J. Math. Soc. Japan
Vol. 27, No. 3, 1975

On some improvements of the Brun-Titchmarsh theorem, III

Dedicated to Professor T. Tatuzawa on his 60th birthday

By Yoichi MOTOHASHI

(Received Aug. 7, 1974)

§ 1. Introduction.

In our preceding papers [5], [6] we have established among other things that, denoting as usual by $\pi(x; q, a)$ the number of primes less than x and congruent to $a \pmod q$, we have the inequality

$$\pi(x; q, a) \leqq \frac{2x}{\varphi(q) \log \frac{x}{\sqrt{q}}} \left(1+O\left(\frac{\log \log x}{\log x}\right)\right), \tag{1}$$

for all $a \pmod q$ and for almost all $a \pmod q$ when $q \leqq x^{2/5}$ and $q \leqq x^{1-\varepsilon}$, respectively. The former case is the first substantial improvement of the Brun-Titchmarsh theorem and also of the recent result of Montgomery and others [4]. The later case is an improvement of a result of Hooley [2].

Roughly speaking, these are concerning the fixed modulus q and moving residue a. And it may be interesting to consider the dual problem in which the residue a is fixed and the modulus q runs over a certain interval. Then we may expect that the Brun-Titchmarsh theorem can be improved for almost all q. The first result in this field has been obtained in the above quoted paper of Hooley. He has proved that, if a is a fixed non-zero integer, K any positive constant and $W \leqq q < 2W$, $(q, a)=1$, then we have

$$\pi(x; q, a) \leqq \begin{cases} \dfrac{(1+\varepsilon)x}{\varphi(q) \log\left\{\left(\dfrac{x^2}{W}\right)^{1/6}\right\}} & \text{for} \quad x^{1/2} \leqq W \leqq x^{4/5} \\[2ex] \dfrac{(1+\varepsilon)x}{\varphi(q) \log \dfrac{x}{W}} & \text{for} \quad x^{4/5} \leqq W \leqq x^{1-\varepsilon}, \end{cases} \tag{2}$$

save for at most $W(\log x)^{-K}$ exceptional values of q.

This problem has certain similarity to the celebrated mean-value prime number theorem of Bombieri [1] (see also A. I. Vinogradov [8]), and the result of Hooley has definite interest, since Bombieri's theorem and even the extended

Riemann hypothesis give no information for $q \geqq x^{1/2}$.

In the same paper Hooley applied his result to the problem of the greatest prime factor of $p+a$ (p a prime number) and obtained an improvement in our estimate [7]. Recently he [3] took up again this problem and found a further improvement. In his new method the complex integration used in the proofs of (1) and (2) is avoided and the whole estimate is reduced to a rather simple application of the large sieve inequality.

The purpose of the present paper is to provide (2) with an additional improvement appealing to this new argument. But we do not use the large sieve at all, and our fundamental tool is the classical theorem of Pólya-Vinogradov (see (18) below). We shall prove

THEOREM. *Let a be a non-zero fixed integer and $K \geqq 2$ be arbitrary, and let $x^{5/6} \leqq W \leqq x(\log x)^{-(6K+165)}$. Then we have for $W \leqq q < 2W$, $(q, a)=1$,*

$$\pi(x; q, a) \leqq \frac{6x}{\varphi(q)\log x}\Big(1+O\Big(\frac{\log\log x}{\log x}\Big)\Big),$$

save for at most $W(\log x)^{-K}$ exceptional values of q.

NOTATIONS. x is a positive variable assumed to be sufficiently large. For any two integers n_1, n_2 the symbol (n_1, n_2) denotes their greatest common divisor. $d(n)$ is the number of divisors of n, and as usual we denote by $\varphi(n)$ and $\mu(n)$ Euler's and Moebius' functions respectively. χ is a Dirichlet character and χ_0 is generally a principal character regardless of its modulus. Finally we remark that the all constants implied by the symbols "O" and "$\ll$" in what follows depend on K at most.

§ 2. Selberg's sieve and the initial transformation of the problem.

Let z be a positive number to be determined optimally later. We set

$$\lambda_d = Y\mu(d)\frac{d}{\varphi(d)}\sum_{\substack{(r,a)=1\\ r\leqq z/d}}\frac{\mu^2(r)}{\varphi(r)}, \qquad (d \leqq z),$$

where

$$Y=\Big\{\sum_{d\leqq z}\frac{\mu^2(d)}{\varphi(d)}\Big\}^{-1}.$$

Then, as it is well-known, we have

$$\lambda_d = O(1), \qquad Y \leqq (\log z)^{-1}. \tag{3}$$

Further we set

$$g(n)=\{\sum_{d\mid n}\lambda_d\}^2=\sum_{h\mid n}\rho_h, \qquad (h\leqq z^2),$$

where denoting by $[d_1, d_2]$ the least common multiple of d_1, d_2

$$\rho_h = \sum_{h=[d_1, d_2]} \lambda_{d_1} \lambda_{d_2} .$$

Here we note that from (3) we have

$$g(n) = O(d(n)^2), \qquad \rho_h = O(d(h)^2). \tag{4}$$

In what follows we use the notation

$$Y(q) = \sum_{(h,q)=1} \frac{\rho_h}{h}. \tag{5}$$

Then it can be shown (see [**2**, § 4]) that

$$Y(q) \leqq \frac{q}{\varphi(q)} Y. \tag{6}$$

Now by the standard application of the Selberg sieve

$$\pi(x; q, a) \leqq \sum_{\substack{n \equiv a (\mathrm{mod}\ q) \\ n \leqq x}} g(n) + O\left(\frac{z}{q}\right). \tag{7}$$

But it should be remarked that this sieving weight $g(n)$ is not the one that is usual in the case of the fixed modulus q, and our choice is made in favor of the moving modulus q. Any way the main term of the right side of (7) is $\frac{x}{q} Y(q)$, and so we are led to consider the variance

$$V(x, z; W) = \sum_{\substack{W \leqq q < 2W \\ (q,a)=1}} \left\{ \sum_{\substack{n \equiv a (\mathrm{mod}\ q) \\ n \leqq x}} g(n) - \frac{x}{q} Y(q) \right\}^2. \tag{8}$$

Henceforth we may restrict the parameters z and W by

$$x^{1/2} \leqq W \leqq x(\log x)^{-(6K+165)} \quad \text{and} \quad z > \left(\frac{x}{W}\right)^{1/2}. \tag{9}$$

The second condition is necessary, since otherwise no improvement would follow. And to make the calculations simple we introduce the following auxiliary function

$$v(x, z; w) = \sum_{\substack{w \leqq q \leqq rw \\ (q,a)=1}} \left\{ \sum_{\substack{n \equiv a (\mathrm{mod}\ q) \\ n \leqq x}} g(n) - \frac{x}{q} Y(q) \right\}^2,$$

where $r = 1 + (\log x)^{-(3K+130)}$. Then we have

$$V(x, z; W) \leqq \frac{1}{\log r} \int_W^{2W} v(x, z; w) \frac{dw}{w}.$$

Now we have

$$v(x, z; w) = \sum_{\substack{w \leqq q \leqq rw \\ (q,a)=1}} \left\{ \sum_{k \leqq \frac{1}{q}(x-a)} g(a+qk) - \frac{x}{q} Y(q) \right\}^2$$

$$\ll \sum_{\substack{w \leqq q \leqq rw \\ (q,a)=1}} \left\{ \left(\sum_{k \leqq \frac{x}{w}} g(a+qk) - \frac{x}{w} Y(q) \right)^2 + \left(\sum_{\frac{1}{q}(x-a) < k \leqq \frac{x}{w}} g(a+qk) \right)^2 + \left(\frac{x}{q} Y(q) - \frac{x}{w} Y(q) \right)^2 \right\}$$

$$=v_1(x, z; w)+G_1+G_2, \qquad \text{say}.$$

For G_1 we have from (4)

$$G_1 \ll \frac{x}{w}(r-1)\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}}\ \sum_{\frac{1}{q}(x-a)<k\leqq\frac{x}{w}} d^4(a+qk)$$

$$\ll \frac{x}{w}(r-1)\sum_{x-a<n\leqq rx} d^4(a+n)d(n).$$

And this sum is

$$\ll\Big\{\sum_{x-a<n\leqq rx} d^8(a+n)\Big\}^{\frac{1}{2}}\Big\{\sum_{x-a<n\leqq rx} d^2(n)\Big\}^{\frac{1}{2}} \ll (r-1)x(\log x)^{130}.$$

Thus we have

$$G_1 \ll \frac{x^2}{w}(r-1)(\log x)^{-3K}.$$

Also we have

$$G_2 \ll \frac{x^2}{w}(r-1)(\log x)^{-3K}.$$

Hence noticing $(r-1)/\log r \ll 1$ we get

$$V(x, z; W) \ll \frac{1}{\log r}\int_W^{2W} v_1(x, z; w)\frac{dw}{w}+O\Big(\frac{x^2}{W}(\log x)^{-3K}\Big), \tag{10}$$

which implies that the problem has been reduced to the estimation of $v_1(x, z; w)$. We decompose this into three parts

$$\begin{aligned} v_1(x, z; w) &= \sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}}\Big\{\sum_{k_1, k_2\leqq\frac{x}{w}} g(a+k_1q)g(a+k_2q) \\ &\qquad -2\frac{x}{w}Y(q)\sum_{k\leqq\frac{x}{w}} g(a+kq)+\Big(\frac{x}{w}Y(q)\Big)^2\Big\} \\ &= J_1(x, z; w)-2\frac{x}{w}J_2(x, z; w)+\Big(\frac{x}{w}\Big)^2\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}} Y(q)^2, \qquad \text{say.}\end{aligned} \tag{11}$$

§ 3. Estimation of $J_2(x, z; w)$.

First we note that

$$\sum_{k\leqq\frac{x}{w}} g(a+kq)=\sum_{(h,q)=1}\rho_h\sum_{\substack{kq\equiv -a\ (\mathrm{mod}\ h)\\ k\leqq\frac{x}{w}}} 1,$$

and so classifying h according to the greatest common divisor $\delta=(a, h)$, we have

$$\sum_{k\leqq\frac{x}{w}} g(a+kq)=\sum_{\delta\mid a}\ \sum_{\left(h,\frac{a}{\delta}q\right)=1}\rho_{\delta h}\sum_{\substack{kq\equiv-\frac{a}{\delta}\ (\mathrm{mod}\ h)\\ k\leqq\frac{x}{\delta w}}} 1 \tag{12}$$

$$=\sum_{\delta\mid a}T(\delta,q)\,,\qquad\text{say}.$$

In the sum $T(\delta, q)$ we have $\left(h, \frac{a}{\delta}\right)=1$, and we may use the expression

$$\begin{aligned}\sum_{\substack{kq\equiv-\frac{a}{\delta}(\mathrm{mod}\ h)\\ k\leqq\frac{x}{\delta w}}}1&=\frac{1}{\varphi(h)}\Big\{\sum_{\substack{(k,h)=1\\ k\leqq\frac{x}{\delta w}}}1+\sum_{\chi\neq\chi_0(\mathrm{mod}\ h)}\bar{\chi}\Big(-\frac{a}{\delta}\Big)\sum_{k\leqq\frac{x}{\delta w}}\chi(kq)\Big\}\\ &=\frac{1}{\varphi(h)}\{P_0(\delta,h)+P(\delta,h\,;q)\}\,,\qquad\text{say}.\end{aligned}$$

Thus we have

$$\begin{aligned}T(\delta,q)&=\sum_{\left(h,\frac{a}{\delta}q\right)=1}\frac{\rho_{\delta h}}{\varphi(h)}\{P_0(\delta,h)+P(\delta,h\,;q)\}\\ &=T_0(\delta,q)+T_1(\delta,q)\,,\qquad\text{say}.\end{aligned}\tag{13}$$

Here we use the well-known fact

$$P_0(\delta,h)=\frac{\varphi(h)}{h\delta w}x+O(d(h))\,,$$

and we get

$$T_0(\delta,q)=\frac{x}{w}\sum_{\left(h,\frac{a}{\delta}q\right)=1}\frac{\rho_{\delta h}}{\delta h}+O((\log x)^8)\,,$$

since we have (4). Then by the definition (5) of $Y(q)$ we have

$$\sum_{\delta\mid a}T_0(\delta,q)=\frac{x}{w}Y(q)+O((\log x)^8)\,.\tag{14}$$

Next as for $T_1(\delta, q)$ we note

$$\begin{aligned}\sum_{\substack{w\leqq q\leqq rw\\ (q,a)=1}}Y(q)T_1(\delta,q)&=\sum_{\left(h,\frac{a}{\delta}\right)=1}\frac{\rho_{\delta h}}{\varphi(h)}\sum_{\substack{w\leqq q\leqq rw\\ (q,ah)=1}}P(\delta,h\,;q)Y(q)\\ &=\sum_{\left(h,\frac{a}{\delta}\right)=1}\frac{\rho_{\delta h}}{\varphi(h)}\sum_{\chi\neq\chi_0(\mathrm{mod}\ h)}\bar{\chi}\Big(-\frac{a}{\delta}\Big)\Big\{\sum_{k\leqq\frac{x}{\delta w}}\chi(k)\Big\}\Big\{\sum_{\substack{w\leqq q\leqq rw\\ (q,a)=1}}\chi(q)Y(q)\Big\}.\end{aligned}\tag{15}$$

For the last factor we have from (5)

$$\sum_{\substack{w\leqq q\leqq rw\\ (q,a)=1}}\chi(q)Y(q)=\sum_{h}\frac{\rho_h}{h}\sum_{\substack{w\leqq q\leqq rw\\ (q,a)=1}}\chi(q)=Y\sum_{\substack{w\leqq q\leqq rw\\ (r,a)=1}}\chi(q)\,.\tag{16}$$

Further we have

$$\begin{aligned}\sum_{\substack{w\leqq q\leqq rw\\ (q,a)=1}}\chi(q)&=\sum_{w\leqq q\leqq rw}\chi(q)\sum_{\substack{l\mid a\\ l\mid q}}\mu(l)\\ &=\sum_{l\mid a}\mu(l)\chi(l)\sum_{\frac{w}{l}\leqq q\leqq\frac{r}{l}w}\chi(q)\,.\end{aligned}\tag{17}$$

Thus combining (15) with (16) and (17), we get

$$
\begin{aligned}
&\Big|\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}} Y(q)T_1(\delta, q)\Big| \\
&\ll \sum_{l\mid a}\sum_{h\leqq z^2}\frac{d(\delta h)^2}{h}\sum_{\chi\neq\chi_0(\mathrm{mod}\,h)}\Big|\sum_{k\leqq\frac{x}{\delta w}}\chi(k)\Big|\Big|\sum_{\frac{w}{l}\leqq q\leqq\frac{r}{l}w}\chi(q)\Big|,
\end{aligned}
$$

since we have (4). Here we quote the well-known result of Pólya-Vinogradov, which states that for any non-principal character χ (mod f) and for any real numbers ξ_1 and ξ_2 we have

$$
\Big|\sum_{\xi_1\leqq n\leqq\xi_2}\chi(n)\Big|\ll f^{\frac{1}{2}}\log f. \tag{18}
$$

Applying this to the last two factors of the above expression we find

$$
\Big|\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}} Y(q)T_1(\delta, q)\Big|\ll z^4(\log x)^5,
$$

which, with (12), (13) and (14), gives

$$
J_2(x, z\,; w)=\frac{x}{w}\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}} Y(q)^2+O\{z^4(\log x)^5+w(r-1)(\log x)^8\}. \tag{19}
$$

§ 4. Estimation of $J_1(x, z\,; w)$.

By the abreviations introduced in the preceding paragraph we have

$$
\begin{aligned}
J_1(x, z\,; w)&=\sum_{\substack{\delta_1\mid a\\\delta_2\mid a}}\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}} T(\delta_1, q)T(\delta_2, q)\\
&=\sum_{\substack{\delta_1\mid a\\\delta_2\mid a}}\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}}\{T_0(\delta_1, q)T_0(\delta_2, q)+T_0(\delta_1, q)T_1(\delta_2, q)\\
&\qquad\qquad+T_1(\delta_1, q)T_0(\delta_2, q)+T_1(\delta_1, q)T_1(\delta_2, q)\}\\
&=U_1(x, z\,; w)+U_2(x, z\,; w)+U_3(x, z\,; w)+U_4(x, z\,; w),\quad\text{say.}
\end{aligned} \tag{20}
$$

From (14) we get easily

$$
\begin{aligned}
U_1(x, z\,; w)&=\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}}\{\sum_{\delta\mid a}T_0(\delta, q)\}^2\\
&=\Big(\frac{x}{w}\Big)^2\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}} Y(q)^2+O\{(r-1)x(\log x)^8\}.
\end{aligned} \tag{21}
$$

As for $U_2(x, z\,; w)$ and $U_3(x, z\,; w)$ we can treat them analogously as in the case of $J_2(x, z\,; w)$. And we show here only the final result

$$
|U_2(x, z\,; w)|,\quad |U_3(x, z\,; w)|\ll\frac{x}{w}z^4(\log x)^{14}. \tag{22}
$$

Now for $U_4(x, z\,; w)$ we note that

$$\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}} T_1(\delta_1, q)T_2(\delta_2, q)=\sum_{\substack{\left(h_1,\frac{a}{\delta_1}\right)=1\\\left(h_2,\frac{a}{\delta_2}\right)=1}} \frac{\rho_{\delta_1 h_1}\rho_{\delta_2 h_2}}{\varphi(h_1)\varphi(h_2)} \sum_{\substack{w\leqq q\leqq rw\\(q,h_1h_2a)=1}} P(\delta_1, h_1; q)P(\delta_2, h_2; q).$$

Also we have

$$\sum_{\substack{w\leqq q\leqq rw\\(q,h_1h_2a)=1}} P(\delta_1, h_1; q)P(\delta_2, h_2; q)$$

$$=\sum_{\substack{\chi_1\neq\chi_0 (\mathrm{mod}\ h_1)\\\chi_2\neq\chi_0 (\mathrm{mod}\ h_2)}} \bar{\chi}_1\Big(-\frac{a}{\delta_1}\Big)\bar{\chi}_2\Big(-\frac{a}{\delta_2}\Big)\Big\{\sum_{k_1\leqq \frac{x}{\delta_1 w}}\chi_1(k_1)\Big\}\Big\{\sum_{k_2\leqq \frac{x}{\delta_2 w}}\chi_2(k_2)\Big\}\Big\{\sum_{\substack{w\leqq q\leqq rw\\(q,a)=1}}\chi_1\chi_2(q)\Big\}$$

$$=\sum_{\chi_1\chi_2=\chi_0}+\sum_{\chi_1\chi_2\neq\chi_0}=R_0(h_1, h_2; \delta_1, \delta_2)+R_1(h_1, h_2; \delta_1, \delta_2)\,, \qquad \text{say.}$$

Thus $U_4(x, z; w)$ is divided into two parts

$$\begin{aligned} U_4(x, z; w)&=\sum_{\substack{\delta_1|a\\\delta_2|a}}\sum_{\substack{\left(h_1,\frac{a}{\delta_1}\right)=1\\\left(h_2,\frac{a}{\delta_2}\right)=1}} \frac{\rho_{\delta_1 h_1}\rho_{\delta_2 h_2}}{\varphi(h_1)\varphi(h_2)}\{R_0(h_1, h_2; \delta_1, \delta_2)+R_1(h_1, h_2; \delta_1, \delta_2)\} \\ &=U_4^{(0)}(x, z; w)+U_4^{(1)}(x, z; w)\,, \qquad \text{say.} \end{aligned} \tag{23}$$

In the sum $R_1(h_1, h_2; \delta_1, \delta_2)$ $\chi_1\chi_2$ can be considered as a non-principal character (mod h_1h_2), and so by (18) we have, using the device (17),

$$|R_1(h_1, h_2; \delta_1, \delta_2)|\ll z^2\log x\Big\{\sum_{\chi_1\neq\chi_0(\mathrm{mod}\ h_1)}\Big|\sum_{k_1\leqq\frac{x}{\delta_1 w}}\chi_1(k_1)\Big|\Big\}\Big\{\sum_{\chi_2\neq\chi_0(\mathrm{mod}\ h_2)}\Big|\sum_{k_2\leqq\frac{x}{\delta_2 w}}\chi_2(k_2)\Big|\Big\},$$

which implies that

$$|U_4^{(1)}(x, z; w)|\ll z^2(\log x)^3\Big\{\sum_{\delta|a}\sum_{h\leqq z^2}\frac{d(h)^2}{h}\sum_{\chi\neq\chi_0(\mathrm{mod}\ h)}\Big|\sum_{k\leqq\frac{x}{\delta w}}\chi(k)\Big|\Big\}^2.$$

This sum on h is

$$\ll\Big\{\sum_{h\leqq z^2}\frac{d(h)^4}{h}\Big\}^{\frac{1}{2}}\Big\{\sum_{h\leqq z^2}\sum_{\chi(\mathrm{mod}\ h)}\Big|\sum_{k\leqq\frac{x}{\delta w}}\chi(k)\Big|^2\Big\}^{\frac{1}{2}}$$

$$\ll(\log x)^8\Big\{\sum_{h\leqq z^2}\Big(h+\frac{x}{\delta w}\Big)\frac{x}{\delta w}\Big\}^{\frac{1}{2}}\ll z^2\Big(\frac{x}{w}\Big)^{\frac{1}{2}}(\log x)^8\,,$$

since we have assumed (9). Thus we get

$$|U_4^{(1)}(x, z; w)|\ll z^6\frac{x}{w}(\log x)^{19}\,. \tag{24}$$

Next in the sum $R_0(h_1, h_2; \delta_1, \delta_2)$ the condition $\chi_1\chi_2=\chi_0$ implies $\chi_1=\bar{\chi}_2$ (mod h_1h_2). And so naturally it must be that

$$\chi_1^*=\bar{\chi}_2^*\,, \qquad h_1^*=h_2^*\,,$$

where χ_i^* (mod h_i^*) is the primitive character which induces χ_i (mod h_i), ($i=1, 2$). Hence we have

$$|R_0(h_1, h_2\,;\,\delta_1, \delta_2)| \ll w \sum_{\substack{h_1=h_1^* m_1 \\ h_2=h_1^* m_2}} \Bigl| \sum_{\substack{(k_1, m_1)=1 \\ k_1 \leq \frac{x}{\delta_1 w}}} \chi_1^*(k_1) \Bigr| \Bigl| \sum_{\substack{(k_2, m_2)=1 \\ k_2 \leq \frac{x}{\delta_2 w}}} \chi_1^*(k_2) \Bigr|$$

$$\ll w \sum_{\substack{h \mid h_1 \\ h \mid h_2 \\ (h_1=hm_1, h_2=hm_2)}} \sum_{\chi \neq \chi_0 (\mathrm{mod}\, h)} \Bigl| \sum_{\substack{(k_1, m_1)=1 \\ k_1 \leq \frac{x}{\delta_1 w}}} \chi(k_1) \Bigr| \Bigl| \sum_{\substack{(k_2, m_2)=1 \\ k_2 \leq \frac{x}{\delta_2 w}}} \chi(k_2) \Bigr| .$$

And we have

$$|U_4^{(0)}(x, z\,;\,w)|$$
$$\ll w(\log x)^2 \sum_{\substack{\delta_1 \mid a \\ \delta_2 \mid a}} \sum_{m_1, m_2 \leq z^2} \frac{d(m_1)^2 d(m_2)^2}{m_1 m_2} \sum_{h \leq z^2} \frac{d(h)^4}{h^2} \sum_{\chi \neq \chi_0 (\mathrm{mod}\, h)} \Bigl| \sum_{\substack{(k_1, m_1)=1 \\ k_1 \leq \frac{x}{\delta_1 w}}} \chi(k_1) \Bigr| \Bigl| \sum_{\substack{(k_2, m_2)=1 \\ k_2 \leq \frac{x}{\delta_2 w}}} \chi(k_2) \Bigr| .$$

We denote this sum on h by $H(\delta, m)$. Then using the device (17) we have

$$H(\delta, m) \ll \sum_{\substack{l_1 \mid m_1 \\ l_2 \mid m_2}} \sum_{h \leq z^2} \frac{d(h)^4}{h^2} \sum_{\chi \neq \chi_0 (\mathrm{mod}\, h)} \Bigl| \sum_{k_1 \leq \frac{x}{\delta_1 l_1 w}} \chi(k_1) \Bigr| \Bigl| \sum_{k_2 \leq \frac{x}{\delta_2 l_2 w}} \chi(k_2) \Bigr|$$

$$= \sum_{\substack{l_1 \mid m_1 \\ l_2 \mid m_2}} \Bigl\{ \sum_{h \leq E} + \sum_{E < h \leq z^2} \Bigr\} = \sum_{\substack{l_1 \mid m_1 \\ l_2 \mid m_2}} \{H_1(\delta, m, l) + H_2(\delta, m, l)\}\,, \qquad \text{say,}$$

where E is to be determined later. To $H_1(\delta, m, l)$ we apply (18), and we get easily

$$H_1(\delta, m, l) \ll E(\log x)^{17}\,.$$

As for $H_2(\delta, m, l)$ we note

$$\sum_{\chi (\mathrm{mod}\, h)} \Bigl| \sum_{k_1 \leq \frac{x}{\delta_1 l_1 w}} \chi(k_1) \Bigr| \Bigl| \sum_{k_2 \leq \frac{x}{\delta_2 l_2 w}} \chi(k_2) \Bigr|$$

$$\ll \Bigl\{ \sum_{\chi (\mathrm{mod}\, h)} \Bigl| \sum_{k_1 \leq \frac{x}{\delta_1 l_1 w}} \chi(k_1) \Bigr|^2 \Bigr\}^{\frac{1}{2}} \Bigl\{ \sum_{\chi (\mathrm{mod}\, h)} \Bigl| \sum_{k_2 \leq \frac{x}{\delta_2 l_2 w}} \chi(k_2) \Bigr|^2 \Bigr\}^{\frac{1}{2}}$$

$$\ll \Bigl(h + \frac{x}{w} \Bigr) \frac{x}{w}\,.$$

This gives

$$H_2(\delta, m, l) \ll \frac{x}{w} \sum_{E \leq h \leq z^2} \Bigl\{ \frac{1}{h} + \frac{x}{wh^2} \Bigr\} d(h)^4$$

$$\ll \frac{x}{w} \Bigl(1 + \frac{x}{Ew} \Bigr) (\log x)^{16}\,.$$

Hence the optimal value of E is x/w $(< z^2)$, and we find

$$H(\delta, m) \ll d(m_1) d(m_2) \frac{x}{w} (\log x)^{17}\,.$$

Inserting this into (25) we get

(26) $$|U_4^{(0)}(x, z\,;\,w)| \ll x(\log x)^{35}\,.$$

Collecting (20), (21), (22), (23), (24) and (26) we obtain

$$J_1(x, z; w) = \left(\frac{x}{w}\right)^2 \sum_{\substack{w \leqq q \leqq rw \\ (q,a)=1}} Y(q)^2 + O\left\{\frac{x}{w}(z^6 + w)(\log x)^{35}\right\}. \tag{27}$$

§ 5. Proof of the Theorem.

Now from (11), (19) and (27) we have

$$v_1(x, z; w) \ll \frac{x}{w}(z^6 + w)(\log x)^{35},$$

which, with (10), gives

$$V(x, z; W) \ll \frac{x}{W}(z^6 + W)(\log x)^{3K+165} + \frac{x^2}{W}(\log x)^{-3K}.$$

Thus if we set

$$z = x^{1/6}(\log x)^{-(K+30)},$$

then we have

$$V(x, z; W) \ll \frac{x^2}{W}(\log x)^{-3K}.$$

By the definition of $V(x, z; W)$ the result means that, save for at most $W(\log x)^{-K}$ exceptional values of q, we have

$$\sum_{\substack{n \equiv a \pmod q \\ n \leqq x}} g(n) = \frac{x}{q} Y(q) + O\left(\frac{x}{W}(\log x)^{-K}\right),$$

and hence from (6) and (7)

$$\pi(x; q, a) \leqq \frac{x}{\varphi(q) \log z}(1 + O((\log x)^{-1}))$$

$$\leqq \frac{6x}{\varphi(q) \log x}\left(1 + O\left(\frac{\log \log x}{\log x}\right)\right).$$

This ends the proof of our theorem.

Added in proof: In the mean time Hooley (Proc. London Math. Soc., (3) **30** (1975), 114-128) obtained a result which supersedes our theorem, by applying a simple variant of Linnik's dispersion method. Both Hooley's and our arguments give an interesting result on the least almost prime number in an arithmetic progression. To this see our paper which will appear in Proc. Japan Acad.

References

[1] E. Bombieri, On the large sieve, Mathematatika, **12** (1965), 201–225.
[2] C. Hooley, On the Brun-Titchmarsh theorem, J. Reine Angew. Math., **225** (1972), 60–79.
[3] C. Hooley, On the largest prime factor of $p+a$, Mathematika, **20** (1973), 135–143.
[4] H. L. Montgomery, Topics in multiplicative number theory, Springer, 1971.
[5] Y. Motohashi, On some improvements of the Brun-Titchmarsh theorem, J. Math. Soc. Japan, **26** (1974), 306–323.
[6] Y. Motohashi, On some improvements of the Brun-Titchmarsh theorem, Res. Inst. Math. Sci. Kyoto Univ. Kōkyūroku, **193** (1973), 97–109, (in Japanese).
[7] Y. Motohashi, A note on the least prime in an arithmetic progression with a prime difference, Acta Arith., **17** (1970), 283–285.
[8] A. I. Vinogradov, On the density conjecture for Dirichlet L-series, Amer. Math. Soc. Transl., (2) **82** (1969), 9–46.

Yoichi MOTOHASHI
Department of Mathematics
College of Science and Technology
Nihon University
Surugadai, Kanda, Chiyoda-ku
Tokyo, Japan