J. Math. Soc. Japan
Vol. 27, No. 4, 1975

An embedding of l^2-manifold pairs in l^2

Dedicated to Professor Kiiti Morita for his 60th birthday

By Katsuro SAKAI

(Received Nov. 20, 1974)

§ 0. Introduction.

An *l^2-manifold* is a *separable* manifold modelled on the separable Hilbert space l^2. A closed subset K of a space X is a *Z-set* in X if for each non-empty homotopically trivial open set U, $U-K$ is non-empty and homotopically trivial. It is known that, in an l^2-manifold pair (M, N), N is a Z-set in M if and only if N is a collared closed subset of M (collared in the sense of M. Brown). Then (M, N) may be considered as a manifold-with-boundary, N being the boundary.

R. D. Anderson raised the problem in [1]: *Under what condition can M be embedded in l^2 such that N is the topological boundary under the embedding*? In this paper, we give an answer to this problem:

THEOREM. *Let (M, N) be an l^2-manifold pair with N a Z-set in M. If one of the following conditions is satisfied, there exists a closed embedding $h: M\to l^2$ such that $bd(h(M))=h(N)$.*

Condition I) *M is contractible (then M is homeomorphic ($\cong$) to l^2).*

Condition II) *$N=N_0\cup N_1$ where N_0 and N_1 are disjoint closed and N_0 is a deformation retract of M.*

The author wishes to thank Prof. Y. Kodama for helpful comments.

§ 1. Techniques of infinite-dimensional topology.

Let α be an open cover of a space X. Then a map $f: X\to X$ is said to be *α-limited* provided that for each $x\in X$ there exists some $U\in\alpha$ such that $x, f(x)\in U$. *If X is a metric space and K is a closed subset of X, then there exists an open cover α of $X-K$ such that each α-limited embedding $f: X-K\to X-K$ can be extended to an embedding $f': X\to X$ such that $f'|K=id$* (Lemma 3 of [2]). Such a cover α of $X-K$ is said to be *normal* (*with respect to K*).

In the following theorems, M and N are l^2-manifolds.

OET (OPEN EMBEDING THEOREM): *M can be embedded as an open subset of l^2* ([7] or Theorem 4 of [8]).

ST (STABILITY THEOREM): $M\times l^2\cong M$ ([4] or [12]).

CT (CLASSIFICATION THEOREM): *Every homotopy equivalence $f: M\to N$ is* *homotopic* ($\simeq$) *to a homeomorphism* (Corollary 3 of [**7**] or Theorem 6 of [**8**]).

HET (HOMEOMORPHISM EXTENSION THEOREM): *For homotopic Z-embeddings* $f \simeq g: X \to M$ (i.e. $f(X)$, $g(X)$ *are Z-sets in* M), *there exists a homeomorphism* $h: M \to M$ *such that* $f = hg$ ([**3**] or Theorem 1 and 2 of [**5**]).

TAE (THEOREM OF APPROXIMATION BY EMBEDDINGS): *Each continuous map* $f: M \to N$ *can be approximated by closed embeddings* $g: M \to N$ *and open embeddings* $h: M \to N$ *such that* $f \simeq g \simeq h$ (Corollary 6 of [**7**] and Theorem C of [**9**]).

TNZ (THEOREM OF NEGLIGIBILITY OF Z-SETS): *Any Z-set* K *in* M *is strongly negligible in* M (i.e. *for each open cover* α *of* M, *there exists an* α*-limited homeomorphism* $h: M \to M-K$) ([**2**] or Corollary of [**5**]).

§ 2. Proof of Theorem.

Our theorem is based on the following Henderson's result in [**6**] for open subset of Hilbert space:

LEMMA. *If* U *is an open subset of* l^2, *then* U *is homeomorphic to an open subset* V *of* l^2 *such that*

(a) $l^2 - V \cong l^2 - cl(V) \cong l^2$,

(b) $V \cong cl(V) \cong bd(V)$, *and*

(c) *there is an open embedding* $k: bd(V) \times R \to l^2$ *such that* $k(x, 0) = x$ *and* $k(bd(V) \times (-\infty, 0)) = V$, *where* R *is the real line.*

We shall give the proof in the following three cases.

I) The case that M is contractible.

In this case, $M \cong l^2$ by the CT. Since N can be considered as an open subset of l^2 (by the OET), then by the lemma of Henderson, there exists an open subset V of l^2 such that

(a) $l^2 - V \cong l^2 - cl(V) \cong l^2$,

(b) $V \cong cl(V) \cong bd(V) \cong N$, and

(c) $bd(V)$ is collared in $l^2 - V$ (then a Z-set in $l^2 - V$).

Let $f: M \to l^2 - V$ be a homeomorphism. Since $f(N)$ and $bd(V)$ are homeomorphic Z-sets in $f(M) = l^2 - V$ ($\cong l^2$), then there exists a homeomorphism $g: f(M) \to f(M) = l^2 - V$ such that $gf(N) = bd(V) = bd(l^2 - V) = bd(gf(M))$.

II)-i The case that N is a deformation retract of M.

This condition is equivalent to the condition which N is a strong deformation retract of M because M is an ANR (see [**10**]), that is, the inclusion $N \subset M$ is a homotopy equivalence.

By the OET, we can consider M as an open subset of l^2. Then there exist an open subset V of l^2 and an open embedding $k: bd(V) \times R \to l^2$ such that

(a) $V \cong cl(V) \cong bd(V) \cong M$,

(b) $k(x, 0)=x$ for each $x\in bd(V)$, and

(c) $k(bd(V)\times(-\infty, 0))=V$ (hence $k(bd(V)\times(-\infty, 0])=cl(V)$).

Let $f: M\to cl(V)$ be a homeomorphism and let $k_t: cl(V)\to cl(V)$ be defined by $k_t(x)=k(p_1k^{-1}(x), (1-t)\cdot p_2k^{-1}(x))$ where $p_1: bd(V)\times(-\infty, 0]\to bd(V)$ and $p_2: bd(V)\times(-\infty, 0]\to(-\infty, 0]$ are projections. By the TAE, there exists a closed embedding $g': N\to bd(V)$ such that $g'\simeq k_1f|N$, then $g'\simeq k_0f|N=f|N$ in $cl(V)$. Since $bd(V)$ is a Z-set in $cl(V)$ and $g'(N)$ is closed in $bd(V)$, then $g'(N)$ is a Z-set in $cl(V)$. By the HET, there exists a homeomorphism $g: cl(V)\to cl(V)$ such that $gf|N=g'$.

Let $d_t: M\to M$ be a strong deformation retraction of M to N. Since $k_1: cl(V)\to bd(V)$ is a retraction and $gfd_tf^{-1}g^{-1}: cl(V)\to cl(V)$ is a strong deformation retraction of $cl(V)$ to $gf(N)=g'(N)\subset bd(V)$, then $k_1gfd_tf^{-1}g^{-1}|bd(V): bd(V)\to bd(V)$ is a strong deformation retraction of $bd(V)$ to $gf(N)$, that is, the inclusion $gf(N)\subset bd(V)$ is a homotopy equivalence. By the CT, there exists a homeomorphism $h': gf(N)\to bd(V)$ which is homotopic to the inclusion. Since $gf(N)$ and $bd(V)$ are Z-sets in $cl(V)$, then there exists a homeomorphism $h: cl(V)\to cl(V)$ which is an extension of h' (by the HET). Then $hgf(N)=bd(V)$. We obtain a desired embedding $hgf: M\to l^2$.

II)-ii. The case that $N=N_0\cup N_1$ where N_0 and N_1 are disjoint closed and N_0 is a deformation retract of M.

Similarly as in the proof of the case II)-i, there exist an open subset V of l^2 and an open embedding $k: bd(V)\times R\to l^2$ such that

(a) $l^2-V\cong l^2-cl(V)\cong l^2$,

(b) $V\cong cl(V)\cong bd(V)\cong M$,

(c) $k(x, 0)=x$ for each $x\in bd(V)$, and

(d) $k(bd(V)\times(-\infty, 0])=cl(V)$.

Let $W=k(bd(V)\times[-1, 0])$, $W_0=k(bd(V)\times\{0\})=bd(V)$ and $W_1=k(bd(V)\times\{-1\})$. Since $l^2\times[0, 1]\cong l^2$ by Klee's theorem (Theorem III.1.3 of [**11**]), then by the ST, $W\cong bd(V)\times[-1, 0]\cong M\times[0, 1]\cong M\times l^2\times[0, 1]\cong M\times l^2\cong M$. Similarly as II)-i, there exists a homeomorphism $f: M\to W$ such that $f(N_0)=W_0$. We may assume that $f(N_1)\subset k(bd(V)\times[-1, -1/2])$. Let $r_t: [-1, 0]\to[-1, 0]$ be defined by

$$r_t(s)=\begin{cases}(1+t)\cdot s & \text{for} \quad -1/2\leqq s\leqq 0\\ (1-t)\cdot(s+1)-1 & \text{for} \quad -1\leqq s\leqq -1/2\end{cases}$$

and let $k_t: W\to W$ be defined by $k_t(x)=k(p_1k^{-1}(x), r_tp_2k^{-1}(x))$ where $p_1: bd(V)\times[-1, 0]\to bd(V)$ and p_2: $bd(V)\times[-1, 0]\to[-1, 0]$ are projections. By the TAE, there exists a closed embedding $g'': N_1\to W_1$ such that $g''\simeq k_1f|N_1$, then $g''\simeq k_0f|N_1=f|N_1$ in W. Let $g': N\to bd(W)=W_0\cup W_1$ be defined by $g'|N_0=f|N_0$ and $g'|N_1=g''$. Since $g'(N)$ and $f(N)$ are Z-sets in W and since

g' is homotopic to $f|N$, then there exists a homeomorphism $g: W \to W$ such that $gf|N = g'$ (by the HET), tnat is, $gf(N_0) = W_0$ and $gf(N_1)$ is a closed subset of W_1.

Since $W - gf(N)$ is open in W, then $W - gf(N)$ is an l^2-manifold and $W_1 - gf(N_1)$ is a Z-set in $W - gf(N)$. Let α be a normal cover of $W - gf(N)$ with respect to $gf(N)$. By the TNZ, there exists an α-limited homeomorphism $h': W - gf(N) \to (W - gf(N)) - (W_1 - gf(N_1))$. Since α is normal, then h' has the extension $h: W \to W - (W_1 - gf(N_1)) = (W - W_1) \cup gf(N_1)$ such that $h|gf(N) = id$. Let $H = (l^2 - V) \cup (W - W_1) \cup k(k^{-1}(gf(N_1)) \times (-\infty, -1])$ (note that $gf(N_1) \subset W_1 = k(bd(V) \times \{-1\})$). It is easy to see that each point of H has an open neighbourhood homeomorphic to l^2, that is, H is an l^2-manifold. Since H is homotopically equivalent to $l^2 - V \cong l^2$, then $H \cong l^2$ by the CT. Let $j: H \to l^2$ be a homeomorphism. We obtain a desired embedding $jhgf: M \to l^2$.

References

[1] R.D. Anderson, Some open questions in infinite-dimensional topology, Proc. of the 3-rd Prague Top. Symp. 1971, 29-35.

[2] R.D. Anderson, D.W. Henderson and J.E. West, Negligible subsets of infinite-dimensional manifolds, Compositio Math., 21 (1969), 143-150.

[3] R.D. Anderson and J.D. McCharen, On extending homeomorphisms to Fréchet manifolds, Proc. Amer. Math. Soc., 25 (1970), 283-289.

[4] R.D. Anderson and R. Schori, Factors of infinite-dimensional manifolds, Trans. Amer. Math. Soc., 142 (1969), 315-330.

[5] T.A. Chapman, Deficiency in infinite-dimensional manifolds, General Topology and Appl., 1 (1971), 263-272.

[6] D.W. Henderson, Open subsets of Hilbert space, Compositio Math., 21 (1969), 312-318.

[7] D.W. Henderson, Infinite-dimensional manifolds are open subsets of Hilbert space, Topology, 9 (1970), 25-33.

[8] D.W. Henderson, Corrections and extensions of two papers about infinite-dimensional manifolds, General Topology and Appl., 1 (1971), 321-327.

[9] D.W. Henderson and R. Schori, Topological classification of infinite-dimensional manifolds by homotopy type, Bull. Amer. Math. Soc., 76 (1970), 121-124.

[10] S-T. Hu, Theory of retracts, Wayne St. Univ. Press, 1965.

[11] V.L. Klee, Convex bodies and periodic homeomorphism in Hilbert space, Trans. Amer. Math. Soc., 74 (1953), 10-43.

[12] R. Schori, Topological stability for infinite-dimensional manifolds, Compositio Math., 23 (1971), 87-100.

Katsuro SAKAI
Department of Mathematics
Faculty of Education
Kagawa University
Takamatsu
Japan