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An extension of a theorem of Myers

Dedicated to Professor S. Kashiwabara on his 60th birthday

By Katsuhiro SHIOHAMA

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§ 0. Introduction.

In this article we shall prove a simple extension of a theorem due to Myers [6], which states; if the sectional curvature of a connected and complete Riemannian manifold M has a positive lower bound then M is compact. He proved also that a connected and complete Riemannian manifold is compact if its Ricci curvature is bounded from below by a positive constant. The latter theorem has been generalized in several ways by Ambrose [1], Calabi [3] and Avez [2].

It is clear that a compact Riemannian manifold has a bounded volume. However as is stated below the converse does not hold in general. In this respect, M. Maeda [5] has shown that a connected and complete Riemannian manifold whose sectional curvature lies in an interval $(0, \alpha]$ is compact if and only if its volume is bounded. Recently Wu [8] proved that a complete, non-compact and orientable n-dimensional hypersurface in a Euclidean $(n+1)$-space has infinite volume if its sectional curvature is non-negative and all positive at one point. Our aim is to prove the following

THEOREM. *Let M be a complete and connected Riemannian n-manifold of non-negative sectional curvature. Then M is compact if and only if its volume is bounded.*

REMARK. In [9] Yau announced the same result by a different method. His proof is based on the existence of a non-trivial convex continuous function on complete open manifold of non-negative curvature, for which he refers to [4].

REMARK. For any positive ε, we can construct an n-dimensional complete, connected and non-compact hypersurface of revolution in a Euclidean $(n+1)$-space which has a bounded volume and its sectional curvature in $[-\varepsilon, \infty)$.

In order to prove our theorem it suffices to show that a connected, complete and non-compact Riemannian n-manifold M of non-negative sectional curvature has an infinite volume. Thus we may restrict our attention to such an M.

§ 1. Notations.

From now on let M be as above. A ray is by definition a geodesic γ: $[0, \infty)\to M$ such that any of its subarc is the shortest connection between the end points. In this article we always parametrize geodesics by arclengths. For a subset X of M, let $B_a(X) := \{y\in M; d(y, X)<a\}$, where $d: M\times M\to R$ is the distance function on M. For each point $x\in M$ we denote by $C(x)$ the cut locus of x. We denote by $\exp_x: M_x\to M$ the exponential map and by $i: M\to R$ the injectivity radius function, where M_x is the tangent space to M at x.

Let $\gamma: [0, \infty)\to M$ be a fixed ray with $\gamma(0)=p$, and $\gamma_t: [0, \infty)\to M$ be $\gamma_t(s)=\gamma(t+s)$. For each $t\geqq 0$, the set $C_{\gamma_t}=M-\bigcup_{t'>0} B_{t'}(\gamma_t(t'))$ is called a supporting half space and is a totally convex set (see [4]). We denote by H_{γ_t} its boundary set. Then we have a filtration $\{C_{\gamma_t}\}_{t\geqq 0}$ of totally convex sets (see Proposition 1.3 of [4]) satisfying; (1) $t_1\leqq t_2$ implies $C_{\gamma_{t_1}}\subset C_{\gamma_{t_2}}$ and $C_{\gamma_{t_1}}=\{y\in C_{\gamma_{t_2}}; d(y, H_{\gamma_{t_2}})\geqq t_2-t_1\}$. In particular $H_{\gamma_{t_1}}=\{y\in C_{\gamma_{t_2}}; d(y, H_{\gamma_{t_2}})=t_2-t_1\}$. (2) $\bigcup_{t\geqq 0} C_{\gamma_t}=M$. (3) $p\in H_{\gamma_0}$.

For each point $x\in M$ $(x\neq p)$ a ray $\sigma: [0, \infty)\to M$, $\sigma(0)=x$, is by definition asymptotic to γ if there are a divergent sequence $\{t_j\}$ and a family of minimizing geodesics $\{\sigma_{x,t_j}\}$ each emanating from x and ending at $\gamma(t_j)$ such that $\lim_{j\to\infty}\sigma'_{x,t_j}(0)=\sigma'(0)$.

§ 2. Properties of asymptotic rays.

LEMMA 2.1. *Let $\lambda: [0, a]\to M$ be a minimizing geodesic joining $p=\lambda(0)$ to $x=\lambda(a)$ and $\sigma: [0, \infty)\to M$ a ray asymptotic to γ emanating from x. Then* $\measuredangle(\sigma'(0), \lambda'(a))\leqq\measuredangle(\gamma'(0), \lambda'(0))$.

PROOF. Let $\{t_j\}$ and $\{\sigma_{x,t_j}\}$ be a divergent sequence and a shortest connections each σ_{x,t_j} joining x to $\gamma(t_j)$ such that $\sigma'(0)=\lim_{j\to\infty}\sigma'_{x,t_j}(0)$. To the geodesic triangle $(\lambda, \sigma_{x,t_j}, \gamma|[0, t_j])$ we apply the angle comparison theorem of Toponogov and let $j\to\infty$.

LEMMA 2.2. *For any $t_0>0$ and any two points $x_1, x_2\in H_{\gamma_{t_0}}$ let $\sigma_1, \sigma_2: [0, \infty)\to M$ be rays emanating from $x_i=\sigma_i(0)$ and asymptotic to γ, both obtained by the same divergent sequence $\{t_j\}$. Then the function $\rho(t) := d(\sigma_1(t), \sigma_2(t))$ is monotone increasing. Moreover $\rho|[t^*, \infty)$ is constant if and only if there exists a totally geodesic flat surface with boundary $\sigma_1([t^*, \infty))$, $\sigma_2([t^*, \infty))$ and a shortest geodesic segment joining $\sigma_1(t^*)$ to $\sigma_2(t^*)$.*

PROOF. From $x_i\in H_{\gamma_{t_0}}$, follows $\lim_{t\to\infty}[d(x_i, \gamma(t))-(t-t_0)]=0$. Let $\{\sigma_{i,t_j}\}$ be the family of shortest connections joining x_i to $\gamma(t_j)$ which satisfy $\sigma_i'(0)=$

$\lim_{j\to\infty}\sigma'_{i,t_j}(0)$. For each $t>t_0$, $\sigma_i(t)\in H_{\gamma_{t_0+t}}$ follows from $\lim_{j\to\infty}\sigma_{i,t_j}(t)=\sigma_i(t)$ and $\lim_{j\to\infty}[d(\sigma_{i,t_j}(t),\ \gamma(t_j))-(t_j-t)]=0$. To a geodesic triangle with vertices $\sigma_{1,t_j}(t)$, $\sigma_{2,t_j}(t)$ and $\gamma(t_j)$ we apply the angle comparison theorem to get (letting $j\to\infty$) both $\measuredangle(\beta'(0),\ \sigma'_1(t))\geqq\pi/2$ and $\measuredangle(-\beta'(\rho(t)),\ \sigma'_2(t))\geqq\pi/2$, where $\beta:[0,\ \rho(t)]\to M$ is a minimizing geodesic joining $\sigma_1(t)$ to $\sigma_2(t)$. Consider a 1-parameter variation $V:(-\varepsilon,\ \varepsilon)\times[0,\ \rho(t)]\to M$ of β such that $V(u,\ 0)=\sigma_1(t+u)$, $V(u,\ \rho(t))=\sigma_2(t+u)$ for $u\in(-\varepsilon,\ \varepsilon)$. Then the first variation formula implies $L'(0)\geqq 0$, where $L(u)$ is the length of the curve $v\to V(u,\ v)$. Clearly $L(u)\geqq\rho(t+u)$ for any $u\in(-\varepsilon,\ 0]$ and $L(0)=\rho(t)$. Therefore for $u\in(-\varepsilon,\ 0]$ we see $\rho(t)-\rho(t+u)\geqq 0$ because of $\liminf_{u\to 0}\dfrac{\rho(t)-\rho(t+u)}{-u}\geqq\lim_{u\to 0}\dfrac{L(0)-L(u)}{-u}\geqq 0$. The last statement follows from the convexity theorem due to Alexandrov (for detail see [7]).

LEMMA 2.3. *For any* $t_0>0$ *and* $x\in H_{\gamma_{t_0}}$, *let* $\sigma:[0,\ \infty)\to M$ *be a ray asymptotic to* γ *such that* $\sigma(0)=x$. *Then we have*

(2.1) *For any* $t>0$, σ_t *is the unique ray emanating from* $\sigma_t(0)$ *and asymptotic to* γ.

(2.2) $C_{\gamma_{t_0+t}}\subset C_{\sigma_t}$ *for any* $t>0$.

PROOF. Let $\{l_j\}$ and $\{\sigma_{t,j}\}$ be a divergent sequence and a family of shortest connections, each $\sigma_{t,j}$ joining $\sigma(t)$ to $\gamma(l_j)$, such that $\{\sigma'_{t,j}(0)\}$ has a limit, say v. Suppose $v\neq\sigma'(t)$. To a geodesic triangle $(\sigma_{x,l_j},\ \sigma_{t,j},\ \sigma|[0,\ t])$ we apply the angle comparison theorem. Looking at the angle at $\sigma(t)$ we can derive a contradiction from $\lim_{j\to\infty}[(l_j-t_0)-d(x,\ \gamma(l_j))]=\lim_{j\to\infty}[l_j-t_0-t-d(\sigma(t),\ \gamma(l_j))]=0$. Next we take a point $z\in M-C_{\sigma_t}$. Then there exist $\alpha>0$ and s_0 such that $(s_0-t)-d(z,\ \sigma(s_0))>\alpha$. Since $C_{\gamma_{s_0+t_0}}$ is convex and $\sigma(t)$ is in it, $\sigma_{t,j}$ has at least one intersection with $H_{\gamma_{s_0+t_0}}$. Let z_j be the intersection furthest from $\sigma(t)$. From (2.1) follows $\lim_{j\to\infty}z_j=\sigma(s_0)$ and hence there is j_0 such that $d(z_j,\ \sigma(s_0))<\alpha/3$, $|d(z_j,\ \sigma(t))-(s_0-t)|<\alpha/3$ and $|d(z_j,\ \gamma(l_j))-l_j+s_0+t_0|<\alpha/3$ for all $j>j_0$. Therefore we have $d(z,\ \gamma(l_j))<l_j-t_0-t-\alpha/3$ for $j>j_0$.

For each $t>0$ let $\mathcal{S}^r_{\gamma_t}$ be the set such that $\mathcal{S}^r_{\gamma_t}=\{\exp_{\gamma(t)}uX;\ 0\leqq u<r,\ X\in M_{\gamma(t)},\ \|X\|=1$ and $\langle X,\ \gamma'(t)\rangle=0\}$ and $\mathcal{S}_{\gamma_t}:=\mathcal{S}^\infty_{\gamma_t}$. Then we have the

LEMMA 2.4. $\mathcal{S}_{\gamma_t}$ *lies in the opposite side of* $\partial B_t(p)$ *with respect to* H_{γ_t}. *If* x *is a point on* $H_{\gamma_t}\cap\partial B_t(p)$ *then there exists a unique ray starting from* x *and asymptotic to* γ.

PROOF. From the argument in Lemma 2.2 we see that any asymptotic ray emanating from $x\in H_{\gamma_t}$ has an intersection with $\mathcal{S}_{\gamma_t}$, and hence $\mathcal{S}_{\gamma_t}\subset M-\mathrm{Int}\,C_{\gamma_t}$. It is evident that $\overline{B_t(p)}\subset C_{\gamma_t}$. Suppose $x\in H_{\gamma_t}\cap\partial B_t(p)$. Then $\lim_{j\to\infty}[(t_j-t)-d(x,\ \gamma(t_j))]=0$. Let $\lambda:[0,\ t]\to M$ be a minimizing geodesic joining $p=\lambda(0)$ to $x=\lambda(t)$. By applying the angle comparison theorem to the triangle with vertices

p, x and $\gamma(t_j)$, we see $\lambda'(t)=\sigma'(0)$. Thus the uniqueness is proved.

REMARK 1. Let y be any point on H_{γ_t} $(t>0)$. Then for any $s>0$ and any asymptotic ray σ starting from y, $\mathcal{S}_{\sigma_s}$ lies in the opposite side of $\partial B_s(y)$ with respect to $H_{\gamma_{t+s}}$.

REMARK 2. If γ is a ray on the complete simply connected hyperbolic space form of constant curvature -1, $\mathcal{S}_{\gamma_t}$ lies between $\partial B_t(p)$ and the horosphere H_{γ_t}, where $p=\gamma(0)$.

REMARK 3. Let $x\in H_{\gamma_t}\cap\mathcal{S}_{\gamma_t}$ and $\beta:[0, b]\to M$ be a geodesic segment joining $\gamma(t)$ to x such that $\beta([0, b])\subset\mathcal{S}_{\gamma_t}$. Then in the same manner as in the proof of Lemma 2.2, there exists a totally geodesic flat surface with boundary $\gamma([t, \infty))$, $\beta([0, b])$ and $\sigma([0, \infty))$, where σ is the asymptotic ray whose initial vector is obtained by the parallel translation of $\gamma'(t)$ along β.

§ 3. Fermi coordinates along a ray.

Let $E_1, \cdots, E_n$ be unit parallel fields along γ such that $E_n(t)=\gamma'(t)$ and $(E_1(t), \cdots, E_n(t))$ is an orthonormal basis for $M_{\gamma(t)}$. Set $W_0:=B_3(\gamma([0, 1]))$ and let κ_0 be the maximum of sectional curvature on $\overline{W}_0$ and i_0 the minimum of the injectivity radius on the set. Let $T_r(\gamma([0, 1])):=\bigcup_{0\leqq t\leqq 1}\mathcal{S}^r_{\gamma_t}$. Since $\gamma([0, \infty))\subset M-C(p)$ there is $r>0$ satisfying

$$B_r(\gamma([0, 1]))\subset M-C(p) \tag{3.1}$$

and

$$r\leqq \mathrm{Min}\left\{\pi/2\sqrt{\kappa_0},\ \frac{1}{2}i_0,\ 1\right\}. \tag{3.2}$$

If r satisfies the above conditions then so does any $r^*\in(0, r]$. Let $\hat{R}_0:=\sup\{r>0;\ r \text{ satisfies (3.1) and (3.2)}\}$. Then clearly $\hat{R}_0$ satisfies them. For any $r\in(0, \hat{R}_0]$, $T_r(\gamma([0, 1]))$ is the disjoint union of $\bigcup_{0\leqq t\leqq 1}\mathcal{S}^r_{\gamma_t}$ and hence we can introduce Fermi coordinates in $T_r(\gamma([0, 1]))$. Indeed, suppose $x=\exp_{\gamma(t_1)}uX=\exp_{\gamma(t_2)}vY$ holds for some $0\leqq t_1\leqq t_2\leqq 1$, $0<u$, $v<\hat{R}_0$ and unit vectors X, Y each perpendicular to $\gamma'(t_1)$, $\gamma'(t_2)$ respectively. The circumference of the triangle $(x, \gamma(t_1), \gamma(t_2))$ is less than $4\hat{R}_0\leqq 2\pi/\sqrt{\kappa_0}$, which enables us to draw the corresponding triangle on the standard n-sphere $S^n(\kappa_0)$ of constant curvature κ_0. From (3.2), each of the edges does not intersect the cut locus of the corresponding vertex. Thus by means of Rauch's theorem, we derive a contradiction from $\hat{R}_0\leqq \pi/2\sqrt{\kappa_0}$. We denote by $\varphi: T_{\hat{R}_0}(\gamma([0, 1]))\to R^n$ the coordinate map, i.e., $\varphi^{-1}(x_1, \cdots, x_n)=\exp_{\gamma(x_n)}\sum_{i=1}^{n-1}x_iE_i(x_n)$ for $0\leqq x_n\leqq 1$ and $\sum_{i=1}^{n-1}x_i^2<\hat{R}_0^2$. For $z\in T_{\hat{R}_0}(\gamma([0, 1]))$, we denote by $dM(z)$ and $dR^n(\varphi(z))$ the volume element of M and R^n at the point.

LEMMA 3.1. *For any* $x_1, x_2 \in \partial B_1(p) \cap T_{\hat{R}_0}(\gamma([0, 1]))$ *such that* $d(x_i, H_{r_1}) < \frac{1}{2}\hat{R}_0$ $(i=1, 2)$, *let* $\sigma_i : [-a_i, \infty) \to M$ $(i=1, 2)$ *be rays each asymptotic to* γ *and emanating from* $x_i = \sigma_i(-a_i)$, *where* $a_i \geqq 0$ *and* $\sigma_i(0) \in H_{r_1}$. *Then we have for any numbers* $r_i \in \left(0, \frac{1}{2}\hat{R}_0\right]$ *such that* $d(\sigma_1(0), \sigma_2(0)) \geqq r_1 + r_2$,

$$T_{r_1}(\sigma_1([0, 1])) \cap T_{r_2}(\sigma_2([0, 1])) = \emptyset . \tag{3.3}$$

PROOF. By means of Lemma 2.4 each $\sigma_i | [0, \infty)$ is the *unique* ray emanating from $\sigma_i(0)$ and asymptotic to γ. Hence the function $\rho(t) := d(\sigma_1(t), \sigma_2(t))$ is monotone increasing. Suppose there is a point q on $T_{r_1}(\sigma_1([0, 1])) \cap T_{r_2}(\sigma_2([0, 1]))$. Then $q = \exp_{\sigma_1(t_1)} r_1^* u_1 = \exp_{\sigma_2(t_2)} r_2^* u_2$ holds for some $0 < r_i^* < r_i$ and unit vectors $u_i \in M_{\sigma_i(t_i)}$, $\langle u_i, \sigma_i'(t_i) \rangle = 0$. Without loss of generality we may assume $t_2 \geqq t_1$. Clearly $q \in W_0$ follows from $d(q, p) < r_1^* + t_1 + \frac{1}{2}\hat{R}_0 + 1 < 3$. From $t_2 - t_1 = d(\sigma_1(t_1), H_{r_{t_2+1}}) \leqq r_1^* + r_2^*$, the circumference of the triangle $(\sigma_1(t_1), \sigma_2(t_2), \sigma_2(t_1))$ is less than $4(r_1^* + r_2^*) \leqq 4\hat{R}_0 \leqq 2\pi/\sqrt{\kappa_0}$. Since $\sigma_2([t_1, t_2]) \subset M - C(\sigma_1(t_1))$ the function $f : [0, \infty) \to R$ defined by $f(t) := d(\sigma_1(t_1), \sigma_2(t))$ is smooth on an open interval containing $[t_1, t_2]$. Suppose $f(t_2) < f(t_1)$. Then there is $t_3 \in (t_1, t_2)$ such that $f'(t_3) < 0$. Hence the edge angles of the triangle $(\sigma_1(t_1), \sigma_2(t_1), \sigma_2(t_3))$ at $\sigma_2(t_3)$ and $\sigma_2(t_1)$ are not smaller than $\pi/2$. Thus we derive a contradiction from Rauch's theorem. Therefore we have $f(t_2) \geqq f(t_1) = \rho(t_1) \geqq \rho(0) \geqq r_1 + r_2$. On the other hand $f(t_2) \leqq r_1^* + r_2^* < r_1 + r_2$ is a contradiction.

Now we take $R_0 \in (0, \hat{R}_0]$ in such a way that $T_{R_0}(\gamma([0, 1]))$ satisfies:

(3.4) For each $x \in \partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$ the minimizing geodesic $\lambda : [0, 1] \to M$ joining p to x has the extension $\lambda | [0, b]$ such that $\lambda(b) \in S_{r_1}^{\hat{R}_0}$ and moreover $\sphericalangle(\lambda'(0), \gamma'(0)) \leqq \pi/6$.

Then we have for any $x \in \partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$ and any ray $\sigma : [-a, \infty) \to M$ asymptotic to γ such that $x = \sigma(-a)$ $(a \geqq 0)$ and $\sigma(0) \in H_{r_1}$,

$$a < \frac{1}{2}\hat{R}_0 . \tag{3.5}$$

This follows immediately from $a = d(x, H_{r_1}) \leqq b - 1 < (1 + \hat{R}_0^2)^{\frac{1}{2}} - 1 < \frac{1}{2}\hat{R}_0^2 < \frac{1}{2}\hat{R}_0$.

PROOF OF THEOREM. For each point $x \in \partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))$ we choose an asymptotic ray $\sigma^x : [-a_x, \infty) \to M$ such that $\sigma^x(-a_x) = x$, $\sigma^x(0) \in H_{r_1}$, and we denote by $\lambda_x : [0, 1] \to M$ the shortest connection joining p to x. We denote by A_r^x the area of $\partial B_1(p) \cap T_r(\sigma^x([-a_x', 1]))$, where a_x' is chosen so that $a_x' = \mathrm{Min}\{a_x'' \in [a_x, 1] ; \partial B_1(p) \cap T_r(\sigma^x([-a_x'', 1]))$ is a connected neighborhood of x on $\partial B_1(p) \cap T_{R_0}(\gamma([0, 1]))\}$. We also denote by $A_r(\sigma^x(t))$ the area of $S_{\sigma_t^x}^r$. Then

$$(3.6)\qquad 1\leqq \lim_{r\to 0}\frac{A_r^x}{A_r(\sigma^x(-a_x))}\leqq\langle\lambda_x'(1),\ \sigma^{x\prime}(-a_x)\rangle^{-1}$$
$$\leqq\langle\lambda_x'(0),\ \gamma'(0)\rangle^{-1}.$$

Thus we can find (making use of Lemma 2.1) R_1^* such that

$$(3.7)\qquad 1\leqq\frac{A_r^x}{A_r(\sigma^x(-a_x))}\leqq 2\langle\lambda_x'(0),\ \gamma'(0)\rangle^{-1}\leqq 4/\sqrt{3}$$

holds for any $x\in\partial B_1(p)\cap T_{R0}(\gamma([0, 1]))$ and any $r\in(0, R_1^*]$.

Setting $W_1:=\bigcup_x B_3(\sigma^x([0, 1]))\cup W_0$, where the union is taken over all points on $\partial B_1(p)\cap T_{R0}(\gamma([0, 1]))$, we see from (3.5) that $d(p, w)\leqq 1+\frac{1}{2}\hat{R}_0+4$ for any $w\in W_1$. Thus $\overline{W}_1$ is compact and hence the sectional curvature and the injectivity radius take the maximum κ_1 and the minimum i_1 on $\overline{W}_1$. Therefore we can find $\hat{R}_1$ such that (a) $0<\hat{R}_1<R_1^*$, (b) $\hat{R}_1\leqq\mathrm{Min}\left\{\pi/2\sqrt{\kappa_1},\ \frac{1}{2}i_1\right\}$, (c) $T_{\hat{R}_1}(\sigma^x[0, 1]))\cap C(\sigma^x(0))=\emptyset$ for any $x\in\partial B_1(p)\cap T_{R0}(\gamma([0, 1]))$.

Next for any $\xi\in(0, 1)$, there exists $\hat{R}_1(\xi)\in(0, \hat{R}_1]$ such that for any $x\in\partial B_1(p)\cap T_{R0}(\gamma([0, 1]))$ and any $y\in\partial B_1(\sigma^x(0))\cap T_{\hat{R}_1(\xi)}(\sigma^x([0, 1]))$, if $\lambda_y:[0, 1]\to M$ is the shortest connection from $\sigma^x(0)$ to y, then

$$(3.8)\qquad \sphericalangle(\lambda_y'(0),\ \sigma^{x\prime}(0))<\frac{1}{2}\cos^{-1}\xi,$$

and

(3.9) λ_y has the extension $\lambda_y|[0, b]$, $(b>1)$ such that $\lambda_y(b)\in S_{\sigma_1^x}^{\hat{R}_1}$.

Let $D^{n-1}(r)$ be the volume of the $(n-1)$-dimensional disk of radius r in R^{n-1}. Then we can choose $R_1(\xi)\in(0, \hat{R}_1(\xi)]$ such that for any $x\in\partial B_1(p)\cap T_{R0}(\gamma[0, 1]))$, we have (making use of Fermi coordinates φ along $\sigma^x|[-a_x, 1]$)

$$(3.10)\qquad dM(z)/dR^n(\varphi(z))\geqq\xi\qquad\text{for any } z\in T_{R_1(\xi)}(\sigma^x([0, 1])),$$

$$(3.11)\qquad \xi\leqq A_r(\sigma^x(t))/D^{n-1}(r)\leqq 1\qquad\text{for any } t\in[-a_x, 1]$$

and

$$(3.12)\qquad \xi\leqq\text{area of }\partial B_1(\sigma^x(0))\cap T_r(\sigma^x([0, 1]))/A_r(\sigma^x(1))$$
$$\text{for any } r\in(0, R_1(\xi)].$$

We are now ready to prove the theorem. It follows from (3.7) and (3.11) that for any $r\in(0, R_1(\xi)]$

the area of $[\partial B_1(p)\cap T_r(\sigma^x([-a_x', 1]))]\leqq\frac{4}{\sqrt{3}}A_r(\sigma^x(-a_x))\leqq\frac{4}{\sqrt{3}}D^{n-1}(r)$.

By means of (3.10) the volume of $[T_r(\sigma^x([0, 1-R_0]))]\geqq(1-R_0)\cdot D^{n-1}(r)\cdot\xi$. Then we can find at most countably many points $\{x_i\}$ on $\partial B_1(p)\cap T_{R0}(\gamma([0, 1]))$, real numbers $\{r_i\}$, $0<r_i\leqq\frac{1}{2}R_1(\xi)$ and asymptotic rays $\sigma^i:[-a_i,\infty)\to M$ such that $\sigma^i(-a_i)=x_i$, $\sigma^i(0)\in H_{r_1}$, and $d(\sigma^i(0),\sigma^j(0))\geqq r_i+r_j$ for $i\neq j$, and such that

$$\partial B_1(p)\cap T_{R_0}(\gamma([0,1]))-\bigcup_{i=1}^{\infty}T_{r_i}(\sigma^i([-a_i',1]))\cap\partial B_1(p) \tag{3.13}$$

has measure zero on $\partial B_1(p)\cap T_{R_0}(\gamma([0,1]))$,

$$T_{r_i}(\sigma^i([0,1]))\cap T_{r_j}(\sigma^j([0,1]))=\emptyset \qquad \text{for } i\neq j. \tag{3.14}$$

Setting $c:=$ the area of $\partial B_1(p)\cap T_{R_0}(\gamma([0,1]))$, we have

$$\sum_{i=1}^{\infty}\mathrm{vol}\,[T_{r_i}(\sigma^i([0,1-R_0]))\geqq\frac{\sqrt{3}}{4}(1-R_0)\cdot c\cdot\xi. \tag{3.15}$$

Next, setting $W_2:=\bigcup_{i=1}^{\infty}\bigcup_{y}B_3(\sigma^y([0,1]))\cup W_1\cup W_0$ where y ranges over all points on $\partial B_1(\sigma^i(0))\cap T_{r_i}(\sigma^i([0,1]))$ and $\sigma^y:[-a_y,\infty)\to M$ is an asymptotic ray such that $\sigma^y(-a_y)=y$, $\sigma^y(0)\in H_{r_2}$, we see that $\overline{W}_2$ is compact. Hence by means of (3.8) there exists R_2^* such that

$$1\leqq\frac{A_r^y}{A_r(\sigma^y(-a_y))}\leqq\xi^{-1} \tag{3.7$'$}$$

for any $y\in\partial B_1(\sigma^i(0))\cap T_{r_i}(\sigma^i([0,1]))$ and any $r\in(0,R_2^*]$. Clearly $\hat{R}_2$ can also be found in the same way as $\hat{R}_1$, and $\hat{R}_0\geqq\hat{R}_1\geqq\hat{R}_2$. In the same manner as we find $\hat{R}_1(\xi)$, we can choose $\hat{R}_2(\xi)$ satisfying (3.8) and (3.9) for any $y\in\partial B_1(\sigma^i(0))\cap T_{r_i}(\sigma^i([0,1]))$, any $z\in\partial B_1(\sigma^y(0))\cap T_{\hat{R}_2(\xi)}(\sigma^y([0,1]))$ and $\lambda_z:[0,1]\to M$ joining $y=\lambda_z(0)$ to $z=\lambda_z(1)$. Thus $R_2(\xi)$ can be chosen so as to satisfy (3.10), (3.11) and (3.12) for any σ^y. Hence for each i there exist at most countably many points $\{x_{ij}\}$ on $\partial B_1(\sigma^i(0))\cap T_{r_i}(\sigma^i([0,1]))$, real numbers $\{r_{ij}\}$, $0<r_{ij}\leqq\frac{1}{2}R_2(\xi)$ and asymptotic rays $\sigma^{ij}:[-a_{ij},\infty)\to M$ such that $\sigma^{ij}(-a_{ij})=x_{ij}$, $\sigma^{ij}(0)\in H_{r_2}$ and $d(\sigma^{ij}(0),\sigma^{ik}(0))\geqq r_{ij}+r_{ik}$ for any j and $k\neq j$, and such that

$$\partial B_1(\sigma^i(0))\cap T_{r_i}(\sigma^i([0,1]))-\bigcup_{j=1}^{\infty}T_{r_{ij}}(\sigma^{ij}([-a_{ij}',1]))\cap B_1(\sigma^i(0)) \tag{3.13$'$}$$

has measure zero on $\partial B_1(\sigma^i(0))\cap T_{r_i}(\sigma^i([0,1]))$

and

$$T_{r_{ij}}(\sigma^{ij}([0,1]))\cap T_{r_{ik}}(\sigma^{ik}([0,1]))=\emptyset \qquad \text{for } j\neq k, \tag{3.14}$$

where $a_{ij}'\in(a_{ij},1)$ is taken so that $\partial B_1(\sigma^i(0))\cap T_{r_{ij}}(\sigma^{ij}([-a_{ij}',1]))$ is a connected open neighborhood of x_{ij} on $\partial B_1(\sigma^i(0))\cap T_{\hat{R}_i}(\sigma^i([0,1]))$. Here we note that each $\sigma^{ij}|[0,\infty)$ is the *unique* asymptotic ray starting from $\sigma^{ij}(0)\in H_{r_2}$ and $\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty}T_{r_{ij}}(\sigma^{ij}([0,1-R_0]))$ is a disjoint union. Thus we obtain

$$\sum_{j=1}^{\infty} \mathrm{vol}\,[T_{r_{ij}}(\sigma^{ij}([0,\,1-R_0]))] \geqq \sum_{j=1}^{\infty}(1-R_0)D^{n-1}(r_{ij})\cdot\xi \ \ \text{(by (3.10))}$$

$$\geqq \sum_{j=1}^{\infty}(1-R_0)\cdot\xi\cdot A_{r_{ij}}(\sigma^{ij}(-a_{ij})) \ \ \text{(by (3.11))}$$

$$\geqq \sum_{j=1}^{\infty}(1-R_0)\xi^2 A_{r_{ij}}^{x_{ij}} \ \ \text{(by (3.7)')}$$

$$\geqq (1-R_0)\cdot\xi^2\cdot\text{area of } [\bigcup_{j=1}^{\infty}\partial B_1(\sigma^i(0))\cap T_{r_{ij}}(\sigma^{ij}([-a'_{ij},\,1]))]$$

$$\geqq (1-R_0)\cdot\xi^2\cdot\text{area of } [\partial B_1(\sigma^i(0))\cap T_{r_i}(\sigma^i([0,\,1]))] \ \ \text{(by (3.13)')}$$

$$\geqq (1-R_0)\cdot\xi^3\cdot A_{r_i}(\sigma^i(1)) \ \ \text{(by (3.12))}$$

$$\geqq (1-R_0)\cdot\xi^4\cdot A_{r_i}(\sigma^i(-a_i))\,.$$

Thus we have

$$\text{(3.15)}' \qquad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathrm{vol}\,[T_{r_{ij}}(\sigma^{ij}([0,\,1-R_0]))] \geqq \frac{\sqrt{3}}{4}(1-R_0)\cdot c\cdot\xi^4\,.$$

Repeating this process N times, we obtain compact subsets $\overline{W}_1, \cdots, \overline{W}_N$ on which $R_1(\xi), \cdots, R_N(\xi)$ are well defined. Letting $i_1, \cdots, i_N$ natural numbers, we have at most countably many points $\{x_{i_1,\cdots,i_N}\}$ on $\partial B_1(\sigma^{i_1,\cdots,i_{N-1}}(0))\cap T_{r_{i_1,\cdots,i_{N-1}}}(\sigma^{i_1,\cdots,i_{N-1}}([0,\,1]))$, real numbers $\{r_{i_1,\cdots,i_N}\}$, $0<r_{i_1,\cdots,i_N}<\frac{1}{2}R_N(\xi)$ and asymptotic rays $\sigma^{i_1,\cdots,i_N}:[-a_{i_1,\cdots,i_N},\,\infty)\to M$ emanating from $x_{i_1,\cdots,i_N}$ such that $\sigma^{i_1,\cdots,i_N}(0)\in H_{r_N}$, e.t.c. Then the above computations imply

$$\text{(3.16)} \qquad \sum_{i_1=1}^{\infty}\cdots\sum_{i_N=1}^{\infty}\mathrm{vol}\,[T_{r_{i_1,\cdots,i_N}}(\sigma^{i_1,\cdots,i_N}([0,\,1-R_0]))]$$

$$\geqq \frac{\sqrt{3}}{4}(1-R_0)\cdot c\cdot\xi^{3N-2}\,.$$

Hence for any number ν, there exist $\xi\in(0,\,1)$ and N such that

$$\mathrm{vol}\,[M] > \frac{\sqrt{3}}{4}(1-R_0)c\cdot\xi\{1+\xi^3+\cdots+\xi^{3(N-1)}\}$$

$$= \frac{\sqrt{3}}{4}(1-R_0)\cdot c\cdot\xi\frac{1-\xi^{3N}}{1-\xi^3} > \nu\,.$$

Thus the proof is complete.

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Katsuhiro SHIOHAMA
Department of Mathematics
University of Tsukuba
Tsukuba, Ibaraki
Japan