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Real hypersurfaces in a complex projective space with constant principal curvatures II

By Ryoichi TAKAGI

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Introduction.

Let $P_n(\boldsymbol{C})$ be a complex projective space of complex dimension n ($\geqq 2$) with the metric of constant holomorphic sectional curvature. We proved in [3] that if M is a connected complete real hypersurface in $P_n(\boldsymbol{C})$ with two constant principal curvatures then M is a geodesic hypersphere. The purpose of this paper is to determine all real hypersurfaces in $P_n(\boldsymbol{C})$ ($n \geqq 3$) with three constant principal curvatures.

To state our result we begin with examples of real hypersurfaces in $P_n(\boldsymbol{C})$ with three constant principal curvatures. Let $\boldsymbol{C}^{n+1}$ be the space of $(n+1)$-tuples of complex numbers $(z_1, \cdots, z_{n+1})$, and π be the canonical projection of $\boldsymbol{C}^{n+1}-\{0\}$ onto $P_n(\boldsymbol{C})$. For an integer m ($2 \leqq m \leqq n-1$) and a positive number s we denote by $M'(2n, m, s)$ a real hypersurface in $\boldsymbol{C}^{n+1}$ defined by

$$\sum_{j=1}^{m} |z_j|^2 = s \sum_{j=m+1}^{n+1} |z_j|^2 , \qquad (z_1, \cdots, z_{n+1}) \neq 0 .$$

For a number t ($0<t<1$) we denote by $M'(2n, t)$ a real hypersurface in $\boldsymbol{C}^{n+1}$ defined by

$$|\sum_{j=1}^{n+1} z_j^2|^2 = t(\sum_{j=1}^{n+1} |z_j|^2)^2 , \qquad (z_1, \cdots, z_{n+1}) \neq 0 .$$

It will be shown that $M(2n-1, m, s) = \pi(M'(2n, m, s))$ ($n \geqq 3$) and $M(2n-1, t) = \pi(M'(2n, t))$ ($n \geqq 2$) are connected compact real hypersurfaces in $P_n(\boldsymbol{C})$ with three constant principal curvatures.

MAIN THEOREM. *If M is a connected complete real hypersurface in $P_n(\boldsymbol{C})$ ($n \geqq 3$) with three constant principal curvatures, then M is congruent to some $M(2n-1, m, s)$ or to some $M(2n-1, t)$, i.e., there exists an isometry g of $P_n(\boldsymbol{C})$ such that $g(M) = M(2n-1, m, s)$ or $g(M) = M(2n-1, t)$.*

In §1 we shall study general properties of a real hypersurface M in $P_n(\boldsymbol{C})$ with constant principal curvatures. In §3, on the assumption that M has three constant principal curvatures, we shall give equations which the almost contact structure of M must satisfy, which are summed up as Lemma 3.4.

§ 1. Preliminaries.

Hereafter let $P_n(C)$ $(n \geqq 2)$ be a complex projective space with the metric of constant holomorphic sectional curvature $4c$ and M be a real hypersurface in $P_n(C)$ with the induced metric. First we shall establish the structure equations of M (for details, cf. [2]). We denote by $F(M)$ the bundle of orthonormal frames of M. An element of $F(M)$ can be expressed as $u=(p: e_1, \cdots, e_{2n-1})$, where p is a point of M and $e_1, \cdots, e_{2n-1}$ is an ordered base of the tangent space $T_p(M)$ of M at p. Hereafter let the indices i, j, k, l run through from 1 to $2n-1$ unless otherwise stated. We denote by θ_i, θ_{ij} and Θ_{ij} the canonical 1-forms, the connection forms and curvature forms on $F(M)$ respectively. Then they satisfy

$$d\theta_i = -\sum_j \theta_{ij} \wedge \theta_j , \qquad \theta_{ij} + \theta_{ji} = 0 , \tag{1.1}$$

$$d\theta_{ij} = -\sum_k \theta_{ik} \wedge \theta_{kj} + \Theta_{ij} . \tag{1.2}$$

Let J be the natural complex structure of $P_n(C)$. For each $u=(p: e_1, \cdots, e_{2n-1}) \in F(M)$ there exists a unique vector e normal to M such that $\{e_1, \cdots, e_{2n-1}, e\}$ is an orthonormal frame of $P_n(C)$ at p compatible with the orientation determined by $\tilde{J}$. Let (J_{ij}, f_k) be the almost contact structure of M, i. e., $\tilde{J}(e_i) = \sum_j J_{ji} e_j + f_i e$. Then (J_{ij}, f_k) satisfies

$$\sum_k J_{ik} J_{kj} = f_i f_j - \delta_{ij} , \qquad \sum_j f_j J_{ji} = 0 , \tag{1.3}$$

$$\sum_i f_i^2 = 1 , \qquad J_{ij} + J_{ji} = 0 .$$

Let ϕ_i be 1-forms on $F(M)$ such that $\sum_i \phi_i \theta_i$ is the second fundamental form of M for e. Then the parallelism of $\tilde{J}$ implies

$$dJ_{ij} = \sum_k (J_{ik} \theta_{kj} - J_{jk} \theta_{ki}) - f_i \phi_j + f_j \phi_i , \tag{1.4}$$

$$df_i = \sum_j (f_j \theta_{ji} - J_{ji} \phi_j) .$$

The equation of Gauss is given by

$$\Theta_{ij} = \phi_i \wedge \phi_j + c\theta_i \wedge \theta_j + c \sum_{k,l} (J_{ik} J_{jl} + J_{ij} J_{kl}) \theta_k \wedge \theta_l . \tag{1.5}$$

The equation of Codazzi is given by

$$d\phi_i = -\sum_j \phi_j \wedge \theta_{ji} + c \sum_{j,k} (f_j J_{ik} + f_i J_{jk}) \theta_j \wedge \theta_k . \tag{1.6}$$

§ 2. Formulas.

In this section we assume that all principal curvatures $x_1, \cdots, x_{2n-1}$ (not necessarily distinct) of M for e are constant. We define a subbundle F' of $F(M)$ by

$$F' = \{u \in F(M) ;\ \phi_i = x_i \theta_i \text{ at } u\}$$

and restrict all differential forms under consideration to F'. Take the exterior derivative of $\phi_i = x_i \theta_i$. Then, using (1.1) and (1.6), we have

$$\sum_j \{(x_i - x_j)\theta_{ij} - c\sum_k (f_i J_{jk} + f_j J_{ik})\theta_k\} \wedge \theta_j = 0 .$$

From this and Cartan's lemma, we have

$$(x_i - x_j)\theta_{ij} = c\sum_k (A_{ijk} + f_i J_{jk} + f_j J_{ik})\theta_k , \tag{2.1}$$

where $A_{ijk} = A_{jik} = A_{ikj}$. In particular,

$$A_{ijk} = -f_i J_{jk} - f_j J_{ik} \quad \text{if} \quad x_i = x_j , \tag{2.2}$$

$$f_i J_{jk} = 0 \quad \text{if} \quad x_i = x_j = x_k . \tag{2.3}$$

In fact, from (2.2) we have

$$0 = A_{ijk} - A_{ikj} = f_k J_{ij} - f_j J_{ik} - 2 f_i J_{jk} \quad \text{if} \quad x_i = x_j = x_k . \tag{2.4}$$

Put $k = i$ in (2.4) to get $f_i J_{ij} = 0$. Hence multiply (2.4) by f_i to get $f_i J_{jk} = 0$.

In order to obtain a further formula let us take the exterior derivative of (2.1) for $x_i \neq x_j$. Then, using (1.1), (1.2), (1.4), (1.5), (2.1) and the identity

$$(x_i - x_j)\sum_k \theta_{ik} \wedge \theta_{kj} = \sum_k (x_i - x_k)\theta_{ik} \wedge \theta_{kj} + \sum_k \theta_{ik} \wedge (x_k - x_j)\theta_{kj} ,$$

we have

$$\begin{aligned} & c\sum_k dA_{ijk} \wedge \theta_k - c\sum_{k,l} (A_{ijk}\theta_{kl} + A_{ikl}\theta_{kj} + A_{jkl}\theta_{ki}) \wedge \theta_l \\ & - c\sum_l x_l (J_{li} J_{jk} + J_{lj} J_{ik})\theta_l \wedge \theta_k \\ & + c\sum_k (x_i f_j f_k \theta_i + x_j f_i f_k \theta_j) \wedge \theta_k - (x_i - x_j)(c + x_i x_j)\theta_i \wedge \theta_j \\ & - c(x_i - x_j)\sum_{k,l} (J_{ik} J_{jl} + J_{ij} J_{kl})\theta_k \wedge \theta_l = 0 . \end{aligned} \tag{2.5}$$

We want to pick out all coefficients of $\theta_i \wedge \theta_j$ in (2.5). To do this we need to know the coefficients of $\theta_i \wedge \theta_j$ in the following sum:

$$\begin{aligned} S = & \ dA_{iji} \wedge \theta_i + dA_{ijj} \wedge \theta_j \\ & - \sum_k (A_{ijk}\theta_{ki} + A_{iki}\theta_{kj} + A_{jki}\theta_{ki}) \wedge \theta_i \\ & - \sum_k (A_{ijk}\theta_{kj} + A_{ikj}\theta_{kj} + A_{jkj}\theta_{ki}) \wedge \theta_j . \end{aligned}$$

However, from (1.4) and (2.2), we have

$$dA_{iji}\wedge\theta_i+dA_{ijj}\wedge\theta_i=-2\sum_k(f_kJ_{ij}\theta_{ki}+f_iJ_{ik}\theta_{kj}-f_iJ_{jk}\theta_{ki})\wedge\theta_i$$
$$-2\sum_k(f_kJ_{ji}\theta_{kj}+f_jJ_{jk}\theta_{ki}-f_jJ_{ik}\theta_{kj})\wedge\theta_j$$
$$+2J_{ij}\sum_k x_kJ_{ki}\theta_k\wedge\theta_i+2J_{ji}\sum_k x_kJ_{kj}\theta_k\wedge\theta_i$$
$$+2(x_if_j^2-x_jf_i^2)\theta_i\wedge\theta_j\,.$$

Consider all terms in S involving θ_{ki} with $x_k=x_i$ and θ_{kj} with $x_k=x_j$. Then it can be easily checked that the sum of such terms vanishes, and so by (2.2) we can find all coefficients of $\theta_i\wedge\theta_j$ in S.

Then from (2.5) we have

$$2c^2\sum_k^{x_k\neq x_i}\frac{(A_{ijk}+f_kJ_{ij}+f_iJ_{kj})^2}{x_k-x_i}$$
$$-2c^2\sum_k^{x_k\neq x_j}\frac{(A_{ijk}+f_kJ_{ji}+f_jJ_{ki})^2}{x_k-x_j}$$
$$-6c(x_i-x_j)J_{ij}^2+3c(x_if_j^2-x_jf_i^2)-(x_i-x_j)(c+x_ix_j)=0 \tag{2.6}$$

if $x_i\neq x_j$.

§ 3. Lemmas.

Hereafter we assume that $\dim M=2n-1\geqq 5$ and that M has three constant principal curvatures x, y, and z. Let $m(x)$, $m(y)$ and $m(z)$ be the multiplicities of x, y and z respectively (so $m(x)+m(y)+m(z)=2n-1$). We shall make use of the following convention on the range of indices:

$$1\leqq a,\ b,\ c\leqq m(x)\,,\qquad m(x)+1\leqq r,\ s,\ t\leqq m(x)+m(y)\,,$$
$$m(x)+m(y)+1\leqq u,\ v,\ w\leqq 2n-1\,.$$

We define a subbundle F'' of F' by

$$F''=\{u\in F'\,;\ \phi_a=x\theta_a,\ \phi_r=y\theta_r,\ \phi_u=z\theta_u\ \text{at}\ u\}\,,$$

and restrict all differential forms under consideration to F''. For simplicity we shall promise that "$f_a=0$" means "$f_a=0$ for *all* a on a nonempty open set of F''", and "$f_a\neq 0$" means "$f_a\neq 0$ for *some* a on a nonempty open set of F''", etc.

LEMMA 3.1. *If* $f_af_rf_u\neq 0$ *then*

$$f_a\sum_r f_rJ_{rb}-f_b\sum_r f_rJ_{ra}=0\,,\qquad f_r\sum_u f_uJ_{us}-f_s\sum_u f_uJ_{ur}=0$$

and $f_u \sum_a f_a J_{av} - f_v \sum_a f_a J_{au} = 0$.

PROOF. From (2.3) we have $J_{ab} = J_{rs} = J_{uv} = 0$. By the symmetry of x, y and z it suffices to prove the first equation. From (1.3) we have

$$\sum_{a,r} f_a J_{ar}(f_r f_b) = \sum_{a,r,u} f_a J_{ar}(J_{ru} J_{ub})$$

$$= \sum_{a,r,u} f_a (J_{ar} J_{ru}) J_{ub} = \sum_a f_a^2 \sum_u f_u J_{ub} = \sum_a f_a^2 \sum_r f_r J_{br} .$$

Square above equation and sum over b to get

$$(\sum_{a,r} f_a f_r J_{ar})^2 \sum_b f_b^2 = (\sum_a f_a^2)^2 \sum_b (\sum_r f_r J_{br})^2 ,$$

which implies

$$\sum_{a>b} (f_a \sum_r f_r J_{rb} - f_b \sum_r f_r J_{ra})^2 = 0 . \qquad \text{Q. E. D.}$$

LEMMA 3.2. $f_a = 0$ *or* $f_r = 0$ *or* $f_u = 0$.

PROOF. Suppose that $f_a \neq 0$, $f_r \neq 0$ and $f_u \neq 0$. If we take the exterior derivative of $J_{ab} = 0$, then, using (1.3), (1.4), (2.1) and (2.2), we have

$$2c(y-z) \sum_u (f_a J_{bu} - f_b J_{au}) J_{uc} - (z-x)(x^2 - yx + 2c)(f_a \delta_{bc} - f_b \delta_{ac}) = 0 , \tag{3.1}$$

$$2c(y-z) \sum_r (f_a J_{br} - f_b J_{ar}) J_{rc} - (x-y)(x^2 - zx + 2c)(f_a \delta_{bc} - f_b \delta_{ac}) = 0 . \tag{3.2}$$

Similarly $dJ_{rs} = 0$ and $dJ_{uv} = 0$ give

$$2c(z-x) \sum_a (f_r J_{sa} - f_s J_{ra}) J_{at} - (x-y)(y^2 - zy + 2c)(f_r \delta_{st} - f_s \delta_{rt}) = 0 , \tag{3.3}$$

$$2c(z-x) \sum_u (f_r J_{su} - f_s J_{ru}) J_{ut} - (y-z)(y^2 - xy + 2c)(f_r \delta_{st} - f_s \delta_{rt}) = 0 , \tag{3.4}$$

$$2c(x-y) \sum_r (f_u J_{vr} - f_v J_{ur}) J_{rw} - (y-z)(z^2 - xz + 2c)(f_u \delta_{vw} - f_v \delta_{uw}) = 0 , \tag{3.5}$$

$$2c(x-y) \sum_a (f_u J_{va} - f_v J_{ua}) J_{aw} - (z-x)(z^2 - yz + 2c)(f_u \delta_{vw} - f_v \delta_{uw}) = 0 . \tag{3.6}$$

Put $c = b$ in (3.1) and (3.2) and sum over b to get

(3.7) $$2c(y-z)(\sum_r f_r^2-\sum_{a,u} J_{au}^2)-(z-x)(x^2-yx+2c)(m(x)-1)=0\,,$$

(3.8) $$2c(y-z)\sum_u f_u^2-\sum_{a,r} J_{ar}^2)-(x-y)(x^2-zx+2c)(m(x)-1)=0$$

since $-\sum_{u,b} f_b J_{au} J_{ub}=\sum_{r,u} f_r J_{au} J_{ur}=f_a \sum_r f_r^2$ etc. Similarly from (3.3)-(3.6) we have

(3.9) $$2c(z-x)(\sum_u f_u^2-\sum_{a,r} J_{ar}^2)-(x-y)(y^2-zy+2c)(m(y)-1)=0\,,$$

(3.10) $$2c(z-x)(\sum_a f_a^2-\sum_{r,u} J_{ru}^2)-(y-z)(y^2-xy+2c)(m(y)-1)=0\,,$$

(3.11) $$2c(x-y)(\sum_a f_a^2-\sum_{r,u} J_{ru}^2)-(y-z)(z^2-xz+2c)(m(z)-1)=0\,,$$

(3.12) $$2c(x-y)(\sum_r f_r^2-\sum_{a,u} J_{au}^2)-(z-x)(z^2-yz+2c)(m(z)-1)=0\,.$$

These equations (3.7)-(3.12) imply that $m(x)=m(y)=m(z)=1$ or $m(x)$, $m(y)$, $m(z)\geqq 2$, but the former is not the case.

Now multiply (3.1) (resp. (3.2)) by J_{cr} (resp. J_{cu}) and sum over c. Then by Lemma 3.1 we have

(3.13) $$(x^2-yx+2c)(f_a J_{br}-f_b J_{ar})=0\,,$$

(3.14) $$(x^2-zx+2c)(f_a J_{bu}-f_b J_{ar})=0\,.$$

Similarly from (3.3) and (3.5), we have

(3.15) $$(y^2-zy+2c)(f_r J_{su}-f_s J_{ru})=0\,,$$

(3.16) $$(z^2-xz+2c)(f_u J_{va}-f_v J_{ua})=0\,.$$

Since $x^2-yx+2c\neq 0$ or $x^2-zx+2c\neq 0$, we may assume $x^2-yx+2c\neq 0$. Then (3.2) and (3.13) imply $x^2-zx+2c=0$ and so $z^2-xz+2c\neq 0$. Hence (3.6) and (3.16) imply $z^2-yz+2c=0$ and so $y^2-zy+2c\neq 0$. Hence (3.4) and (3.15) imply $y^2-xy+2c=0$, which contradicts the previous two equations. Q.E.D.

Owing to Lemma 3.2, we may set $f_a=0$.

LEMMA 3.3. *$f_r=0$ or $f_u=0$.*

PROOF. If we take the exterior derivative of $f_a=0$, then, using (1.3), (1.4), (2.1) and (2.2), we have

(3.17) $$\frac{c}{z-x}\sum_u f_u A_{aru}=-\Big(\frac{2c}{y-x}\sum_s f_s^2+\frac{c}{z-x}\sum_v f_v^2+y\Big)J_{ar}+\frac{c}{y-x}f_r\sum_s f_s J_{sa}\,,$$

(3.18) $$\frac{c}{y-x}\sum_r f_r A_{aru}=-\Big(\frac{c}{y-x}\sum_s f_s^2+\frac{2c}{z-x}\sum_v f_v^2+z\Big)J_{au}+\frac{c}{z-x}f_u\sum_v f_v J_{va}\,.$$

Cancel A_{aru} from (3.17) and (3.18) to get

(3.19) $$\sum_r f_r J_{ra}\left\{\frac{3c(x-z)}{y-x}\sum_r f_r^2+\frac{3c(x-y)}{z-x}\sum_u f_u^2+yx+zx-2yz-c\right\}=0$$

since $\sum_r f_r^2+\sum_u f_u^2=1$. Here we assert $\sum_r f_r J_{ra}=0$. In fact, if $\sum_r f_r J_{ra}\neq 0$, then it follows from (3.19) and the relation $\sum_r f_r^2+\sum_u f_u^2=1$ that $\sum_r f_r^2$ is constant. Taking account of the coefficient of θ_a in $\sum_r f_r df_r=0$, we have $yx+zx-2yz-c=-3(x-y)(x-z)$, which contradicts (3.19). Thus our assertion was proved. Hence

$$0=\sum_{a,r,u} f_r f_u (J_{ra}J_{au})=\sum_r f_r^2 \sum_u f_u^2 . \qquad \text{Q.E.D.}$$

Owing to Lemma 3.3, we may set $f_a=f_r=0$. Now, from $df_a=df_r=0$, we find

(3.20) $$(x^2-zx-c)J_{ab}=0\,,$$

(3.21) $$c\sum_u f_u A_{aru}=-(c+zy-xy)J_{ar}\,,$$

(3.22) $$(z^2-xz+2c)J_{au}=0\,,$$

(3.23) $$(y^2-zy-c)J_{rs}=0\,,$$

(3.24) $$c\sum_u f_u A_{aru}=-(c+zx-yx)J_{ra}\,,$$

(3.25) $$(z^2-yz+2c)J_{ru}=0\,.$$

From (3.21) and (3.24), we have

(3.26) $$(zx+zy-2xy+2c)J_{ar}=0\,.$$

There are two possibilities as follows.

LEMMA 3.4. (I) $J_{ar}=J_{au}=J_{ru}=0$, $J_{ab}\neq 0$, $J_{rs}\neq 0$: $f_a=f_r=0$, $f_u\neq 0$: *both* $m(x)$ *and* $m(y)$ *are even,* $m(z)=1$: $x^2-zx-c=0$, $y^2-zy-c=0$, *or* (II) $J_{ab}=J_{rs}=J_{au}=J_{ru}=0$, $J_{ar}\neq 0$: $f_a=f_r=0$, $f_u\neq 0$: $m(x)=m(y)\geqq 2$, $m(z)=1$: $4c+zx+zy=0$, $c+xy=0$, *in particular,* $(x^2-zx-c)(y^2-zy-c)\neq 0$.

PROOF. First let $x^2-zx-c=0$. Then (3.22) and (3.26) imply $J_{ar}=J_{au}=0$. Taking account of the coefficient of θ_r in $dJ_{au}=0$ we have $\sum_b J_{ab}A_{bru}=0$. This shows $A_{aru}=0$ since $\sum_c J_{ac}J_{bc}=\delta_{ab}$. Put $i=a$ and $j=r$ (resp. $i=r$ and $j=u$) in (2.6) to get $c+xy=0$ (resp. $J_{ru}=0$). Hence $y^2-zy-c=0$. Moreover put $i=a$ and $j=u$ in (2.6). Then, using $x^2-zx-c=0$, we have $m(z)=1$. Since the rank of J is equal to $2n-2$, both matrices (J_{ab}) and (J_{rs}) have maximal rank and so both $m(x)$ and $m(y)$ are even.

Next let $x^2-zx-c\neq 0$. Then (3.20) implies $J_{ab}=0$. We assert $zx+zy-2xy+2c=0$. In fact, if not so, then (3.26) implies $J_{ar}=0$ and so $J_{au}\neq 0$. Hence (3.22) implies $z^2-xz+2c=0$. Since (3.5) was led on the assumption that $J_{uv}=0$, it remains valid for our situation and implies $(f_u J_{vr}-f_v J_{ur})J_{ru}=0$.

Multiply this equation by f_v and sum over v to get $J_{ru}=0$. Then (3.23) implies $y^2-zy-c=0$ since $J_{rs}\neq 0$. On the other hand, taking account of the coefficient of θ_v in $dJ_{ru}=0$, we have $A_{aru}=0$. Put $i=a$ and $j=r$ in (2.6) to get $c+xy=0$, which contradicts the previous two equations. Thus our assertion was proved. Now since $(z^2-xz+2c)(z^2-yz+2c)\neq 0$, (3.22) and (3.25) imply $J_{au}=J_{ru}=0$. Taking account of the coefficient of θ_a in $dJ_{ab}=0$, we have

$$\sum_s J_{rs}A_{asu}=0\,. \tag{3.27}$$

Moreover $dJ_{au}=0$ gives

$$c\sum_s J_{as}(A_{bsu}+f_uJ_{sb})+x(y-z)f_u\delta_{ab}=0\,. \tag{3.28}$$

Multiply (3.28) by J_{ra} and sum over a. Then, using (3.27), we have

$$cA_{aru}=(c+xz-xy)f_uJ_{ar}\,. \tag{3.29}$$

Put $i=a$ and $j=r$ in (2.6). Then, using $zx+zy-2xy+2c=0$ and (3.29), we have $c+xy=0$. Put $i=a$ and $j=u$ in (2.6) and sum over u. Then, using $y=-c/x$ and $z=-4cx/(x^2-c)$, we have $m(z)=1$. Put $i=r$ and $j=u$ in (2.6). Then, using $\sum_a J_{ra}^2+\sum_s J_{rs}^2=1$, we have $J_{rs}=0$. Since the rank of J is equal to $2n-2$, we see $m(x)=m(y)$. The last equation is trivial. Q. E. D.

REMARK. We used the assumption $\dim M\geqq 5$ only to obtain Lemma 3.2. If M is a 3-dimensional real hypersurface in $P_2(\boldsymbol{C})$ with three constant principal curvatures then we have $J_{12}=\varepsilon f_3$, $J_{31}=\varepsilon f_2$ and $J_{23}=\varepsilon f_1$ for $\varepsilon=\pm 1$. The author could not clarify whether on such a hypersurface $f_1f_2f_3\neq 0$ or not.

§ 4. A proof of Main Theorem.

Let $S^m(1/r^2)$ denote the hypersphere in a Euclidean $(m+1)$-space $\boldsymbol{R}^m$ of radius r centered at the origin. We naturally identify $\boldsymbol{C}^{n+1}$ with $\boldsymbol{R}^{2n+2}$ with a complex structure I. In the following we shall consider a hypersurface $M'=\pi^{-1}(M)\cap S^{2n+1}(c)$ in $S^{2n+1}(c)$. Let $\{e_1, \cdots, e_{2n-1}, e\}$ be an orthonormal frame of $P_n(\boldsymbol{C})$ at $p\in M$ compatible with the orientation determined by $\tilde{J}$ such that $(p: e_1, \cdots, e_{2n-1})\in F(M)$ as in § 1 and let $\theta_1, \cdots, \theta_{2n-1}$ be the coframe dual to $e_1, \cdots, e_{2n-1}$. Let $\{e'_1, \cdots, e'_{2n-1}, e'_{2n}, e'\}$ be an orthonormal frame of $S^{2n+1}(c)$ at $p'\in M'$ such that $\pi_* e'_i=e_i$, $\pi_* e'_{2n}=0$ and $\pi_* e'=e$ and let $\theta'_1, \cdots, \theta'_{2n}$ be the coframe dual to $e'_1, \cdots, e'_{2n}$. Then the following Lemma is well-known (cf., e. g., [3] p. 45).

LEMMA 4.1. *If the second fundamental form of M for e is given by $\sum_{i,j}H_{ij}\theta_i\theta_j$ then that of M' for e' is given by $\sum_{i,j}H_{ij}\circ\pi\theta'_i\theta'_j-2\sqrt{c}\sum_i f_i\circ\pi\theta'_i\theta'_{2n}$.*

REMARK. Lemma 4.1 holds without the assumption that all principal curvatures of M are constant.

It follows from Lemma 3.4 and Lemma 4.1 that for case (I) M' has two constant principal curvatures x and y for e' with multiplicities $m(x)+1$ and $m(y)+1$ respectively, and for case (II) M' has four constant principal curvatures x, y, z_1 and z_2 for e' with multiplicities $m(x)$, $m(y)$, 1 and 1 respectively, where $z_i^2-zz_i-c=0$ $(i=1, 2)$.

By Lemma 3.4 we can choose an orthonormal frame $\{e_1', \cdots, e_{2n-1}', e_{2n}', e'\}$ of $S^{2n+1}(c)$ under consideration so that $e_{2n-1}'=I(e')$, $e_{2n}'=I(p')$ and $(p: e_1, \cdots, e_{2n-1})\in F''$.

Case (I). By a theorem of E. Cartan [**1**, p. 180] there are two $\boldsymbol{R}$-linear subspaces $\boldsymbol{R}_x=\boldsymbol{R}^{m(x)+2}$ and $\boldsymbol{R}_y=\boldsymbol{R}^{m(y)+2}$ of $\boldsymbol{R}^{2n+2}$ such that

$$\boldsymbol{R}^{2n+2}=\boldsymbol{R}_x\oplus\boldsymbol{R}_y \quad \text{(orthogonal direct sum)}$$

and

$$M'=S^{m(x)+1}(x^2+c)\times S^{m(y)+1}(y^2+c)\,.$$

Thus the eigenspace for the principal curvature x (resp. y) in $T_{p'}(M')$ coincides with $T_{p'(x)}(S^{m(x)+1}(x^2+c))$ (resp. $T_{p'(y)}(S^{m(y)+1}(y^2+c))$), where $p'=p'(x)+p'(y)$, $p'(x)\in\boldsymbol{R}_x$, $p'(y)\in\boldsymbol{R}_y$. We want to show that I makes $\boldsymbol{R}_x$ (so also $\boldsymbol{R}_y$) invariant. By Lemma 3.4 we see that I makes the subspace of $\boldsymbol{R}_x$ spanned by e_a' invariant. Hence it suffices to show that $I(p'(x))$ is in a direction of principal curvature x. The vector e' normal to M' can be written as

$$e'=\cot\theta p'(x)-\tan\theta p'(y)$$

for a number θ such that $\sin^{-2}\theta=x^2+c$. Then we have $x=-\sqrt{c}\cot\theta$ and $y=-\sqrt{c}\tan\theta$. It follows from Lemma 4.1 that a vector $\cos\theta I(e')+\sin\theta I(p')$ is in a direction of principal curvature x, which is equal to $\sin^{-1}\theta I(p'(x))$. Now since both $\boldsymbol{R}_x$ and $\boldsymbol{R}_y$ are $\boldsymbol{C}$-linear subspaces of $\boldsymbol{C}^{n+1}$, there is a unitary transformation g' of $\boldsymbol{C}^{n+1}$ such that $g'(M')=M'(2n, m(x)/2+1, \tan^2\theta)$. Then g' induces an isometry g of $P_n(\boldsymbol{C})$ such that $g(M)=M'(2n-1, m(x)/2+1, \tan^2\theta)$. This completes the half of Main Theorem.

Case (II). We know already the following

(1) A space $M'(2n, t)\cap S^{2n+1}(c)$ is a connected compact hypersurface in S^{2n+1} having 4 constant principal curvatures with multiplicities $n-1$, $n-1$, 1 and 1, and it admits a transitive group of isometries isomorphic to $SO(2)\times SO(n+1)$ ([**4**]).

(2) A space $M(2n-1, t)$ is a connected compact real hypersurface in $P_n(\boldsymbol{C})$ having 3 constant principal curvatures with multiplicities $n-1$, $n-1$ and 1 ([**3**]).

(3) There exist an element h' of $O(2n+2)$ and a number t_0 such that $h'(M')=M'(2n, t_0)$ ([**4**]).

It follows from (1) and (3) that the almost contact structure of $M(2n-1, t_0)$ satisfies (II) of Lemma 3.4 and $M(2n-1, t_0)$ has 3 constant principal curvatures

x, y and z with multiplicities $n-1$, $n-1$ and 1 respectively. Since h'_* preserves directions of principal curvatures z_1 and z_2, we find $h'_*(I(p'))=\pm I(h'(p'))$ and $h'_*(I(e'))=\pm I(h'_*(e'))$ for each $p'\in M'$. This means that h' induces an isometry h'' of M onto $M(2n-1, t_0)$, and that the dual mapping of h''_* sends the second fundamental form of $M(2n-1, t_0)$ for $\pi_* h''_* e'$ to that of M for e. Hence by Theorem 3.2 in [2] there exists an isometry h of $P_n(\boldsymbol{C})$ such that $h(M)=M(2n-1, t_0)$. This completes the proof of Main Theorem.

References

[1] E. Cartan, Familles de surfaces isoparamétriques dans les espaces à courbure constante, Ann. Mat. Pura Appl., **17** (1938), 177–191.

[2] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math., **10** (1973), 495–506.

[3] R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures, J. Math. Soc. Japan, **27** (1975), 43–53.

[4] R. Takagi, A class of hypersurfaces with constant principal curvatures in a sphere, to appear in J. Differential Geometry.

Ryoichi TAKAGI
Department of Mathematics
Faculty of Science
Tokyo University of Education
Otsuka, Bunkyo-ku
Tokyo, Japan