J. Math. Soc. Japan
Vol. 27, No. 4, 1975

On normal connection of Kaehler submanifolds

By Bang-yen CHEN[1] and Huei-shyong LUE

(Received Nov. 8, 1974)

§ 1. Introduction.

Let M be an n-dimensional Riemannian manifold with Levi-Civita connection ∇. Then the curvature tensor R of M is given by $R(X, Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$ for any tangent vector fields X and Y. Let $E_1, \cdots, E_n$ be an orthonormal frame on M. Then the Ricci tensor $S(X, Y)$ and the scalar curvature ρ are given respectively by

$$S(X, Y)=\sum_{i=1}^{n} R(E_i, X; Y, E_i), \qquad \rho=\frac{1}{n}\sum_{i=1}^{n} S(E_i, E_i),$$

where $R(E_i, X; Y, E_i)=g(R(E_i, X)Y, E_i)$ and g is the metric tensor of M.

Let $x: M\to\tilde{M}$ be an isometric immersion of M into an m-dimensional Riemannian manifold $\tilde{M}^m$ with connection $\tilde{\nabla}$ and metric tensor $\tilde{g}$. Then the second fundamental form h of M in $\tilde{M}$ is given by $\tilde{\nabla}_X Y=\nabla_X Y+h(X, Y)$. Let N be a normal vector field of M in $\tilde{M}$, we write

$$\tilde{\nabla}_X N=-A_N(X)+D_X N,$$

where $-A_N(X)$ and $D_X N$ denote the tangential and normal components of $\tilde{\nabla}_X N$. Then we have $g(A_N(X), Y)=\tilde{g}(h(X, Y), N)$. D is called the *normal connection* of M in $\tilde{M}^m$. A local normal vector field $N\neq 0$ is called a *parallel section* if $DN=0$. Let $R^{\perp}$ be the curvature tensor associated with D, i.e., $R^{\perp}(X, Y)=D_X D_Y-D_Y D_X-D_{[X,Y]}$. Then the normal connection D is *flat* if $R^{\perp}$ vanishes identically. The normal connection is flat if the (real) codimension is one. If the (real) codimension is higher, then the normal connection is not flat in general.

In this paper, we shall study the normal connection of a Kaehler submanifold M in another Kaehler manifold $\tilde{M}$. In § 3, we shall prove that the normal connection of M in $\tilde{M}$ is flat only when the Ricci tensors of M and $\tilde{M}$ are equal on the tangent bundle of M. Moreover, we shall prove that if M and $\tilde{M}^m$ are both compact and $\tilde{M}$ is flat then the normal connection is flat when and only when the first Chern class $c_1(\nu)$ of the normal bundle ν is trivial. In

1) Work done under partial support by NSF Grant GP-36684.

§4, we shall prove that the complex projective line in a complex sphere $Q_n = SO(n+2)/SO(2)XSO(n)$ is the only Kaehler submanifold of Q_n whose normal bundle admits a parallel section. Moreover, the complex projective line in Q_2 is the only Kaehler submanifold in Q_n with flat normal connection.

§2. Basic formulas.

Let M^n be a complex n-dimensional Kaehler manifold with complex structure J and metric tensor g. Then the curvature tensor R of M^n satisfies the following formulas.

$$R(JX, JY) = R(X, Y), \qquad R(X, Y)JZ = JR(X, Y)Z \tag{2.1}$$

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \tag{2.2}$$

$$R(X, Y; Z, W) = R(Z, W; X, Y) = -R(Y, X; Z, W) = -R(X, Y; W, Z). \tag{2.3}$$

Let M^n be isometrically immersed in a complex m-dimensional Kaehler manifold $\tilde{M}^m$ as a complex submanifold. Let $\tilde{J}$, $\tilde{R}$ and $\tilde{g}$ be the complex structure, the curvature tensor and the metric tensor of $\tilde{M}^m$, respectively. Then the equations of Gauss and Ricci are given respectively by

$$\tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) + \tilde{g}(h(X, Z), h(Y, W)) - \tilde{g}(h(Y, Z), h(X, W)), \tag{2.4}$$

$$\tilde{R}(X, Y; N, N') = R^{\perp}(X, Y; N, N') - g([A_N, A_{N'}](X), Y), \tag{2.5}$$

where X, Y, Z, W are vector fields tangent to M^n and N, N' are vector fields normal to M^n. Moreover, we have

$$A_{\tilde{J}N} = JA_N \qquad \text{and} \qquad JA_N = -A_N J, \tag{2.6}$$

from which we have trace $h = 0$.

§3. Ricci tensor and normal connection.

Let M^n be a Kaehler submanifold in another Kaehler manifold $\tilde{M}^m$ as in §2. Suppose N be a parallel section in normal bundle ν. Then $R^{\perp}(X, Y)N = 0$ for all vector fields X, Y tangent to M^n. From the equation of Ricci, we find

$$\tilde{R}(X, Y; N, \tilde{J}N) = -g([A_N, A_{\tilde{J}N}](X), Y). \tag{3.1}$$

Hence, by using (2.6), we have

$$\tilde{R}(X, Y; N, \tilde{J}N) = 2g(JA_N^2(X), Y). \tag{3.2}$$

Let $H_B(X, N)$ denote the holomorphic bisectional curvature for the pair (X, N). Then we have

$$H_B(X, N)=\tilde{R}(X, JX; \tilde{J}N, N)/g(X, X)\tilde{g}(N, N).$$

From (3.2) we have the following Proposition.

PROPOSITION 1. *Let M^n be a Kaehler submanifold of a Kaehler manifold $\tilde{M}^m$. If there is a unit tangent vector X such that, for all unit normal vectors N, the holomorphic bisectional curvatures $H_B(X, N)$ are positive, then the normal bundle admits no parallel section.*

In [**5**] Smyth proved that the normal connection of a Kaehler hypersurface M^n in $\tilde{M}^{n+1}$ is flat if and only if $S(X, Y)=\tilde{S}(X, Y)$ for all X, Y in TM^n. In this section we shall prove the following.

THEOREM 2. *Let M^n be a Kaehler submanifold of a Kaehler manifold $\tilde{M}^m$. If the normal connection of M^n in $\tilde{M}^m$ is flat, then the Ricci tensors S and $\tilde{S}$ of M^n and $\tilde{M}^m$ satisfy the following relation: $S(X, Y)=\tilde{S}(X, Y)$ for all $X, Y \in TM^n$, TM^n being the tangent bundle of M^n.*

PROOF. Let M^n be an n-dimensional Kaehler submanifold of an m-dimensional Kaehler manifold $\tilde{M}^m$ with flat normal connection. Then, by Proposition 1.1 in [**1**, p. 99], there exist locally $2m-2n$ mutually orthogonal unit normal vector fields $N_1, N_2, \cdots, N_{2m-2n}$ such that $DN_r=0$ for all $r=1, 2, \cdots, 2m-2n$. Since $\tilde{M}^m$ is Kaehlerian, $\tilde{\nabla}\tilde{J}=0$, we see that $N_1, N_2, \cdots, N_{m-n}, \tilde{J}N_1, \cdots, \tilde{J}N_{m-n}$ are orthonormal parallel sections in the normal bundle. From the definition of Ricci tensors and the equation of Gauss, we have

$$S(X, Y)=\tilde{S}(X, Y)-\sum_{\alpha=1}^{m-n}\{\tilde{R}(N_\alpha, X; Y, N_\alpha)+\tilde{R}(\tilde{J}N_\alpha, X; Y, \tilde{J}N_\alpha)\} \tag{3.3}$$

$$-\sum_{A=1}^{2n}\tilde{g}(h(E_A, X), h(E_A, Y)),$$

where $E_1, \cdots, E_{2n}$ is an orthonormal frame of M^n. On the other hand, since N_α, $\alpha=1, \cdots, m-n$ are parallel, (3.2) implies

$$\tilde{R}(X, Y, N_\alpha, JN_\alpha)=2g(JA^2_{N_\alpha}(X), Y). \tag{3.4}$$

By (2.2) and (2.3), we have

$$\tilde{R}(X, JY; N_\alpha, \tilde{J}N_\alpha)=R(N_\alpha, JY; X, JN_\alpha)-\tilde{R}(N_\alpha, X; JY, \tilde{J}N_\alpha). \tag{3.5}$$

Hence, by using (2.1) and (2.3), we have

$$\tilde{R}(X, JY; N_\alpha, \tilde{J}N_\alpha)=-[\tilde{R}(\tilde{J}N_\alpha, X; Y, \tilde{J}N_\alpha)+\tilde{R}(N_\alpha, X; Y, N_\alpha)]. \tag{3.6}$$

Moreover, from (2.6), we may find

$$\sum_{A=1}^{2n} \tilde{g}(h(E_A, X), h(E_A, Y)) = 2\sum_{\alpha=1}^{m-n} g(A_\alpha^2(X), Y), \tag{3.7}$$

where $A_\alpha = A_{N_\alpha}$. Combining (3.3), (3.4), (3.6) and (3.7), we find $S(X, Y)=\tilde{S}(X, Y)$ for all vector fields X, Y tangent to M^n. This completes the proof.

A Kaehler manifold M^n is called an Einstein space if there exists a function ρ on M^n such that $S(X, Y)=\rho g(X, Y)$ for all tangent vectors X and Y. The function ρ is the *scalar curvature* of M^n. If $n>1$, ρ is constant.

A Kaehler manifold M^n is called a *complex space form* of holomorphic curvature c if the curvature tensor R satisfies

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\}. \tag{3.8}$$

From Theorem 2, we have immediately the following

THEOREM 3. *Let M^n be a Kaehler submanifold of a Kaehler-Einstein manifold $\tilde{M}^m$. If the normal connection is flat, then M^n is also Einstein. Moreover, M^n and $\tilde{M}^m$ have the same scalar curvature.*

Let M^n and $\tilde{M}^m$ be both compact. If $m>n+1$, then $S(X, Y)=\tilde{S}(X, Y)$ for all $X, Y \in TM^n$ seems to be too weak to conclude the flatness of the normal connection. However we have the following.

THEOREM 4. *Let M^n be a compact Kaehler submanifold of a compact Kaehler manifold $\tilde{M}^m$. Then we have*

(a) *$S(X, Y)=\tilde{S}(X, Y)$ for all $X, Y \in TM^n$ implies $c_1(\nu)=0$, where $c_1(\nu)$ denotes the first Chern class of the normal bundle ν.*

(b) *If $\tilde{M}^m$ is flat, then the normal connection is flat if and only if $c_1(\nu)$ is zero.*

PROOF. Let Φ be the fundamental 2-form on M^n, i.e., a closed 2-form defined by

$$\Phi(X, Y) = \frac{1}{2} g(JX, Y).$$

Let $\tilde{\gamma}$ (respectively, γ) be the Ricci 2-form of $\tilde{M}^m$ (respectively, M^n) i.e., a closed 2-form defined by

$$\tilde{\gamma}(\tilde{X}, \tilde{Y}) = \frac{1}{4\pi}\tilde{S}(\tilde{J}\tilde{X}, \tilde{Y}) \left(\text{respectively, } \gamma(X, Y) = \frac{1}{4\pi} S(JX, Y)\right). \tag{3.9}$$

Then the first Chern class $c_1(T\tilde{M}^m)$ of $T\tilde{M}^m$ is represented by $\tilde{\gamma}$ (respectively, $c_1(TM^n)$ of TM^n is represented by γ).

Now suppose that $S=\tilde{S}$ on TM^n, then, equation (3.9) implies $\tilde{\gamma}|_{M^n}=\gamma$. Hence we have

$$c_1(T\tilde{M}^m|_{M^n}) = c_1(TM^n). \tag{3.10}$$

On the other hand, since $T\tilde{M}^m|_{M^n}=TM^n\oplus\nu$, we find

$$c_1(T\tilde{M}^m|_{M^n})=c_1(TM^n)+c_1(\nu)\,. \tag{3.11}$$

Substituting (3.10) into (3.11), we get $c_1(\nu)=0$. This proves (a).

Now, suppose that $\tilde{M}^m$ is flat and $c_1(\nu)=0$. Then, by (3.9) and (3.11), we have $c_1(TM^n)=0$. Hence, there exists a 1-form η such that

$$\gamma=d\eta\,. \tag{3.12}$$

Let Λ be the operator of interior product by Φ. Applying Λ to both sides of (3.12) we have

$$n\rho=4\pi\Lambda d\eta\,. \tag{3.13}$$

Let δ be the codifferential operator and C the operator defined by $C\alpha=(\sqrt{-1})^{r-s}\,\alpha$, where α is a form of type (r, s). Then by using the well-known identity $d\Lambda-\Lambda d=\delta C-C\delta$, we have $\Lambda d\eta=-\delta C\eta$ since $d\Lambda\eta=C\delta\eta=0$. Thus we find

$$\int_{M^n}\rho*1=0\,. \tag{3.14}$$

On the other hand, the flatness of $\tilde{M}^m$ and the equation (3.3) imply

$$n\rho=-\|h\|^2\,,$$

where $\|h\|$ is the length of h. Hence, by using (3.14), we find $\rho=h=0$, from which we find $R^\perp=0$. The remaining part of this theorem is trivial. This proves the theorem.

§4. Kaehler submanifold in Q_n with parallel normal sections.

Let $P_{m+1}(c)$ be an $(m+1)$-dimensional complex projective space with holomorphic sectional curvature 4. Let $z_0, z_1, \cdots, z_{m+1}$ be homogeneous coordinates in $P_{m+1}(c)$. Then the complex sphere Q_m is a complex hypersurface of $P_{m+1}(c)$ defined by the equation

$$z_0^2+z_1^2+\cdots+z_{m+1}^2=0\,.$$

It is well-known that the Hermitian symmetric space $SO(m+2)/SO(2)\times SO(m)$ is complex analytically isometric to the complex sphere Q_m.

THEOREM 5. *Let M^n be an n-dimensional Kaehler submanifold of Q_m.*

(a) *If the normal bundle of M^n in Q_m admits a parallel section, then $n=1$, i.e., M^n is a holomorphic curve in Q_m.*

(b) *If the normal connection of M^n in Q_m is flat, then $n=1$ and $m=2$. Moreover, M^1 is a linear curve in $P_3(c)$.*

PROOF. (a) Let N be a parallel section in the normal bundle. Then, for any vector X tangent to M^n, equation (3.2) implies that

$$\tilde{R}(X, JX; N, \tilde{J}N)=2g(A_N(X), A_N(Y)) . \tag{4.1}$$

On the other hand, let $\tilde{A}$ be the operator associated with the second fundamental form of the immersion of Q_m into $P_{m+1}(c)$. Then (3.8) and the equation of Gauss imply that

$$\begin{aligned}\tilde{R}(X, JX; N, \tilde{J}N)=2\{\tilde{g}(X, \tilde{A}(N))^2+\tilde{g}(JX, \tilde{A}(N))^2\}\\ -2g(X, X)\tilde{g}(N, N) .\end{aligned} \tag{4.2}$$

Hence from (4.1) and (4.2) we get

$$\tilde{g}(X, \tilde{A}(N))^2+\tilde{g}(JX, \tilde{A}(N))^2=g(X, X)\tilde{g}(N, N)+g(A_N(X), A_N(X)) . \tag{4.3}$$

Since N has nonzero constant length, (4.3) implies that

$$\tilde{g}(X, \tilde{A}(N))^2+\tilde{g}(JX, \tilde{A}(N))^2\neq 0$$

for any nonzero vector X tangent to M^n. This is clearly impossible if $n\geqq 2$.

(b) If the normal bundle of M^n in Q_m is flat, then there exists $2m-2n$ local parallel sections. Hence, from part (a), we see that $n=1$. On the other hand, from Theorem 2, we have

$$S(X, X)=\tilde{S}(X, X) \tag{4.4}$$

for all vector X tangent to M^1. Since Q_m is Einstein with $\tilde{S}(X, X)=2mg(X, X)$. Hence, M^1 is of constant holomorphic sectional curvature $2m$. On the other hand, if we regard Q_m as a hypersurface in $P_{m+1}(C)$, then, by the equation of Gauss, we find that $m=2$, and M^1 is a linear curve in $P_3(C)$.

REMARK 1. Q_2 is complex analytically isometric to $P_1(C)\times P_1(C)$. Hence, if we regard $P_1(C)$ as a Kaehler submanifold of Q_2 in a natural way, then the normal connection of $P_1(C)$ in Q_2 is flat. Let Q_2 be imbedded in Q_m as a totally geodesic submanifold ($m>2$). Then the normal bundle of $P_1(C)$ in Q_m admits a parallel section.

REMARK 2. The normal bundle of Kaehler submanifolds in a complex space form of holomorphic sectional curvature $c\neq 0$ admits no parallel section (Chen-Ogiue [2]). (For hypersurface case, see Kon [3], Nomizu-Smyth [4] and Smyth [5].)

References

[1] B.-Y. Chen, Geometry of submanifolds, Marcel Dekker, New York, 1973.

[2] B.-Y. Chen and K. Ogiue, Some extrinsic results for Kaehler submanifolds, Tamkang J. Math., 4 (1973), 207–213.

[3] M. Kon, Kaehler immersions with trivial normal connection, TRU Math., **9** (1973), 29-33.
[4] K. Nomizu and B. Smyth, Differential geometry of complex hypersurface, II, J. Math. Soc. Japan, **20** (1968), 498-521.
[5] B. Smyth, Differential geometry of complex hypersurfaces, Ann. of Math., **85** (1967), 246-266.

Bang-yen CHEN
Department of Mathematics
Michigan State University
East Lansing, Michigan 48824
U. S. A.

Huei-shyong LUE
Department of Mathematics
Michigan State University
East Lansing, Michigan 48824
U. S. A.
Current address:
Department of Mathematics
National Tsing-Hua University
Hsinchu, Taiwan