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A simplified proof of a theorem of Kato on linear evolution equations

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In [2], T. Kato proved some basic and important theorems about systems $\{U(t, s); 0 \leqq s \leqq t \leqq T\}$ of bounded linear transformations associated with a linear evolution equation

$$du/dt + A(t)u = f(t), \quad 0 \leqq t \leqq T.$$

Here, f is a given function from $[0, T]$ into a Banach space X, $A(t)$ is a given, in general unbounded, linear operator in X, and the unknown function u is from $[0, T]$ into X. These theorems were strengthened and made more useful in [3], and the proofs were simplified by using a device due to Yosida, [5], [6]. The theorems of [3] were further generalized in Kato's subsequent paper [4]. For the most part, the proofs in [3] are quite easy to follow; in fact, remarkably so, considering the strength of the results. However, the proof of one of the theorems, [3; Theorem 6.1], is considerably more complicated than the others. We give a simplified proof of this theorem that extends to the more general case treated in [4]. We will give the proof first in the simpler setting of [3] and then point out how it extends to [4].

Unless otherwise specified, notation and terminology is the same as in [3]. In particular, X and Y are Banach spaces, with Y densely and continuously imbedded in X, and for each $t \in [0, T]$, $A(t)$ is a linear operator in X such that $-A(t)$ is the infinitesimal generator of a class C_0 semigroup (see [1] or [6]) of linear transformations in X. Assume, as in [3; Theorem 4.1], that:

(i) $\{A(t)\}$ is *stable*; i.e., there are constants M, β such that

$$\left\| \prod_{j=1}^{k} (A(t_j)+\lambda)^{-1} \right\| \leqq M(\lambda-\beta)^{-k}$$

for $\lambda > \beta$ and $0 \leqq t_1 \leqq \cdots \leqq t_k \leqq T$, $k = 1, 2, \cdots$.

(ii) Y is *$A(t)$-admissible* for each t (the semigroup generated by $-A(t)$ leaves Y invariant and forms a semigroup of class C_0 in Y), and if $\tilde{A}(t)$ is the part of $A(t)$ in Y, then $\{\tilde{A}(t)\}$ is stable.

(iii) $Y \subset D(A(t))$ for each t, and $A(t)$ is norm continuous from $[0, T]$ into $B(Y, X)$.

Let $\{U(t, s); 0 \leqq s \leqq t \leqq T\} \subset B(X)$ be the *evolution operator* for the family

$\{A(t)\}$; see [3; Theorem 4.1]. The principal properties of $\{U(t, s)\}$ are:

(a) $U(t, s)$ is strongly continuous (X) in s, t, and $U(s, s)=1$.

(b) $U(t, r)=U(t, s)U(s, r)$, $r\leqq s\leqq t$.

(c) $(D_t^+U(t, s)y)_{t=s}=-A(s)y$, $y\in Y$, $0\leqq s<T$.

(d) $(\partial/\partial s)U(t, s)y=U(t, s)A(s)y$, $y\in Y$, $0\leqq s\leqq t\leqq T$.

More can be said about $\{U(t, s)\}$ if we replace (ii) by the stronger condition:

(ii″) There is a family $\{S(t)\}$ of isomorphisms from Y onto X such that $S(t)$ is strongly continuously differentiable (Y, X), and

$$S(t)A(t)S(t)^{-1}=A(t)+B(t),\ B(t)\in B(X),$$

where B is strongly continuous (X).

Kato considers in [3] an intermediate condition (ii′) such that (ii″)$\Rightarrow$(ii′) $\Rightarrow$(ii).

THEOREM ([3; Theorem 6.1]). *Suppose* $\{A(t)\}$ *satisfies* (i), (ii″), *and* (iii). *Then the evolution operator* $\{U(t, s)\}$ *satisfies the following conditions* (*which correspond to* (e), (f′), *and* (h) *in* [3]).

(e) $U(t, s)Y\subset Y$ *for* $0\leqq s\leqq t\leqq T$.

(f) $U(t, s)Y$ *is strongly continuous* (Y) *jointly in* s *and* t.

(g) *For each fixed* $y\in Y$ *and* $s\in[0, T)$,

$$(\partial/\partial t)U(t, s)y=-A(t)U(t, s)y$$

for $s\leqq t\leqq T$, *and this derivative is continuous* (X).

PROOF. We want to prove (e) and (f); (g) then follows readily from (iii), (a), (b), and (c).

Let $\dot{S}(t)=dS(t)/dt$, and let $C(t)=\dot{S}(t)S(t)^{-1}$, as in [3]. If we let $W(t, s)$ be as in [3, Theorem 6.1], then it follows that

$$W(t, r)x=U(t, r)x-\int_r^t W(t, s)[B(s)-C(s)]U(s, r)x\,ds \tag{1}$$

for $x\in X$ and $0\leqq r\leqq t\leqq T$. It is simpler, however, to let $W(t, s)$ be defined by the equation (1). That is, we can regard (1) as a Volterra-type integral equation to be solved for the operator function $W(t, s)$. It follows from standard iteration techniques that this equation has a unique solution $W(t, s)\in B(X)$, and that this solution is strongly continuous (X), jointly in s and t.

Now define $Q(t, s)=U(t, s)S(s)^{-1}$. We will prove that $Q(t, s)=S(t)^{-1}W(t, s)$, which will establish (e) and (f).

First we note that $S(\cdot)$ is Lipschitz continuous in $B(Y, X)$; this follows from the fact that $S(\cdot)$ is strongly continuously differentiable (Y, X). It follows that $S(\cdot)^{-1}$ is Lipschitz continuous in $B(X, Y)$. From the continuity of $S(\cdot)^{-1}$ and strong differentiability of $S(\cdot)$, if follows that $S(\cdot)^{-1}$ is strongly continuously differentiable (X, Y), and that

$$(d/ds)[S(s)^{-1}x] = -S(s)^{-1}\dot{S}(s)S(s)^{-1}x \tag{2}$$

for $x \in X$ and $0 \leqq s \leqq T$. It follows from (2) and the properties of $U(t, s)$ that

$$\begin{aligned}(\partial/\partial s)Q(t, s)x &= U(t, s)A(s)S(s)^{-1}x - U(t, s)S(s)^{-1}\dot{S}(s)S(s)^{-1}x \\ &= U(t, s)A(s)S(s)^{-1}x - Q(t, s)C(s)x\end{aligned} \tag{3}$$

for each $x \in X$. Since $A(s)S(s)^{-1}Y \subset Y$ by (ii″) and (iii), then we have

$$A(s)S(s)^{-1}y = S(s)^{-1}S(s)A(s)S(s)^{-1}y = S(s)^{-1}[A(s)+B(s)]y$$

for each $y \in Y$. Thus

$$(\partial/\partial s)Q(t, s)y = Q(t, s)[A(s)+B(s)-C(s)]y \tag{4}$$

for $y \in Y$ and $0 \leqq s \leqq t \leqq T$.

Now, as in [3], define $A_n(t)$ by $A_n(t) = A([nt/T]T/n)$, and let $\{U_n(t, s)\}$ be the approximating evolution operator constructed from $\{A_n(t)\}$. Then $U_n(t, s)Y \subset Y$, and if $y \in Y$, then

$$(\partial/\partial s)U_n(s, r)y = A_n(s)U_n(s, r)y \tag{5}$$

for all but finitely many s. It follows from (4) and (5) that for each $y \in Y$, we have

$$(\partial/\partial s)Q(t, s)U_n(s, r)y = Q(t, s)[A(s)+B(s)-C(s)-A_n(s)]U_n(s, r)y \tag{6}$$

for all but finitely many s. Integrating both sides of (6) from $s=r$ to $s=t$, we get

$$\begin{aligned}&S(t)^{-1}U_n(t, r)y - Q(t, r)y \\ &= \int_r^t Q(t, s)[A(s)+B(s)-C(s)-A_n(s)]U_n(s, r)y\,ds\end{aligned} \tag{7}$$

for each $y \in Y$. Letting $n \to \infty$, we obtain as in the proof of [3, Theorem 4.5]

$$S(t)^{-1}U(t, r)y - Q(t, r)y = \int_r^t Q(t, s)[B(s)-C(s)]U(s, r)y\,ds\,, \tag{8}$$

$$Q(t, r)y = S(t)^{-1}U(t, r)y - \int_r^t Q(t, s)[B(s)-C(s)]U(s, r)y\,ds \tag{9}$$

for each $y \in Y$. By continuity, the equation (9) holds on all of X.

Now define $Z(t, s) = S(t)^{-1}W(t, s)$. It follows from (1) that

$$Z(t, r)x = S(t)^{-1}U(t, r)x - \int_r^t Z(t, s)[B(s)-C(s)]U(s, r)x\,ds \tag{10}$$

for each $x \in X$. Regarding $Z(t, s)$ as an unknown operator, and $B(s)$, $C(s)$, $S(s)$, and $U(t, s)$ as given, the equation (10) has a unique solution. Therefore, $Q(t, s) = S(t)^{-1}W(t, s)$.

In [**4**], Kato introduces the concept of *quasi-stability*, which is similar to stability, but weaker. Condition (i) is replaced by:

(i′) $\{A(t)\}$ is quasi-stable.

Condition (ii″) is replaced by:

(ii‴) There is a family $\{S(t)\}$ of isomorphisms of Y onto X such that

$$S(t)A(t)S(t)^{-1}=A(t)+B(t),\ B(t)\in B(X)$$

a. e. on $[0, T]$, where $B(\cdot)$ is strongly measurable with $\|B(\cdot)\|_X$ upper-integrable on $[0, T]$. Furthermore, there is a strongly measurable function $\dot{S}:[0, T]\to B(Y, X)$ with $\|\dot{S}(\cdot)\|_X$ upper-integrable on $[0, T]$, such that S is an indefinite strong integral of $\dot{S}$.

If we assume that $\{A(t)\}$ satisfies (i′), (ii‴), and (iii), then it is proven in [**4**], in a manner similar to that of [**3**], that there is a unique evolution operator $\{U(t, s)\}$ associated with $\{A(t)\}$, having the properties (a) through (d). The proof we have given here for properties (e) and (f) goes over with little change. That is, the argument extends without any formal changes, but the reasoning necessary to justify each step is more subtle.

We can still define W and Q as before. The analysis necessary to apply standard iteration techniques to the equation (1), and thus to justify our definition of W as the unique solution of (1) is discussed by Kato in [**4**; pp. 652, 653]. As before, (e) and (f) will be established if we prove that $Q(t, s)=S(t)^{-1}W(t, s)$.

In order to conclude that (2) holds almost everywhere, we need to know that $S(\cdot)^{-1}$ is at least strongly continuous (X, Y). Observe, as in [**4**], that

$$\|S(t)-S(s)\|\leqq\overline{\int_s^t}\|\dot{S}(r)\|dr,\qquad 0\leqq s\leqq t\leqq T, \tag{11}$$

where $\overline{\int}$ denotes the upper integral. It follows that $S(\cdot)$, and thus $S(\cdot)^{-1}$, is continuous in operator norm. Thus (2), (3), and (4) hold almost everywhere.

We can also conclude from (11) that $S(\cdot)$ is absolutely continuous in operator norm. Since $S(\cdot)^{-1}$ is bounded in operator norm, it follows that $S(\cdot)^{-1}$ also absolutely continuous in operator norm; i. e.,

$$\sum\|S(t_j)-S(s_j)\|_{X,Y},\qquad \sum\|S(t_j)^{-1}-S(s_j)^{-1}\|_{X,Y}$$

both approach zero as $\sum(t_j-s_j)\to 0$.

More care must be exercised in defining $\{A_n(t)\}$ and $\{U_n(t, s)\}$. We can define $A_n(t)=A(t_{n_k})$ for $t\in I_{n_k}$, where $\{I_{n_k}\}$ is a partition of $[0, T]$ into intervals, and $t_{n_k}\in I_{n_k}$. This can be done in such a way that the evolution operators $\{U_n(t, s)\}$ constructed from the step functions $\{A_n(t)\}$ stay bounded in $B(X)$ and $B(Y)$ and converge as before to $\{U(t, s)\}$. This is how Kato constructs $\{U(t, s)\}$ in [4]; see [4; p. 651].

Equation (5) holds as before, and (6) holds almost everywhere. From (5), it follows that the function $U_n(\cdot, r)y$ is Lipschitz continuous in X. Since $S(\cdot)^{-1}$ is absolutely continuous in $B(Y, X)$, then $S(\cdot)^{-1}U_n(\cdot, r)y$ is absolutely continuous in Y. It follows from (a), (d), and (iii) that $U(t, \cdot)$ is Lipschitz continuous in $B(Y, X)$. Thus $Q(t, \cdot)U_n(\cdot, r)y$ is absolutely continuous in X, and by [1; Theorem 3.8.6, p. 88], we can obtain (7) by integrating both sides of (6) from $s=r$ to $s=t$. Equations (8), (9), and (10) follow much as before. The fact that (10) has a unique solution is again not quite so elementary as in the setting of [3], but the standard iteration techniques with still apply, as they do for the equation (1).

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