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On a tensor product C^*-algebra associated with the free group on two generators

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Let G be the free group on two generators, and L^2 the Hilbert space of square summable complex valued functions on G. Let $\mathcal{L}$ and $\mathcal{R}$ be the C^*-algebras generated respectively by the left and right regular representations of G on L^2 and let $\mathfrak{A}$ be the C^*-algebra generated by $\mathcal{L}$ and $\mathcal{R}$ jointly. In [**1**] the authors provided a formula for computing the norm of certain operators in $\mathcal{L}$. In this paper the results of [**1**] are applied to the study of $\mathfrak{A}$, which may be regarded as a C^*-tensor product. (See the remark preceding Lemma 4.) We prove that $\mathfrak{A}$ contains the compact operators $\mathcal{C}$ in L^2 (Theorem 1) as its only closed two-sided ideal (Theorem 3), and that there is a derivation of $\mathfrak{A}$ into $\mathcal{C}$ which is not inner (Example 5). This investigation was suggested by Jun Tomiyama and Masamichi Takesaki at the Japan-U.S. Seminar on C^*-Algebras and Applications to Physics in Kyoto in May of 1974. Some related papers are listed in the references.

§1. Notation and Terminology.

Let S be a non-empty set. By $L^2(S)$ we mean the vector space of square summable complex valued functions on S. We prefer, however, to write the elements of $L^2(S)$ as (generally) infinite linear combinations, identifying the complex valued function f on S with the vector $\sum_{w\in S} f(w)w$. Thus we have

$$L^2(S)=\{\sum_{w\in S}\lambda_w w \mid \sum_{w\in S}|\lambda_w|^2<\infty\}.$$

$L^2(S)$ is a Hilbert space with inner product

$$(\sum_{w\in S}\lambda_w w,\ \sum_{w\in S}\mu_w w)=\sum_{w\in S}\lambda_w\bar{\mu}_w,$$

and resulting l_2 norm

$$\|\sum_{w\in S}\lambda_w w\|_2=(\sum_{w\in S}|\lambda_w|^2)^{\frac{1}{2}}.$$

By $L(S)$ we mean the subspace of $L^2(S)$ spanned by S; i.e., $L(S)$ consists of

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all finite linear combinations $\sum_{i=1}^{n} \alpha_i x_i$ with x_i in S.

Let G be the free group on two generators. For simplicity of reference we will abbreviate $L^2(G)$ to L^2 and $L(G)$ to L. G acts on L^2 from either the left or right. For x in G and $\Lambda = \sum_{w \in G} \lambda_w w$ in L^2, let

$$L_x(\Lambda) = \sum_{w \in G} \lambda_w x w\,, \qquad R_x(\Lambda) = \sum_{w \in G} \lambda_w w x^{-1}\,.$$

These are the left and right regular representations of G on L^2. Each extends by linearity to an action of L on L^2. For $A = \sum_{i=1}^{n} \alpha_i x_i$ in L,

$$L_A = \sum_{i=1}^{n} \alpha_i L_{x_i}\,, \qquad R_A = \sum_{i=1}^{n} \alpha_i R_{x_i}\,.$$

For each $A = \sum_{i=1}^{n} \alpha_i x_i$ in L, L_A and R_A are bounded operators on L^2, with operator norm satisfying

$$\|L_A\| = \|R_A\| \leqq \sum_{i=1}^{n} |\alpha_i|\,.$$

$\mathcal{L}$ and $\mathcal{R}$ denote the completions in operator norm of $\{L_A \mid A \in L\}$ and $\{R_A \mid A \in L\}$ respectively, and $\mathfrak{A}$ is the closed subalgebra of $\mathcal{B}$, the bounded operators on L^2, generated by $\mathcal{L} \cup \mathcal{R}$. $\mathfrak{A}$ is the principal object of study in this paper.

In L^2 we have a convolution operation. For $A = \sum_{x \in G} \alpha_x x$ and $\Lambda = \sum_{u \in G} \lambda_u u$,

$$A\Lambda = \sum_{w \in G} \Big(\sum_{x \in G} \alpha_x \lambda_{x^{-1}w}\Big) w\,.$$

$A\Lambda$ is always well defined in the sense that each coefficient is finite (in fact $\leqq \|A\|_2 \|\Lambda\|_2$ by the Schwarz inequality). But $A\Lambda$ is not generally in L^2. When $A\Lambda \in L^2$ for every $\Lambda \in L^2$ we say that A is a convolver of L^2.

Clearly each $A \in L$ is a convolver and

$$L_A(\Lambda) = A\Lambda$$

for each Λ in L^2. More generally, if $\varphi \in \mathcal{L}$, then $A = \varphi(e)$ is a convolver (e is the identity of G), and

$$\varphi(\Lambda) = A\Lambda$$

for each Λ in L^2. This follows from [7, p. 788-9] but may easily be verified directly. Let

$$\mathcal{U} = \{\varphi(e) \mid \varphi \in \mathcal{L}\}\,.$$

For each $A \in \mathcal{U}$ let L_A be the linear operator given by

$$L_A(\Lambda) = A\Lambda\,.$$

For $A \in \mathcal{U}$ define the operator norm of A by

$$\|A\| = \|L_A\| .$$

Then

$$\mathcal{L} = \{L_A \mid A \in \mathcal{U}\} ,$$

and the mapping $A \rightarrow L_A$ is an isometry of $\mathcal{U}$ (with operator norm) onto $\mathcal{L}$.

$\mathcal{U}$ represents $\mathcal{R}$ in a similar manner. For $A = \sum_{x \in G} \alpha_x x$ in L^2, let $\hat{A} = \sum_{x \in G} \alpha_x x^{-1}$. For $A \in \mathcal{U}$ define the operator R_A on L^2 by

$$R_A(\Lambda) = \Lambda \hat{A} .$$

Then

$$\mathcal{R} = \{R_A \mid A \in \mathcal{U}\}$$

and the mapping $A \rightarrow R_A$ is an isometry of $\mathcal{U}$ onto $\mathcal{R}$. (For $\theta \in \mathcal{R}$, $\theta = R_A$ where $A = \widehat{\theta(e)}$.)

Thus in a sense $\mathcal{U}$ is an abstract formulation of either regular representation of G on L^2. It also provides a convenient way to describe the algebra $\mathfrak{A}$, namely, as the closure in $\mathcal{B}$ of

$$\{\sum_{i=1}^{n} L_{A_i} R_{B_i} \mid A_i, B_i \in \mathcal{U}\} .$$

Tensor product spaces play an important role in our study of $\mathfrak{A}$. Let $L \otimes L$ denote the usual algebraic tensor product of L with itself. Each element of $L \otimes L$ can be expressed uniquely in the form

$$\sum_{i=1}^{n} \lambda_i x_i \otimes y_i$$

with $x_i, y_i \in G$. In particular for $A = \sum_{i=1}^{n} \alpha_i x_i$ and $B = \sum_{j=1}^{t} \beta_j y_j$ in L,

$$A \otimes B = \sum_{i=1}^{n} \sum_{j=1}^{t} \alpha_i \beta_j x_i \otimes y_j .$$

In $L \otimes L$ we have the usual l_2 norm. For $\Lambda = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$,

$$\|\Lambda\|_2 = (\sum_{i=1}^{n} |\lambda_i|^2)^{\frac{1}{2}} .$$

We note that $\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$ for each $A, B \in L$.

$L^2 \otimes L^2$ denotes the completion of $L \otimes L$ in the l_2 norm. This may be formally represented

$$L^2 \otimes L^2 = \{ \sum_{x, y \in G} \lambda_{x,y} x \otimes y \mid \sum_{x, y \in G} |\lambda_{x,y}|^2 < \infty\} ,$$

with

$$\| \sum_{x, y \in G} \lambda_{x,y} x \otimes y \|_2 = (\sum_{x, y \in G} |\lambda_{x,y}|^2)^{\frac{1}{2}} .$$

$L\otimes L$ acts on $L^2\otimes L^2$ from the left. For $u, v\in G$, and $\Lambda=\sum_{x,y\in G}\lambda_{x,y}x\otimes y$ in $L^2\otimes L^2$,

$$(u\otimes v)\Lambda=\sum_{x,y\in G}\lambda_{x,y}(ux)\otimes(vy).$$

This leads to the usual operator norm on $L\otimes L$ which we call the α-norm.

$$\|A\|_\alpha=\sup\{\|A\Lambda\|_2 \mid \Lambda\in L^2\otimes L^2,\ \|\Lambda\|_2=1\}.$$

This is a cross-norm on $L\otimes L$, meaning that

$$\|A\otimes B\|_\alpha\leqq\|A\|\|B\|$$

for each A, $B\in L$. (See [**9**, p. 111].) Thus we may extend by continuity to an action of $\mathcal{U}\otimes\mathcal{U}$ on $L^2\otimes L^2$. ($\mathcal{U}\otimes\mathcal{U}$ is the algebraic tensor product of $\mathcal{U}$ with itself.) $\mathcal{U}\otimes_\alpha\mathcal{U}$ denotes the closure of $\mathcal{U}\otimes\mathcal{U}$ in the algebra of all bounded operators on $L^2\otimes L^2$.

We are now prepared to prove some theorems.

§ 2. Results.

Recall that $\mathcal{C}$ denotes the algebra of compact operators on L^2.

THEOREM 1. $\mathcal{C}\subset\mathfrak{A}$.

To prove Theorem 1 it is sufficient to show that $\mathfrak{A}$ is irreducible and that $\mathcal{C}\cap\mathfrak{A}\neq\{0\}$. (See [**2**, 4.1.10].) The irreducibility of $\mathfrak{A}$ is a consequence of [**7**, pp. 788–9]. To complete the proof we will show that $\mathfrak{A}$ contains the orthogonal projection P of L^2 onto the one-dimensional subspace of L^2 spanned by e, the identity of G. To that end fix an integer $n\geqq 3$ and let X be a free subset of G of cardinality n (meaning that X freely generates a subgroup of G). Define $A\in\mathfrak{A}$ by

$$A=\frac{1}{n^2}\sum_{x\in X}\sum_{y\in X}L_{x^{-1}y}R_{x^{-1}y}.$$

We shall show $\|A-P\|<4/n$. Since $n\geqq 3$ is arbitrary, it follows that $P\in\mathfrak{A}$. The short proof of the following lemma was suggested to us by Marek Borejko. We first establish some notation.

Let $D=\{x^{-1}y: x, y\in X\}$ and let S be the subgroup of G generated by D. Let T be an abelian subgroup of S and let S/T denote the left coset space. Let ϕ be the representation of S on $L^2(S/T)$ defined by left multiplication and extend ϕ to $L(S)$. Let $B=\sum_{x\in X}\sum_{y\in X}x^{-1}y$ and $\bar{B}=\phi(B)$.

LEMMA 2. $\|\bar{B}\|\leqq 4(n-1)$.

PROOF OF LEMMA 2. Since T is abelian, the trivial representation on T is weakly contained (in the sense of [**3**]) in the left regular representation of T. By Theorem 4.2 of [**3**] and [**6**, p. 121] ϕ is weakly contained in the left regular representation of S. Thus $\|\bar{B}\| \leqq \|\sum_{x \in X} \sum_{y \in X} L_{x^{-1}y}\| = 4(n-1)$, where the last equality is Theorem IV. J of [**1**].

PROOF OF THEROEM 1. For each word w of G let $G_w = \{zwz^{-1} | z \in S\}$ and let $H_w = L(G_w)$. It is apparent that L is the direct sum of the distinct orthogonal subspaces H_w, each of which is invariant under $A-P$. Thus it suffices to show that $A-P$ restricted to H_w is of norm $< 4\sqrt{3}/n$ for each $w \in G$. Since $(A-P)(e) = 0$ we need only consider $w \neq e$, in which case $A-P = A$ on H_w.

Fix $w \neq e$ in G, and let $T = \{z \in S | zwz^{-1} = w\}$. In any free group elements which commute with a given non-trivial element also commute with each other. Thus T is an abelian subgroup of S. For each $y, z \in S$, $ywy^{-1} = zwz^{-1}$ if and only if $yT = zT$. Thus the mapping $\theta: H_w \to L(S/T)$ defined by $\theta(zwz^{-1}) = zT$ is an isometry. Moreover,

$$A | H_w = \frac{1}{n^2} \theta^{-1} \bar{B} \theta .$$

Thus by Lemma 2 we have

$$\|A | H_w\| = \frac{1}{n^2} \|\bar{B}\| < 4/n ,$$

and Theorem 1 is proved.

THEOREM 3. *$\mathcal{C}$ is the only proper non-zero closed two-sided ideal in $\mathfrak{A}$.*

We first need some notation and a lemma.

Define a linear mapping $\theta: \mathcal{U} \otimes \mathcal{U} \to \mathfrak{A}$ by

$$\theta(\sum_{i=1}^{n} A_i \otimes B_i) = \sum_{i=1}^{n} L_{A_i} R_{B_i} .$$

It is clear that

$$\begin{aligned} &\theta((A_1 + A_2) \otimes B - A_1 \otimes B - A_2 \otimes B) \\ &= \theta(A \otimes (B_1 + B_2) - A \otimes B_1 - A \otimes B_2) \\ &= \theta(\lambda(A \otimes B) - (\lambda A) \otimes B) \\ &= \theta(\lambda(A \otimes B) - A \otimes (\lambda B)) \\ &= 0 \end{aligned}$$

for all appropriate $A, A_1, A_2, B, B_1, B_2, \lambda$. Thus θ is well defined. Moreover $\mathcal{U}$ is central simple [8] and therefore $\mathcal{U} \otimes \mathcal{U}$ is simple [**4**, p. 91]. Then θ is an isomorphism. Thus θ induces a norm on $\mathcal{U} \otimes \mathcal{U}$ given by

$$\|\sum_{i=1}^{n} A_i \otimes B_i\| = \|\sum_{i=1}^{n} L_{A_i} R_{B_i}\| .$$

This is a C^*-cross norm on $\mathcal{U} \otimes \mathcal{U}$. But the α-norm on $\mathcal{U} \otimes \mathcal{U}$ is the minimal C^*-cross norm on $\mathcal{U} \otimes \mathcal{U}$ [**9**, p. 116]. Thus

$$\|\sum_{i=1}^{n} A_i \otimes B_i\|_\alpha \leqq \|\theta(\sum_{i=1}^{n} A_i \otimes B_i)\| .$$

Let φ be the inverse mapping of θ. Then φ is a $*$-isomorphism of a dense $*$-subalgebra of $\mathfrak{A}$ onto a dense $*$-subalgebra of $\mathcal{U} \otimes_\alpha \mathcal{U}$, and $\|\varphi\| = 1$. By [**2**, p. 18] φ extends to a $*$-homomorphism of $\mathfrak{A}$ onto $\mathcal{U} \otimes_\alpha \mathcal{U}$.

REMARK. Via the isomorphism θ, $\mathfrak{A}$ can be regarded as a C^*-tensor product of $\mathcal{U}$ with itself; i. e., $\mathcal{U}$ is the completion of $\mathcal{U} \otimes \mathcal{U}$ with respect to a C^*-cross norm on $\mathcal{U} \otimes \mathcal{U}$.

We now come to the heart of the argument.

LEMMA 4. *The kernel of* φ *is* $\mathcal{C}$.

PROOF OF LEMMA 4. $\mathcal{C}$ has no non-trivial closed ideals [**2**, 4.1] so either $\varphi(\mathcal{C}) = 0$ or φ is 1-1 on $\mathcal{C}$. $\mathcal{U}$ is simple and therefore $\mathcal{U} \otimes_\alpha \mathcal{U}$ is simple [**9**, p. 117]. Since $\varphi(\mathcal{C})$ is an ideal of $\mathcal{U} \otimes_\alpha \mathcal{U}$ and contains no unit, $\varphi(\mathcal{C}) = 0$.

Conversely, fix A in the kernel of φ and $\varepsilon > 0$. There is a $B = \sum_{i=1}^{n} \beta_i L_{x_i} R_{y_i}$ in $\mathfrak{A}$ with $\|A - B\| < \varepsilon$. Since $\varphi(A) = 0$ and $\|\varphi\| = 1$, we have $\|\varphi(B)\|_\alpha < \varepsilon$. To complete the proof we will find $C \in \mathcal{C}$ such that $\|B - C\| < \sqrt{2}\|\varphi(B)\|_\alpha$. Because $\mathcal{C}$ is closed and $\varepsilon > 0$ is arbitrary, this implies that $A \in \mathcal{C}$.

Before proceeding we must introduce some special notation associated with G as the free group on two generators, say a and b. Each element $w \neq e$ in G can be written uniquely in the form $w = w_1^{\varepsilon_1} w_2^{\varepsilon_2} \cdots w_t^{\varepsilon_t}$ where $w_1, \cdots, w_t \in \{a, b\}$, and $\varepsilon_1, \cdots, \varepsilon_t \in \{-1, 1\}$, and for each $1 \leqq i < t$ either $w_i \neq w_{i+1}$ or $\varepsilon_i = \varepsilon_{i+1}$. We call any such product a reduced product. If $w = w_1^{\varepsilon_1} \cdots w_t^{\varepsilon_t}$ is a reduced product then t is the length of w, denoted $|w|$. In particular $|e| = 0$. For each integer $i \geqq 1$, let

$$S_i = \{w \in G \mid |w| < i\} \quad \text{and} \quad T_i = \{w \in G \mid |w| \geqq i\} .$$

Let $w = w_1^{\varepsilon_1} \cdots w_t^{\varepsilon_t}$ be a reduced product. For each $0 \leqq i \leqq t$ let

$$f_i(w) = w_1^{\varepsilon_1} w_2^{\varepsilon_2} \cdots w_i^{\varepsilon_i} \qquad (f_0(w) = e)$$

and

$$g_i(w) = w^{-1} f_i(w) = w_t^{-\varepsilon_t} w_{t-1}^{-\varepsilon_{t-1}} \cdots w_{i+1}^{-\varepsilon_{i+1}} \qquad (g_t(w) = e) .$$

We note that $f_i, g_i : T_i \to G$ and for each $w \in T_i$ we have

$$f_i(w) g_i(w)^{-1} = w .$$

Returning now to the problem at hand, we must find a $C \in \mathcal{C}$ such that $\|B - C\| < \sqrt{2}\|\varphi(B)\|_\alpha$, where $B = \sum_{i=1}^{n} \beta_i L_{x_i} R_{y_i}$. Let

$$p = \max \{|x_i|, |y_i| \mid 1 \leqq i \leqq n\} .$$

Let P be the orthogonal projection of L^2 onto $L^2(S_{6p})$, and let $C = BP$. Then

C is certainly in $\mathcal{C}$. Note that $B-C=0$ on $L^2(S_{6p})$ and $B-C=B$ on $L^2(T_{6p})$. Thus

$$\|B-C\|=\sup\{\|B\Lambda\|_2 \mid \Lambda\in L(T_{6p}),\ \|\Lambda\|_2=1\}.$$

Now fix $\Lambda=\sum_{i=1}^{n}\lambda_i w_i$ in $L(T_{6p})$ with $\|\Lambda\|_2=1$. We may presume that the w_i are distinct. For each $z\in G$, let

$$I(z)=\{(i,j)\mid 1\leqq i\leqq n,\ 1\leqq j\leqq t,\ x_i w_j y_i^{-1}=z\},$$

and let $H=\{z\in G\mid I(z)\neq\emptyset\}$. H is finite. For each $z\in H$ let

$$\mu_z=\sum_{(i,j)\in I(z)}\beta_i\lambda_j.$$

Then

$$B\Lambda=\sum_{i=1}^{n}\beta_i\sum_{j=1}^{t}\lambda_j x_i w_j y_i^{-1}=\sum_{z\in H}\mu_z z,$$

so

$$\|B\Lambda\|_2=(\sum_{z\in G}|\mu_z|^2)^{\frac{1}{2}}.$$

We will now construct a $\Gamma\in L\otimes L$ with $\|\Gamma\|_2=1$ such that $\|B\Lambda\|_2\leqq\sqrt{2}\|\varphi(B)\Gamma\|_2$. It will then follow that $\|B-C\|\leqq\sqrt{2}\|\varphi(B)\|_\alpha$ as desired.

For each $z\in G$ let K_z be the subspace of $L\otimes L$ spanned by $\{u\otimes v\mid uv^{-1}=z\}$. Note that the K_z constitute a decomposition of $L\otimes L$ into orthogonal subspaces. For each $1\leqq j\leqq t$ define $\Gamma_j\in K_{w_j}$ by

$$\Gamma_j=\frac{1}{\sqrt{4p}}\sum_{k=p}^{5p-1}f_k(w_j)\otimes g_k(w_j)$$

and define $\Gamma\in L\otimes L$ by

$$\Gamma=\sum_{j=1}^{t}\lambda_j\Gamma_j.$$

Clearly $\|\Gamma_j\|_2=1$ for each j. Since the subspaces K_{w_j} are orthogonal, $\|\Gamma\|_2=(\sum_{j=1}^{t}|\lambda_j|^2)^{\frac{1}{2}}=\|\Lambda\|_2=1$.

Let $z\in H$ and $(i,j)\in I(z)$. Then

$$(x_i\otimes y_i)\Gamma_j=\frac{1}{\sqrt{4p}}\sum_{k=p}^{5p-1}(x_i f_k(w_j))\otimes(y_i g_k(w_j)).$$

Note that $x_i w_j y_i^{-1}=z$ and $|w_j|\geqq 6p$. Thus for each $p\leqq k\leqq 5p-1$, $x_i f_k(w_j)$ is an "initial portion" of z whose length depends only on the amount of cancellation in the product $x_i w_j$ when x_i and w_j are written as reduced products. This is independent of k for all $k\geqq p$. Thus there exists an integer $r(i,j)$ with $|r(i,j)|\leqq p$ such that

$$x_i f_k(w_j)=f_{k+r(i,j)}(z)$$

for all $p \leqq k \leqq 5p-1$. Also

$$y_i g_k(w_j) = g_{k+r(i,j)}(z)$$

for each k, since

$$\begin{aligned} y_i g_k(w_j) &= y_i w_j^{-1} f_k(w_j) \\ &= (y_i w_j^{-1} x_i^{-1})(x_i f_k(w_j)) \\ &= z^{-1} f_{k+r(i,j)}(z) \\ &= g_{k+r(i,j)}(z) . \end{aligned}$$

Then

$$(x_i \otimes y_i)\Gamma_j = \frac{1}{\sqrt{4p}} \sum_{k=p+r(i,j)}^{5p-1+(i,j)} f_k(z) \otimes g_k(z) .$$

In particular we note that $(x_i \otimes y_i)\Gamma_j \in K_z$ for each $(i, j) \in I(z)$. Now let Q_z denote the orthogonal projection of K_z onto the subspace spanned by $\{f_k(z) \otimes g_k(z) \mid 2p \leqq k \leqq 4p-1\}$, and let

$$\Delta_z = \frac{1}{\sqrt{4p}} \sum_{k=2p}^{4p-1} f_k(z) \otimes g_k(z) .$$

Then $\|\Delta_z\|_2^2 = \frac{1}{2}$ and $Q_z((x_i \otimes y_i)\Gamma_j) = \Delta_z$ for each $(i, j) \in I(z)$.

Finally we estimate $\|\varphi(B)\Gamma\|_2$.

$$\begin{aligned} \varphi(B)\Gamma &= \sum_{i=1}^{n} \beta_i \sum_{j=1}^{t} \lambda_j (x_i \otimes y_i)\Gamma_j \\ &= \sum_{z \in H} \Big(\sum_{(i,j) \in I(z)} \beta_i \lambda_j x_i \otimes y_i \Gamma_j \Big) . \end{aligned}$$

Since $x_i \otimes y_i \Gamma_j \in K_z$ for each $(i, j) \in I(z)$,

$$\begin{aligned} \|\varphi(B)\Gamma\|_2^2 &= \sum_{z \in H} \Big\| \sum_{(i,j) \in I(z)} \beta_i \lambda_j x_i \otimes y_i \Gamma_j \Big\|_2^2 \\ &\geqq \sum_{z \in H} \Big\| Q_z \Big(\sum_{(i,j) \in I(z)} \beta_i \lambda_j x_i \otimes y_i \Gamma_j \Big) \Big\|_2^2 \\ &= \sum_{z \in H} \Big\| \sum_{(i,j) \in I(z)} \beta_i \lambda_j \Delta_z \Big\|_2^2 \\ &= \sum_{z \in H} \|\mu_z \Delta_z\|_2^2 \\ &= \frac{1}{2} \sum_{z \in H} |\mu_z|^2 \\ &= \frac{1}{2} \|BA\|_2^2 . \end{aligned}$$

Thus

$$\|BA\|_2 \leqq \sqrt{2} \|\varphi(B)\Gamma\|_2 ,$$

and the lemma is proved.

The proof of Theorem 3 is now a triviality.

PROOF OF THEOREM 3. As noted earlier, $\mathcal{V}\otimes_\alpha\mathcal{V}$ is simple. The kernel of φ is simple and $\mathfrak{A}$ is irreducible. Thus the kernel of φ is the only non-trivial two-sided closed ideal of $\mathfrak{A}$.

Our final result is an example associated with the algebra $\mathfrak{A}$.

EXAMPLE 5. $\mathfrak{A}$ has a derivation which is not inner.

To construct this derivation we need an auxiliary operator on L^2. For each $w\in G$ define the real number β_w as follows. $\beta_e=1$. For $w\neq e$ there is a unique non-negative integer i such that $2^i\leqq|w|<2^{i+1}$, where $|w|$ is the length of w as previously defined. Define

$$\beta_w=\begin{cases}\dfrac{|w|}{2^i}-1 & \text{if } i \text{ is even}\\[2ex] 2-\dfrac{|w|}{2^i} & \text{if } i \text{ is odd}.\end{cases}$$

The numbers β_w have these properties.

(1) $0\leqq\beta_w\leqq1$ for all $w\in G$.

(2) $\beta_w=0$ if $|w|=2^i$ for some even i.

(3) $\beta_w=1$ if $|w|=2^i$ for some odd i.

(4) $|\beta_w-\beta_v|\leqq\dfrac{1}{2^i}$ if $|w|$, $|v|\geqq2^i$ and $||w|-|v||=1$.

Now define the linear operator B on L^2 by

$$B\Big(\sum_{w\in G}\lambda_w w\Big)=\sum_{w\in G}\lambda_w\beta_w w\,.$$

Clearly B is a bounded operator on L^2 with $\|B\|=1$, and $B^*=B$. To complete the construction we need two key facts about B which we present as lemmas.

LEMMA 6. $B\notin\mathfrak{A}$.

PROOF. Let $A=\sum_{i=1}^n\alpha_i L_{x_i}R_{y_i}$. We may presume without loss of generality that the pairs (x_i, y_i) are distinct and that $x_1=y_1=e$ (with α_1 possibly 0). We shall show that $\|A-B\|\geqq\dfrac{1}{2}$, thus establishing that $B\notin\mathfrak{A}$.

Let $2\leqq i\leqq n$. If either x_i or y_i is e then the other is not e and clearly $x_iwy_i^{-1}\neq w$ for every $w\in G$. Suppose $x_i, y_i\neq e$. Then there are at most two words w of any given length such that $x_iwy_i^{-1}=w$. To see this, suppose that $x_iwy_i^{-1}=w$ and $x_ivy_i^{-1}=v$. Then $x_i=wy_iw^{-1}=vy_iv^{-1}$. Then $v^{-1}w$ commutes with y_i. If $H=\{z\in G|zy_i=y_iz\}$ then $v^{-1}w\in H$ so $vH=wH$. Conversely if $x_iwy_i^{-1}=w$ and $vH=wH$ then $x_ivy_i^{-1}=v$. Thus $\{w\in G|x_iwy_i^{-1}=w\}$ is either empty or is a coset of the abelian subgroup H, and every such coset contains at most two words of any given length.

For each $t \geqq 1$ there are $4 \cdot 3^{t-1}$ words of length t. For all but at most $2n$ words w of length t, $x_i w y_i^{-1} \neq w$ for all $2 \leqq i \leqq n$. Thus we can choose words v, w such that $\beta_v = 0$, $\beta_w = 1$ and $x_i v y_i^{-1} \neq v$, $x_i w y_i^{-1} \neq w$ for all $2 \leqq i \leqq n$. Then

$$\|A-B\| \geqq \|(A-B)v\|_2 \geqq |\alpha_1 - \beta_v| = |\alpha_1|$$

and

$$\|A-B\| \geqq \|(A-B)w\|_2 \geqq |\alpha_1 - \beta_w| = |\alpha_1 - 1| .$$

Thus $\|A-B\| \geqq \frac{1}{2}$.

LEMMA 7. *$BA - AB \in \mathcal{C}$ for all $A \in \mathfrak{A}$.*

PROOF. Recall that a and b denote the free generators of G. Let $D = L_{a^{-1}} B L_a - B$. Recall that S_k denotes the finite dimensional subspace of L^2 spanned by $\{z \in G \mid |z| < k\}$, and T_k is its orthogonal complement. Let P_k denote the orthogonal projection of L^2 onto S_k. Then $DP_k \in \mathcal{C}$.

Let i be a positive integer and $k \geqq 2^i + 1$. Note that $D - DP_k = 0$ on S_k and $D - DP_k = D$ on T_k. Thus

$$\|D - DP_k\| = \sup \{\|DA\|_2 \mid A \in T_k,\ \|A\|_2 = 1\} .$$

For each $w \in T_k$, $Dw = L_{a^{-1}} B L_a w - Bw = (\beta_{aw} - \beta_w) w$. Moreover $|w|, |aw| \geqq 2^i$ and $||w| - |aw|| = 1$. Thus $|\beta_{aw} - \beta_w| \leqq 1/2^i$. Then for each $A \in T_k$,

$$\|DA\|_2 \leqq \|A\|_2 / 2^i ,$$

so

$$\|D - DP_k\| \leqq 1/2^i .$$

Thus $D \in \mathcal{C}$. Then

$$BL_a - L_a B = L_a D \in \mathcal{C}$$

and

$$BL_{a^{-1}} - L_{a^{-1}} B = -DL_{a^{-1}} \in \mathcal{C} .$$

Proceeding in similar fashion we can show that $BL_{x^\varepsilon} - L_{x^\varepsilon} B$ and $BR_{x^\varepsilon} - R_{x^\varepsilon} B \in \mathcal{C}$ for $x = a, b$ and $\varepsilon = 1, -1$. For any $u, v \in G$,

$$BL_{uv} - L_{uv} B = (BL_u - L_u B) L_v + L_u (BL_v - L_v B) .$$

Thus by the obvious induction on $|w|$, $BL_w - L_w B \in \mathcal{C}$ for every $w \in G$. Similarly $BR_w - R_w B \in \mathcal{C}$ for all w. Finally

$$BL_u R_v - L_u R_v B = (BL_u - L_u B) R_v + L_u (BR_v - R_v B) .$$

Thus $BL_u R_v - L_u R_v B \in \mathcal{C}$ for every $u, v \in G$. Then

$$B\left(\sum_{i=1}^{n} \alpha_i L_{u_i} R_{v_i}\right) - \left(\sum_{i=1}^{n} \alpha_i L_{u_i} R_{v_i}\right) B$$

is in $\mathcal{C}$ for all $u_i, v_i \in G$. By continuity, $BA - AB$ is in $\mathcal{C}$ for all $A \in \mathfrak{A}$.

PROOF OF EXAMPLE 5. Define $\varphi:\mathfrak{A}\rightarrow C$ by

$$\varphi(A)=BA-AB\,.$$

φ is clearly a derivation and $\varphi(\mathfrak{A})\subset C$ by Lemma 7. Suppose φ were an inner derivation. Then there would be a $C\in\mathfrak{A}$ such that $\varphi(A)=CA-AC$ for all $A\in\mathfrak{A}$. This would imply that $B-C$ commutes with each $A\in\mathfrak{A}$. Since $\mathfrak{A}$ is irreducible, $B-C$ would be a multiple of the identity, which is in $\mathfrak{A}$, and therefore $B\in\mathfrak{A}$, contradicting Lemma 6. Thus φ is not inner.

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