exists such a T is easily seen from the explicit expression of $\Delta(T)$ and $\rho(T,w)$.

Next let us consider a problem whether, for any given T, there exists a positive r such that the integral equation (IE) has a solution in Ψ_T^γ for every $a\in\mathcal{D}(A^{1/2})$ with $\|A^{1/2}a\|\leqq r$. If we notice that $\|A^{1/2}a\|$ does not appear in the first one of the three inequalities above and that $\chi(T)\to 0$ when $\|A^{1/2}a\|\to 0$, we see that we can construct a solution in question by a finite number of steps of time-intervals.

Thus we have proved the following

THEOREM 1. (1) *For every given $r>0$, we can choose $T>0$ such that for* *every* $a\in\mathscr{D}(A^{1/2})$ *with* $\|A^{1/2}a\|\leqq r$, *there exists a solution of the integral equation* (IE) *in the interval* $[0, T]$ *which belongs to the class* Ψ_T^γ.

(2) *For every given* $T>0$ *we can choose* $r>0$ *such that there exists a solution of* (IE) *on* $[0, T]$ *which belongs to the class* Ψ_T^γ *for every* $a\in\mathscr{D}(A^{1/2})$ *with* $\|A^{1/2}a\|\leqq r$.

REMARK. We have not mentioned the uniqueness of the solution. However, it is not difficult to prove the uniqueness of the solution in the class Ψ_T^γ.

§4. Differentiability of the evolution operator S_t.

First let us recall the definition of the Fréchet derivative.

DEFINITION. Let X, Y be Banach spaces and Φ be a continuous mapping defined in a neighbourhood U of an element $a\in X$ with values in Y. A bounded linear operator A from X to Y is called the Fréchet derivative of Φ at a if

$$\Phi(a+h)-\Phi(a)=Ah+o(\|h\|) \qquad \text{as } a+h \text{ tends to } a \text{ in } U.$$

And when this is the case, Φ is said to be Fréchet differentiable at a.

We now return to the integral equation (IE) and define the evolution operator $S_t:\mathscr{D}(A^{1/2})\to\mathscr{D}(A^{1/2})$ by $S_t a=u(t)$ where $u(t)$ is the solution of (IE) with initial data a. By Theorem 1, we know that for any given $T>0$ we can find a neighbourhood U of $0\in\mathscr{D}(A^{1/2})$ where S_t is defined everywhere. Hence we can talk about the Fréchet differentiability of S_t at 0. We fix U and T above. For $h\in U$, S_t satisfies the following integral equation by definition

$$S_t h=e^{-tA}h-\int_0^t e^{-(t-s)A}P[(S_s h\cdot\nabla)S_s h+(w\cdot\nabla)S_s h+(S_s h\cdot\nabla)w]ds.$$

An inspection indicates that the Fréchet derivative of S_t at $0\in\mathscr{D}(A^{1/2})$ which we denote by DS_t must satisfy the following integral equation if it exists

$$DS_t h=e^{-tA}h-\int_0^t e^{-(t-s)A}P[(w\cdot\nabla)DS_s h+(DS_s h\cdot\nabla)w]ds. \tag{4.1}$$

In order to integrate the equation (4.1) we define an operator Γ in the function space Ψ_T^γ by

$$\Gamma(f)(t)=e^{-tA}h-\int_0^t e^{-(t-s)A}P[(w\cdot\nabla)f(s)+(f(s)\cdot\nabla)w]ds.$$

The operator Γ is well-defined since we have the following estimate

$$\max\{\sup_{0<s\leqq t}s^{\gamma-1/2}\|A^\gamma\Gamma(f)(s)\|,\ \sup_{0<s\leqq t}\|A^{1/2}\Gamma(f)(s)\|\} \tag{4.2}$$
$$\leqq\|A^{1/2}h\|+c(t^{1-\gamma}K_f(t)+t^{1/2}M_f(t))$$

with positive constant c depending only on γ and w (the Couétte flow). And we have for any $f_1, f_2 \in \Psi_\tau^\gamma$ $(0<\tau \leqq T)$,

$$\|\|\Gamma(f_1)-\Gamma(f_2)\|\| \leqq c_1\tau^{1-\gamma}\|A^\gamma w\|(1+c_2\tau^{\gamma-1/2})\|\|f_1-f_2\|\| \tag{4.3}$$

with positive constants c_1, c_2 depending only on γ. We choose $\tau>0$ sufficiently small so that

$$c_1\tau^{1-\gamma}\|A^\gamma w\|(1+c_2\tau^{\gamma-1/2})<1\,. \tag{4.4}$$

Then there exists uniquely in Ψ_τ^γ a solution of $f=\Gamma(f)$ which we denote by $f(t\,;h)$.

Next we shall prove, making use of $f(t\,;h)$, that there exists $0<\tau'\leqq\tau$ such that $S_{\tau'}$ is Fréchet differentiable at 0. We set $g(t\,;h)=S_t h$ then we have

$$\begin{aligned} f(t\,;h)-g(t\,;h)&=\int_0^t e^{-(t-s)A}P[(g(s\,;h)\cdot\nabla)g(s\,;h)]ds \\ &+\int_0^t e^{-(t-s)A}P[(w\cdot\nabla)(f(s\,;h)-g(s\,;h))+([f(s\,;h)-g(s\,;h)]\cdot\nabla)w]ds. \end{aligned} \tag{4.5}$$

It is enough to prove that there exists $\tau'>0$ such that

$$\|A^{1/2}(f(\tau'\,;h)-g(\tau'\,;h))\|/\|A^{1/2}h\| \longrightarrow 0 \qquad \text{as} \quad \|A^{1/2}h\|\to 0\,.$$

Estimating (4.5), we have

$$\begin{aligned} \|\|f(t\,;h)-g(t\,;h)\|\| &\leqq c_3 t^{1-\gamma}\|A^\gamma w\|(1+c_4 t^{\gamma-1/2})\|\|f(t\,;h)-g(t\,;h)\|\| \\ &\quad +c_5 t^{1-\gamma}\|A^\gamma w\|L(t)^2 \end{aligned}$$

with positive constants c_3, c_4, c_5 depending only on γ where the norm $\|\|\cdot\|\|$ is for functions f, $g\in\Psi_\tau^\gamma$, and

$$L(t)=\max\{\sup_{0<s\leqq t} s^{\gamma-1/2}\|A^\gamma g(s\,;h)\|,\ \sup_{0<s\leqq t}\|A^{1/2}g(s\,;h)\|\}\,.$$

If we choose $\tau'>0$ such that

$$c_3\tau'^{1-\gamma}\|A^\gamma w\|(1+c_4\tau'^{\gamma-1/2})<\theta<1\,, \tag{4.6}$$

we have

$$\|\|f(t\,;h)-g(t\,;h)\|\|<(1-\theta)^{-1}c_5\tau'^{1-\gamma}\|A^\gamma w\|L(\tau')^2\,.$$

This implies the desired relation

$$f(\tau'\,;h)-g(\tau'\,;h)=o(\|A^{1/2}h\|)\,,$$

since the inequality $L(\tau')\leqq c_6(\tau')\|A^{1/2}h\|$ is obvious from the property of Theorem 1. Thus we have proved that $S_{\tau'}$ is Fréchet differentiable at 0. By the estimates (4.2) and (4.4), we see that τ' is determined only by γ and w and so, by the chain rule for Fréchet derivatives, we have the following

THEOREM 2. *For every $T>0$ the evolution operator S_t from a neighbourhood of $0\in\mathcal{D}(A^{1/2})$ into $\mathcal{D}(A^{1/2})$ is Fréchet differentiable at $0\in\mathcal{D}(A^{1/2})$.*

§ 5. The eigenvalue problem.

In this section we study the eigenvalue problem of the linear operator $E_t = DS_t$ which is the Fréchet derivative at zero of the evolution operator S_t. What we wish to know is whether E_t has an eigenvalue whose absolute value is greater than 1 or not.

This problem can be reduced to the question whether the eigenvalue problem

$$-Au - P\nabla_u w - P\nabla_w u = \lambda u \tag{5.1}$$

admits real positive eigenvalues or not. It is easy to verify that (5.1) is equivalent to

$$\Delta u - \nabla_u w - \nabla_w u - \operatorname{grad} p = \lambda u, \quad \operatorname{div} u = 0 \tag{5.2}$$

with a suitable scalar function p. We adopt the cylindrical coordinate system (r, ϕ, z), where $u = (u_r, u_\phi, u_z) = u_r \frac{\partial}{\partial r} + \frac{u_\phi}{r} \frac{\partial}{\partial \phi} + u_z \frac{\partial}{\partial z}$ and the Couëtte flow w is expressed as $w = (w_r, w_\phi, w_z) = \left(\alpha + \frac{\beta}{r^2}\right) \frac{\partial}{\partial \phi}$. We recall that $\alpha = (R_2^2 \Omega_2 - R_1^2 \Omega_1)/(R_2^2 - R_1^2)$, $\beta = R_1^2 R_2^2 (\Omega_1 - \Omega_2)/(R_2^2 - R_1^2)$. In the cylindrical coordinate system, (5.2) is expressed as

$$\left\{\begin{aligned}
&\left(\Delta - \frac{1}{r^2}\right) u_r - \frac{2}{r^2} \frac{\partial u_\phi}{\partial \phi} - \mathfrak{N}(w) u_r + \frac{w_\phi u_\phi}{r} - \mathfrak{N}(u) w_r + \frac{u_\phi w_\phi}{r} - \frac{\partial p}{\partial r} = \lambda u_r \\
&\left(\Delta - \frac{1}{r^2}\right) u_\phi + \frac{2}{r^2} \frac{\partial u_r}{\partial \phi} - \mathfrak{N}(w) u_\phi - \frac{w_\phi u_r}{r} - \mathfrak{N}(u) w_\phi - \frac{u_\phi w_r}{r} - \frac{1}{r} \frac{\partial p}{\partial \phi} = \lambda u_\phi \\
&\Delta u_z - \mathfrak{N}(w) u_z - \mathfrak{N}(u) w_z - \frac{\partial p}{\partial z} = \lambda u_z \\
&\frac{1}{r} \frac{\partial}{\partial r}(r u_r) + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0
\end{aligned}\right. \tag{5.3}$$

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$ and $\mathfrak{N}(v) = v_r \frac{\partial}{\partial r} + \frac{v_\phi}{r} \frac{\partial}{\partial \phi} + v_z \frac{\partial}{\partial z}$. We seek u and p which are independent of the variable ϕ. Then (5.3) reduces to

$$\left\{\begin{aligned}
&\left(\Delta - \frac{1}{r^2}\right) u_r + 2\left(\alpha + \frac{\beta}{r^2}\right) \frac{\partial u_\phi}{\partial z} - \frac{\partial p}{\partial r} = \lambda u_r \\
&\left(\Delta - \frac{1}{r^2}\right) u_\phi - 2\alpha \frac{\partial u_r}{\partial z} = \lambda u_r \\
&\Delta u_z - \frac{\partial p}{\partial z} = \lambda u_z \\
&\frac{1}{r} \frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0 .
\end{aligned}\right. \tag{5.4}$$

Introducing a stream function f by the relations $ru_r=\frac{\partial}{\partial z}(rf)$ and $ru_z=-\frac{\partial}{\partial r}(rf)$, we obtain from (5.4)

$$
\begin{aligned}
&L(L-\lambda)f+2\left(\alpha+\frac{\beta}{r^2}\right)u_\phi=0\\
(5.5)\qquad&\\
&(L-\lambda)u_\phi-2\alpha f=0
\end{aligned}
$$

where we write $L=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\frac{1}{r^2}$. On f we impose an additional boundary condition that $f, \frac{\partial}{\partial r}f=0$ for $r=R_1, R_2$. We set $f(r, z)=\hat{f}(r)\cos\sigma z$, $u_\phi(r, z)=\hat{u}(r)\sin\sigma z$ with $\sigma=\frac{2\pi}{h}$. The boundary condition is reduced to the condition $f(R_i)=f'(R_i)=u(R_i)=0$ for $i=1, 2$. Hence we have the following system of linear ordinary differential equations

$$
(5.6)\qquad \begin{cases}
(\mathcal{L}-\sigma^2)(\mathcal{L}^2-\sigma^2-\lambda)f(r)+2\left(\alpha+\frac{\beta}{r^2}\right)\sigma u(r)=0\\
(\mathcal{L}-\sigma^2-\lambda)u(r)+2\alpha f(r)=0\\
f(R_i)=f'(R_i)=u(R_i)=0\,, \qquad i=1, 2
\end{cases}
$$

where we write $\mathcal{L}=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}$. In order to investigate the system (5.6), we consider the following two boundary value problems for ordinary differential operators $\mathcal{L}-\mu$ for $\mu\geqq 0$.

$$
\text{(BVP-1)}\qquad \begin{cases}
(\mathcal{L}-\mu)g(r)=\varphi(r)\,, & r\in(R_1, R_2)\\
g(R_i)=0\,, \quad i=1, 2
\end{cases}
$$

$$
\text{(BVP-2)}\qquad \begin{cases}
(\mathcal{L}-\mu_1)(\mathcal{L}-\mu_2)g(r)=\varphi(r)\,, & r\in(R_1, R_2)\\
g(R_i)=g'(R_i)=0\,, \quad i=1, 2
\end{cases}
$$

with μ, $\mu_i\geqq 0$, $i=1, 2$.

The next lemma is useful.

LEMMA 4. *Let $G(r, r'\,;\mu)$ and $H(r, r'\,;\mu_1, \mu_2)$ be the Green functions for* (BVP-1) *and* (BVP-2) *respectively. Then G and H are negative valued and positive valued respectively almost everywhere.*

This lemma may be proved by an explicit construction of the kernels making use of the Bessel functions but it can be proved also by repetition of elementary discussions. So we omit the proof.

We now return to the system (5.6). From (5.6) we have

$$
(5.7)\qquad f(r)=-\int_{R_1}^{R_2}H(r, r'\,;\sigma^2, \sigma^2+\lambda)2\sigma\left(\alpha+\frac{\beta}{r'^2}\right)u(r')dr'
$$

and

$$u(r)=-\int_{R_1}^{R_2}G(r, r'\,;\,\sigma^2+\lambda)2\alpha f(r')dr'\,. \tag{5.8}$$

Hence we have

$$f(r)=\int_{R_1}^{R_2}K(r, s\,;\,\sigma, \lambda)f(s)ds \tag{5.9}$$

where $K(r, s\,;\,\sigma, \lambda)=4\sigma\int_{R_1}^{R_2}H(r, r'\,;\,\sigma^2, \sigma^2+\lambda)\alpha\left(\alpha+\frac{\beta}{r'^2}\right)G(r', s\,;\,\sigma^2+\lambda)dr'$. Now the problem is reduced to that whether the integral operator K defined by the kernel $K(r, s\,;\,\sigma, \lambda)$ has 1 as its eigenvalue or not. For that purpose the following lemma of Jentzsch (See Schmeidler [**4**]) is useful.

LEMMA 5. *Let $K(r, s)$ be a continuous kernel on the interval $[R_1, R_2]$ which is positive almost everywhere. Then the integral operator $Kf(r)=\int_{R_1}^{R_2}K(r, s)f(s)ds$ has a positive eigenvalue.*

What we have to do next is to investigate the sign of the function $k(r)=\alpha\left(\alpha+\frac{\beta}{r^2}\right)=\frac{R_2^2\Omega_2-R_1^2\Omega_1}{R_2^2-R_1^2}\left[(R_2^2\Omega_2-R_1^2\Omega_1)+\frac{R_1^2R_2^2(\Omega_1-\Omega_2)}{r^2}\right]$. For a fixed $\omega=(\omega_1, \omega_2)\in S^1$ (the 1-sphere) we set $(\Omega_1, \Omega_2)=\rho(\omega_1, \omega_2)$, $\rho\geqq 0$, and

$$l(r\,;\,\rho, \omega)=\rho^2\frac{R_2^2\omega_2-R_1^2\omega_1}{R_2^2-R_1^2}\left[(R_2^2\omega_2-R_1^2\omega_1)+\frac{R_1^2R_2^2(\omega_1-\omega_2)}{r^2}\right].$$

Then the kernel K satisfies

$$\begin{aligned}K(r, s\,;\,\lambda)&=4\sigma\int_{R_1}^{R_2}H(r, r'\,;\,\sigma^2, \sigma^2+\lambda)l(r'\,;\,\rho, \omega)K(r', s\,;\,\sigma^2, \lambda)dr'\\&\equiv\rho^2L(r, s\,;\,\lambda, \omega)\,.\end{aligned}$$

Hence, making use of Lemma 4 and Lemma 5, we have

THEOREM 3. *For every fixed $\omega=(\omega_1, \omega_2)\in S^1$, such that $(R_2^2\omega_2-R_1^2\omega_1)(\omega_2-\omega_1)>0$ there exists $\rho(\omega)>0$ such that for every $\rho\equiv\Omega_1^2+\Omega_2^2>\rho(\omega)$ the corresponding Couétte flow is unstable under infinitesimal perturbations.*

It suffices only to note that the integral equation (5.9) is reduced to

$$f(r)=\rho^2\int_{R_1}^{R_2}L(r, s\,;\,\lambda, \omega)f(s)ds\,.$$

§ 6. Isolatedness of the Couétte flow.

What we are going to do in this section is to investigate whether or not 0 is an isolated solution of the equation

$$Au+P(u\cdot\nabla)u+P(w\cdot\nabla)u+P(u\cdot\nabla)w=0 \tag{E}$$

where w is the Couétte flow. We take γ with $\frac{3}{4}<\gamma<1$ and work in the Hilbert space $\mathcal{D}(A^\gamma)$. Assume that $0\in\mathcal{D}(A^{1/2})$ is not an isolated solution of (E). Then there exist solutions u_n $(\neq 0)$ of (E) for $n=1, 2, \cdots$ such that $\lim_{n\to\infty}\|A^\gamma u_n\|=0$. Set $\phi_n=u_n/\|A^\gamma u_n\|$. Then we have

$$A\phi_n+\|A^\gamma u_n\|P((\phi_n\cdot\nabla)\phi_n+P(w\cdot\nabla)\phi_n+P(\phi_n\cdot\nabla)w)=0\,. \tag{6.1}$$

As for the second term, we see that

$$\|P(\phi_n\cdot\nabla)\phi_n\|\leqq c_\gamma\|A^\gamma\phi_n\|\|A^{1/2}\phi_n\|\leqq c_\gamma\|A^{1/2-\gamma}\|\|A^\gamma\phi_n\|^2=c_\gamma\|A^{1/2-\gamma}\|$$

and so $\lim_{n\to\infty}\|A^\gamma u_n\|P(\phi_n\cdot\nabla)\phi_n=0$. Hence $\lim_{n\to\infty}\{A\phi_n+P(w\cdot\nabla)\phi_n+P(\phi_n\cdot\nabla)w\}=0$ strongly in L^2_σ. And so by the boundedness of $A^{\gamma-1}$ we have

$$\lim_{n\to\infty}\{A^\gamma\phi_n+A^{\gamma-1}[P(w\cdot\nabla)\phi_n+P(\phi_n\cdot\nabla)w]\}=0\,. \tag{6.3}$$

By the compactness of the inclusions $\mathcal{D}(A^\gamma)\to\mathcal{D}(A^{1/2})\to L^2_\sigma$ and the equality $\|\nabla\varphi\|=\|A^{1/2}\varphi\|$ for $\varphi\in\mathcal{D}(A^{1/2})$ we see there exists a subsequence $\{\phi_{n'}\}$ such that $\lim_{n'\to\infty}[P(w\cdot\nabla)\phi_{n'}+P(\phi_{n'}\cdot\nabla)w]$ exists strongly in L^2_σ and this implies that $\phi_\infty=\lim_{n'\to\infty}\phi_{n'}$ exists strongly in $\mathcal{D}(A^\gamma)$. Hence we have

$$\begin{cases} A^\gamma\phi_\infty+A^{\gamma-1}[P(w\cdot\nabla)\phi_\infty+P(\phi_\infty\cdot\nabla)w]=0 \\ \|A^\gamma\phi_\infty\|=1\,. \end{cases} \tag{6.3}$$

This implies that the operator B defined by $B\varphi=-A^{-1}[P(w\cdot\nabla)\varphi+P(\varphi\cdot\nabla)w]$ in L^2_σ with domain of definition $\mathcal{D}(A^{1/2})$ has 1 as its eigenvalue.

We now recall the explicit form of w.

$$w=\frac{w_\phi}{r}\frac{\partial}{\partial_\phi}=\rho\frac{1}{R_2^2-R_1^2}\Big[R_2^2\omega_2-R_1^2\omega_1+\frac{R_1^2R_2^2(\omega_1-\omega_2)}{r}\Big]\frac{\partial}{\partial_\phi}$$

where $\rho\geqq 0$, $\omega=(\omega_1, \omega_2)\in S^1$. We set $w=\rho w_\omega$. Then

$$B\varphi=-\rho A^{-1}[P(w_\omega\cdot\nabla)\varphi+P(\varphi\cdot\nabla)w_\omega]\equiv\rho\hat{B}\varphi\,.$$

By the compactness of the operator A^{-1} and the equality $\|\nabla\varphi\|=\|A^{1/2}\varphi\|$ for $\varphi\in\mathcal{D}(A^{1/2})$ we can prove that $\hat{B}$ can be extended to the whole space L^2_σ and the resulting operator is a compact operator. Hence by means of the Riesz-Schauder theorem we have

THEOREM 4. *The Couétte flow is an isolated solution in* $W_2^\gamma(G_h)$ $\left(r>\frac{3}{4}\right)$. *For a fixed* $(\omega_1, \omega_2)\in S^1$, *the isolatedness holds except for a countable set of values of* $\Omega_1^2+\Omega_2^2$.

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