J. Math. Soc. Japan
Vol. 27, No. 3, 1975

Orthogonality and the numerical range

By Mary R. EMBRY

(Received June 3, 1974)

§1. Introduction.

In this paper we shall expand upon results and techniques developed in [2] to investigate certain geometric relationships between a complex Hilbert space X and the numerical range of a continuous linear operator A on X. In Section 2 we present a version of the Cauchy-Schwartz inequality valid in the boundary of the numerical range of A. In Section 3 we study the action on elements z of $W(A)$ induced by the action of A on elements x of X such that $\langle Ax, x\rangle/\|x\|^2=z$.

The *numerical range* of A is the set of complex numbers, $W(A)=\{\langle Ax, x\rangle :$ $x\in X$ and $\|x\|=1\}$, where $\langle\,,\rangle$ is the given inner product on X and $\|\ \|$ is the associated norm. Basic properties of the numerical range are discussed in [4]. In particular the Hausdorff-Toeplitz theorem is proven: $W(A)$ is convex. We use the following terminology: z is an *extreme* point of $W(A)$ if $z\in W(A)$ and z is not in the interior of any line segment lying in $W(A)$; L is a *line of support* for $W(A)$ if $W(A)$ lies in one of the two closed half-planes determined by L and L contains at least one point of the closure of $W(A)$; b and c are *adjacent extreme points* of $W(A)$ if the line segment joining b and c lies in the boundary of $W(A)$; c is a *corner* of $W(A)$ if c is an extreme point of $W(A)$ and there exist more than one line of support for $W(A)$ passing through c.

We define the set M_z for each complex z by $M_z=\{x: x\in X$ and $\langle Ax, x\rangle$ $=z\|x\|^2\}$.

§2. A Cauchy-Schwartz inequality.

Consider a line of support L for $W(A)$ and the associated set in X, $N=$ $\{x: \langle Ax, x\rangle=z\|x\|^2, z\in L\}$. In [2] we proved that N is a closed linear subspace of X and that A behaves very much like an Hermitian operator on N. More precisely

LEMMA 2.1. *Let L be a line of support of $W(A)$ and $N=\{x: \langle Ax, x\rangle=$ $z\|x\|^2, z\in L\}$. Let $\theta=0$ if L is horizontal; otherwise θ is the measure of the acute angle between L and the x-axis. Then*

i) *N is a closed linear subspace of X, and*

ii) *for each* z *in* L

$$N=\{x: e^{i\theta}(A-z)x=e^{-i\theta}(A^*-z^*)x\}\,.$$

Thus we see that the compression of $e^{i\theta}(A-z)$ $(z\in L)$ to N is Hermitian; that is, if P is the orthogonal projection of X onto N, then $Pe^{i\theta}(A-z)P$ is an Hermitian operator on X. One consequence of Lemma 2.1 is that if $x\in N$, then $Ax\in N$ if and only if $A^*x\in N$. Furthermore if $x\in N$ and $Ax=zx$, then necessarily $z\in L$ and by ii) $A^*x=z^*x$. Thus the standard argument shows that if x is an eigenvector associated with the boundary of $W(A)$ and y is an eigenvector for some other eigenvalue, then x and y are orthogonal. This was first observed by C. H. Meng in [5].

THEOREM 2.2. *Let* L *be a line of support for* $W(A)$ *and* $N=\{x:\langle Ax, x\rangle =z\|x\|^2,\ z\in L\}$. *Let* b *be an element of* L *such that either* b *is an extreme point of* $W(A)$ *or* $b\notin W(A)$. *Then for all* x *and* y *in* N

$$|\langle(A-b)x, y\rangle|^2\leqq\langle(A-b)x, x\rangle\langle y, (A-b)y\rangle\,.$$

PROOF. Let θ be as defined in Lemma 2.1 and let P be the projection of X onto N. Then $Pe^{i\theta}(A-b)P$ is an Hermitian operator on X. Further since b is an extreme point of $W(A)$ or $b\notin W(A)$ we may assume that $Pe^{i\theta}(A-b)P$ is nonnegative-definite. Thus by the generalized Cauchy-Schwartz inequality

$$|\langle Pe^{i\theta}(A-b)Px, y\rangle|^2\leqq\langle Pe^{i\theta}(A-b)Px, x\rangle\langle Pe^{i\theta}(A-b)Py, y\rangle$$

for all x and y in X. Thus for all x and y in N

$$|\langle(A-b)x, y\rangle|^2\leqq\langle e^{i\theta}(A-b)x, x\rangle\langle e^{i\theta}(A-b)y, y\rangle\,.$$

But by Lemma 2.1 ii) $\langle e^{i\theta}(A-b)y, y\rangle=\langle e^{-i\theta}(A^*-b^*)y, y\rangle=e^{-i\theta}\langle y, (A-b)y\rangle$. Substitution of this expression in the right-hand member of the last inequality leads to the desired conclusion.

If z is an extreme point of $W(A)$, the set $M_z=\{x:\langle Ax, x\rangle=z\|x\|^2\}$ is a closed linear subspace of X. This was proved by Stampfli in [7]. It is interesting to note that this is a consequence of Theorem 2.2.

COROLLARY 2.3 (Stampfli). *If* b *is an extreme point of* $W(A)$, *then* $M_b=\{x:\langle Ax, x\rangle=b\|x\|^2\}$ *is linear.*

PROOF. By Theorem 2.2 $\langle(A-b)x, y\rangle=0$ for all y in N and x in M_b. If x_1 and x_2 are in M_b, then by Lemma 2.1, $z_1x_1+z_2x_2\in N$ for all complex z_1 and z_2. Thus $\langle(A-b)x_i, z_1x_1+z_2x_2\rangle=0$ for $i=1, 2$ and consequently $z_1x_1+z_2x_2\in M_b$.

In the proof of Corollary 2.3 we also showed the following:

COROLLARY 2.4. *If* b *is an extreme point of* $W(A)$, *then* $(A-b)M_b$ *is orthogonal to* N *where* $N=\{x:\langle Ax, x\rangle=z\|x\|^2,\ z\in L\}$, L *a line of support for* $W(A)$ *passing through* b.

COROLLARY 2.5. *Let* b, N *and* L *be as given in Corollary* 2.4. *If* $x\in M_b$

and $Ax \in N$, then $Ax = bx$ and $A^*x = b^*x$.

PROOF. If $Ax \in N$, then since N is linear, $Ax - bx \in N$. But by Corollary 2.4 $Ax - bx$ is orthogonal to N. Consequently $Ax = bx$ and by Lemma 2.1, $A^*x = b^*x$.

If A is an Hermitian operator and b is an extreme point of $W(A)$, then $M_b = \{x : Ax = bx \text{ and } A^*x = b^*x\}$. This well-known fact is generalized in each of Corollaries 2.3, 2.4 and 2.5.

Theorem 2.2 can also be used to prove that M_b and M_c are orthogonal if b and c are adjacent extreme points of $W(A)$. However this is a result of the following more general theorem on angles between vectors associated with different points on a given line of support of $W(A)$.

THEOREM 2.6. *Let b and c be adjacent extreme points of $W(A)$ and let $d = tb + (1-t)c$, $0 \leqq t \leqq 1$. If $x \in M_b$ and $y \in M_d$, then*

$$|\langle x, y\rangle| \leqq \sqrt{t}\, \|x\|\, \|y\| .$$

In particular M_b is orthogonal to M_c.

PROOF. We may assume that $b = 0$ and $c = 1$. Then $d = (1-t)$. Let $x \in M_b$ and $y \in M_d$. Then by Lemma 2.1 i) for r real $x + ry \in \{x : \langle Ax, x\rangle = s\|x\|^2, 0 \leqq s \leqq 1\}$. Thus $\langle A(x+ry), x+ry\rangle \leqq \|x+ry\|^2$. By Corollary 2.4, and Lemma 2.1 $\langle Ax, x+ry\rangle = 0$ and $\langle A^*x, x+ry\rangle = 0$. Therefore $\langle A(x+ry), x+ry\rangle = r^2\langle Ay, y\rangle = r^2(1-t)\|y\|^2$. Substituting in the preceding inequality, we arrive at $r^2(1-t)\|y\|^2 \leqq \|x+ry\|^2$ for all real r. Standard algebraic techniques now yield $|\mathrm{Re}\langle x, y\rangle| \leqq \sqrt{t}\|x\|\|y\|$. Since this inequality is valid for all x in M_b, it is valid for λx, $|\lambda| = 1$, and judicious choice of λ results in $|\langle x, y\rangle| \leqq \sqrt{t}\|x\|\|y\|$.

The proof of Theorem 2.6 actually yields a result stronger than the one stated. Rather than requiring b and c to both be extreme we only need require that b be extreme and that c be on a line L of support for $W(A)$ through b such that $W(A) \cap L$ is contained in the closed line segment from b to c.

§ 3. The numerical range orbit.

In our study of the relationship between the numerical range of A and the action of A on X the following question arose: if $x \in M_z$ ($\langle Ax, x\rangle = z\|x\|^2$) and $A^n x \in M_w$ ($\langle A^{n+1}x, A^n x\rangle = w\|A^n x\|^2$), can one draw any conclusion about the relation between z and w? The results appear to be limited to special cases and because of the possibility that A be nilpotent, a slightly different approach proved to be more profitable. For each $x \neq 0$ define $z(x) = \langle Ax, x\rangle / \|x\|^2$, $G^0(x) = x$ and $G(x) = Ax - z(x)x$. Inductively we define $G^n(x) = G(G^{n-1}x)$ for all x such that $G^{n-1}x \neq 0$. In [6] Salinas calls $z : x \to \langle Ax, x\rangle / \|x\|^2$ the *numerical range function*. By the *numerical range orbit of x* we mean the set $W(x) = \{z(G^n x) : G^n x \neq 0\}$. We list several easily proved facts about the function G:

1. $G(x)=0$ if and only if x is an eigenvector of A;
2. $G(x)$ is orthogonal to x;
3. If $z(x)$ is an extreme point of $W(A)$, then $G(x)$ is orthogonal to $N=\cup\{M_w : w$ and $z(x)$ on a common line of support for $W(A)\}$ (Corollary 2.4);
4. $G(\lambda x)=\lambda G(x)$ for all complex $\lambda\neq 0$;
5. $G(x+y)=G(x)+G(y)$ if and only if $z(x+y)=z(x)=z(y)$;
6. G is linear on M_z if z is an extreme point of $W(A)$;
7. If $z(x)$ is on a line of support L and x is not an eigenvector of A, then $z(Ax)\in L$ if and only if $z(G(x))\in L$ (Lemma 2.1).

In [6] Salinas studied the differentiability of the numerical range function and the function E_A defined by $E_A(x)=Ax-\langle Ax, x\rangle x$ which is closely related to our function G. From his results it follows that G is differentiable on $X-\{0\}$.

To have a complete catalogue in this section of our results on the numerical range orbit of A we restate Corollary 2.5.

THEOREM 3.1. *If $z(x)$ is an extreme point of $W(A)$ and either $Ax=0$ or $z(Ax)$ is on a line of support for $W(A)$ through $z(x)$, then $G(x)=0$.*

Our second result concerns the general situation in which $z(x)$ and $z(Ax)$ are on the same line of support for $W(A)$.

THEOREM 3.2. *Let $z(x)$ be on the boundary of $W(A)$ and L a line of support for $W(A)$ through $z(x)$. If $Ax=0$ or $z(Ax)\in L$, then either*

i) $G(x)=0$

or ii) $|z(x)-b|<|z((A-b)x)-b|$ *for each point b of L for which b is an extreme point of $W(A)$ or $b\notin W(A)$.*

PROOF. Note that if $Ax=0$, then $z(x)=0$ and hence $G(x)=0$. Henceforth we assume $Ax\neq 0$ and to simplify the argument we assume that L is the real line, $b=0$, and $L\cap W(A)\subset[0, \infty)$. Assume that $G(x)\neq 0$ and recall this implies that x is not an eigenvector of A. By Lemma 2.1 if $z(Ax)$ is real, then $z(Ax-wx)$ is real for all real w. Moreover since x is not an eigenvector of A, then by Theorem 3.1 $z(x)$ is not an extreme point of $W(A)$. Thus $z(x)>0$.

Applying Theorem 2.2 with $b=0$ and $y=Ax$ we obtain $\|Ax\|^4\leqq\langle Ax, x\rangle\langle Ax, A^2x\rangle=z(x)\|x\|^2 z(Ax)\|Ax\|^2$. Since $Ax\neq 0$, we have $\|Ax\|^2\leqq z(x)z(Ax)\|x\|^2$. On the other hand the Cauchy-Schwartz inequality yields $|z(x)|^2\|x\|^4=|\langle Ax, x\rangle|^2\leqq\|Ax\|^2\|x\|^2$. Combining these last two inequalities we have $|z(x)|^2\|x\|^4\leqq z(x)z(Ax)\|x\|^4$ or equivalently $z(x)\leqq z(Ax)$. A review of the preceding argument shows that equality can hold only if x is an eigenvector of A, contradicting our assumption that $G(x)\neq 0$.

These last two results effectively describe the action of A on x in case x and Ax both map into the same boundary line under the numerical range function. To simplify the picture let us assume L is the real axis and that 0 is an extreme point of $W(A)$. Then consider x such that $z(x)\in L$. Either $Ax=0$ (and hence $z(x)=0$) or $z(Ax)$ is not real or $z(Ax)\geqq z(x)$, equality holding if and only if $Ax=z(x)x$. In particular, if x is not an eigenvector of A and $z(A^n x)\in L$ for each n, then $\{z(A^n x)\}$ is a strictly increasing sequence of real numbers.

We turn now to consideration of the situation in which each of $z(x)$ and $z(G(x))$ is on the boundary of $W(A)$, but not necessarily on the same line of support. In Theorem 3.4 we shall see that this occurs only if the line of support for $W(A)$ through $z(x)$ is parallel or equal to the line of support through $z(G(x))$. Furthermore in Theorem 3.5 we shall see that if each of $z(x)$, $z(G(x))$ and $z(G^2(x))$ is on the boundary of $W(A)$, then necessarily $z(x)$ and $z(G^2(x))$ are on the same line of support. These results and Theorem 3.1 were reported at the winter meeting of the AMS, 1974 [**3**].

LEMMA 3.3. *Let L_1 and L_2 be nonparallel lines of support of $W(A)$, $L_1\cap L_2=\{c\}$, and $N_j=\{x: \langle Ax, x\rangle=z\|x\|^2,\ z\in L_j\}$, $j=1, 2$. Then $(A-c)N_1$ is orthogonal to N_2.*

PROOF. Let θ_j be associated with L_j as in Lemma 2.1 and let $x_j\in N_j$, $j=1, 2$. By Lemma 2.1 $e^{i\theta_j}(A-c)x_j=e^{-i\theta_j}(A^*-c^*)x_j$, $j=1, 2$. A simple manipulation shows that $e^{i\theta_1}\langle(A-c)x_1, x_2\rangle=e^{i(2\theta_2-\theta_1)}\langle(A-c)x_1, x_2\rangle$. Since L_1 and L_2 are nonparallel, $e^{2i\theta_1}\neq e^{2i\theta_2}$ and hence $\langle(A-c)x_1, x_2\rangle=0$.

THEOREM 3.4. *If $G(x)\neq 0$ and each of $z(x)$ and $z(G(x))$ is in the boundary of $W(A)$, then the line of support of $W(A)$ through $z(x)$ is parallel (or equal) to the line of support through $z(G(x))$.*

PROOF. Assume the two lines of support are nonparallel and intersect in point c. By Lemma 3.3 $(A-c)x$ is orthogonal to $G(x)$. Since x is always orthogonal to $G(x)$, we also have $(A-z(x))x$ orthogonal to $G(x)$. Since $G(x)=(A-z(x))x$, this means that $G(x)=0$, contradicting our hypothesis. Hence the lines must be parallel.

We note that a very general converse to Theorem 3.4 is true: if L_1 and L_2 are distinct parallel lines of support of $W(A)$, then the corresponding associated subspaces of X are orthogonal. We prove this as follows: assume L_1 and L_2 are horizontal. Let $N_j=\{x: \langle Ax, x\rangle=z\|x\|^2,\ z\in L_j\}$. Then by Lemma 2.1, $N_j=\{x: \operatorname{Im} Ax=b_j x\}$ where L_j is defined by $z=b_j i$, b_j real. Thus N_1 and N_2 are eigenspaces of the Hermitian operator $\operatorname{Im} A$ and consequently orthogonal.

Observe that Theorem 3.4 implies that if $G(x)\neq 0$, then neither $z(x)$ nor $z(G(x))$ is a corner of $W(A)$. Donoghue [**1**] showed that if $z(x)$ is a corner of $W(A)$, then $G(x)=0$. We note here that if c is a corner of $W(A)$, then $M_c\cap \operatorname{range} G=\{\theta\}$.

LEMMA 3.5. *If $z(x)$ is in the boundary of $W(A)$ and $G(x)\neq 0$, then $e^{2i\theta}\langle G^2(x), x\rangle=\|G(x)\|^2$, where θ is the measure of the angle between the x-axis* *and the line of support through* $z(x)$.

PROOF. By definition of $G^2(x)$ and $G(x)$, $G^2(x)=(A-z(x))G(x)+(z(G(x))-z(x))G(x)$. Recall that x is orthogonal to $G(x)$ and if $z(x)$ is on the boundary of $W(A)$, $e^{i\theta}(A-z(x))x=e^{-i\theta}(A^*-z(x)^*)x$. Therefore

$$e^{i\theta}\langle G^2(x), x\rangle=\langle e^{i\theta}(A-z(x))G(x), x\rangle$$
$$=\langle G(x), e^{i\theta}(A-z(x))x\rangle=e^{-i\theta}\|G(x)\|^2 .$$

THEOREM 3.6. *Let* $z(x)$ *be in the boundary of* $W(A)$ *and assume* $G(x)\neq 0$. *Then*

i) $G(x)$ *is not an eigenvector of* A

and

ii) *if* $z(G(x))$ *is in the boundary of* $W(A)$, *then either* $z(G^2(x))$ *is in the interior of* $W(A)$ *or* $z(G^2(x))$ *is on the line of support of* $W(A)$ *passing through* $z(x)$.

PROOF. By Lemma 3.5 $G^2(x)\neq 0$ and consequently $G(x)$ is not an eigenvector of A. Assume that $z(G(x))$ and $z(G^2(x))$ are in the boundary of $W(A)$. Let L_j be the line of support of $W(A)$ through $z(G^j(x))$, $j=0, 1, 2$. By Theorem 3.4 L_1 is parallel to each of L_0 and L_2. Thus L_0 and L_2 are parallel (or equal). As we have noted previously if $L_0\neq L_2$ the associated subspaces of X are orthogonal. However by Lemma 3.5 x and $G^2(x)$ are not orthogonal and consequently $L_0=L_2$.

We are now in the position to make several observations about the numerical range orbit of x, $W(x)=\{z(G^n(x)) : G^n(x)\neq 0\}$.

COROLLARY 3.7. *If* $W(x)$ *is contained in the boundary of* $W(A)$ *and* $G(x)\neq 0$, *then* $G^n(x)\neq 0$ *for any* n. *Furthermore in this case* $z(G^{2n}(x))\in L_0$ *and* $z(G^{2n+1}(x))\in L_1$ *for each* n *where* L_0 *is the line of support for* $W(A)$ *through* $z(x)$ *and* L_1 *the line of support through* $z(G(x))$.

PROOF. Applying Theorem 3.6 we see that if $G^n(x)\neq 0$, then $G^n(x)$ is not an eigenvector of A and hence $G^{n+1}(x)\neq 0$. Also if $z(G^n(x))\in L$, then $z(G^{n+2}(x))\in L$.

We observe that in Corollary 3.7 either $L_0=L_1$ and $W(x)$ is entirely contained in the line of support through $z(x)$ or $L_0\neq L_1$ and the elements of $W(x)$ oscillate between these two parallel lines. If A is a hyponormal operator and $z(x)$ is in the boundary of $W(A)$, then it follows from [**7**, Lemma 3] that $W(x)$ is contained in the line of support through $z(x)$. An example of the second type behavior is found in any nonnormal operator on two dimensional space. More precisely if A is a nonnormal operator on two dimensional space, then $W(A)$ is an ellipse and if $z(x)$ is in the boundary of $W(A)$, then $G(x)\neq 0$. Thus since $G(x)$ is orthogonal to x, $z(G(x))$ must be the point on the opposite side of $W(A)$ from $z(x)$, having line of support parallel to the one through

$z(x)$. In this case we also have $G^2(x)=\lambda x$. This will always be the case if $W(A)\cap L=\{z(x)\}$, $G(x)\neq 0$, and $W(x)$ is contained in the boundary of $W(A)$. If $z(x)$ is a nonextreme boundary point and $W(x)$ is contained in the boundary of $W(A)$, it is not necessarily the case that $G^2(x)=\lambda x$. However, the following sequential generalization is valid:

COROLLARY 3.8. *If $W(x)$ is contained in the boundary of $W(A)$ and $G(x)\neq 0$, then*

i) $\lim \|y_{n+2}-e^{-2i\theta}y_n\|=0$ *where $y_n=G^n(x)/\|G^n(x)\|$ and θ is the measure of the angle between the x-axis and the line of support through $z(x)$,*

and

ii) *if some subsequence $\{G^{n_k}(x)\}$ converges to a nonzero y, then $G^2(y)=\lambda y$ for some complex λ.*

PROOF. By Lemma 3.5 $\|G(x)\|^2=e^{2i\theta}\langle G^2(x), x\rangle$. Therefore $\|G^{n+1}(x)\|^2=e^{2i\theta}\langle G^{n+2}(x), G^n(x)\rangle\leqq\|G^{n+2}(x)\|\|G^n(x)\|$. Thus the sequence of real numbers $r_n=\|G^{n+1}(x)\|/\|G^n(x)\|$ is monotone increasing. Since r_n is bounded above by $\|A\|$, we see that r_n converges to a positive real number L and consequently $r_n/r_{n+1}\to 1$.

Let $y_n=G^n(x)/\|G^n(x)\|$. Then $\|e^{-2i\theta}y_n-y_{n+2}\|^2=2-2\,\mathrm{Re}\,e^{-2i\theta}\langle y_n, y_{n+2}\rangle=2-2\,\mathrm{Re}\,e^{2i\theta}\langle G^{n+2}(x), G^n(x)\rangle/\|G^{n+2}(x)\|\ \|G^n(x)\|=2-2\|G^{n+1}(x)\|^2/\|G^{n+2}(x)\|\|G^n(x)\|=2-2\,r_n/r_{n+1}\to 0$. Thus i) is established. Assertion ii) follows from i) and the continuity of G on $X-\{0\}$.

References

[1] W. F. Donoghue, Jr., On the numerical range of a bounded operator, Michigan Math. J., **4** (1957), 261-263.

[2] M. R. Embry, The numerical range of an operator, Pacific J. Math., **32** (1970), 647-650.

[3] M. R. Embry, The numerical range of an operator, II, Notices Amer. Math. Soc., **21** (1974), A-196.

[4] P. R. Halmos, A Hilbert Space Problem Book, Van Nostrand, Princeton, 1967.

[5] C. H. Meng, On the numerical range of an operator, Proc. Amer. Math. Soc., **14** (1963), 167-171.

[6] N. Salinas, On tne η function of Brown and Pearcy and the numerical function of an operator, Canad. J. Math., **23** (1971), 565-578.

[7] J. G. Stampfli, Extreme points of the numerical range of a hyponormal operator, Michigan Math. J., **13** (1966), 87-89.

Mary R. EMBRY
Department of Mathematics
University of North Carolina at Charlotte
Charlotte, North Carolina 28223
U. S. A.