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Micro-hyperbolic pseudo-differential operators I

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§ 0. Introduction.

Bony and Schapira [1] has proved that the Cauchy problem is well-posed for hyperbolic operators with variable coefficients in the framework of hyperfunctions. In their paper they took up the defining functions of hyperfunctions, and by applying their refined version of the Cauchy-Kovalevsky theorem they proved that the solutions with the initial data which are the defining functions of hyperfunction data become also the defining functions of sought for hyperfunction solutions. In this paper, their results are extended to the case of micro-hyperbolic pseudo-differential operators; namely we will prove that in the framework of microfunctions the Cauchy problem is well-posed for the pseudo-differential operators of that type. This result implies the result of Bony and Schapira about the hyperbolic differential operators.

The essential step in our argument is in the construction of the elementary solution for the Cauchy problem. As a by-product of this method, we obtain a rather wide class of solvable pseudo-differential operators, whose null solutions propagate in one-sided direction along the "bicharacteristics" (if it exists).

Generalization of the results in this paper to the system of pseudo-differential equations will be dealt with in the subsequent paper.

A pseudo-differential operator $P(x, D_x)$ is said to be partially micro-hyperbolic at $(x_0, \sqrt{-1}\xi_0)$ with respect to the direction $\langle\vartheta, dx\rangle+\langle\rho, d\xi\rangle$ (see § 1 for the precise definition) if $P_m(x+\sqrt{-1}\varepsilon\rho, \sqrt{-1}\xi+\varepsilon\vartheta)$ never vanishes for every (x, ξ) near (x_0, ξ_0) and $0<\varepsilon\ll 1$. We reduce it, by means of a quantized contact transformation, to an operator of the form $P=D_1-A(x, D')$ where A is a matrix of pseudo-differential operators of order $\leqq 1$ commuting with x_1 and such that all the eigenvalues of the principal symbol $A_1(x, \sqrt{-1}\xi')$ have non-negative real part. Then we construct a formal solution $G(x, D')=\sum_{\alpha} a_\alpha(x)D'^\alpha$ such that $PG=0$ and $G|_{x_1=0}=1$ in § 2. In § 3 and § 4, we will show that GD_1^{-1} can be realized as a microfunction, which becomes an elementary solution of P. In this way, we obtain the following theorems (Theorem 5.2 and Theorem 5.5).

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THEOREM. *If P is partially micro-hyperbolic, then P has an elementary solution, whose support is contained in a half space.*

THEOREM. *If $P(x, D)=D_1-A(z, D')$ is micro-hyperbolic with respect to the x_1-direction, that is, if all the eigenvalues of $A_1(x, \sqrt{-1}\xi')$ are purely imaginary, then we obtain an elementary solution $E(x, y')$ for the Cauchy problem:*

$$\begin{cases} PE(x, y')=0 \\ E(x, y')|_{x_1=0}=\delta(x'-y') . \end{cases}$$

If P is a pseudo-differential operator with constant coefficients, we can construct an elementary solution without any difficulty by using plane waves (See Andersson [**1**], Kawai [**2**], Sato-Kashiwara-Kawai [**1**] Chap. I). If P is not with constant coefficients, we cannot expect that P has always a good phase function because we do not assume that P is with simple characteristics. Therefore, we must abandon the usual method of construction of the elementary solution by using the phase function of P. We think the employment of the phase function in constructing elementary solutions is a means for "obtaining the boundary value" of pseudo-differential operators defined in a complex domain, which we shall perform in this paper.

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§ 1. Definition of a micro-hyperbolic pseudo-differential operator and a partially micro-hyperbolic pseudo-differential operator.

First we will give the notations which are used in this paper. We denote by M a real analytic manifold of dimension n and by X a complex neighbourhood of M. We will denote by L the real analytic manifold $S_M^*X=\sqrt{-1}S^*M$, which is the conormal spherical bundle of M in X. Let Λ be a complex neighborhood of L. The canonical map from L to the cotangential projective bundle P^*X of X, can be extended to a holomorphic map from Λ to P^*X. Since this map is a local isomorphism, we often identify Λ with P^*X. $\mathcal{P}_X$ (resp. $\mathcal{P}_X^f$) denotes the sheaf of rings of pseudo-differential operators (resp. of finite order) and often abbreviated to $\mathcal{P}$ (resp. $\mathcal{P}^f$). (See Sato-Kawai-Kashiwara [**1**], which will be referred to as S-K-K).

Since $\mathcal{P}$ (resp. $\mathcal{P}^f$) is a sheaf of rings on P^*X, we can consider it as a sheaf of rings on Λ. Note that the sheaf of microfunctions $\mathcal{C}_M$, which is a sheaf on L, is a $\mathcal{P}$- (resp. $\mathcal{P}^f$-) Module. We will denote by ${}^L\widetilde{\Lambda}$ the real monoidal transform of Λ with center L.

DEFINITION 1.1. Let $\mathfrak{M}$ be a system of pseudo-differential equations (that is, an admissible $\mathcal{P}$-Module), V be the support of $\mathfrak{M}$ and $x+\sqrt{-1}v0$ be a point of $\sqrt{-1}SL=S_L\Lambda$. We say that $\mathfrak{M}$ is partially micro-hyperbolic at $x+\sqrt{-1}v0$ if $x+\sqrt{-1}v0$ is not contained in the closure of $V-L$ in ${}^L\widetilde{\Lambda}$. If v satisfies the relation $\langle v, \omega_L\rangle=0$ (ω_L is the canonical 1-form on L) and $\mathfrak{M}$ is partially micro-hyperbolic at $x+\sqrt{-1}v0$ and at $x-\sqrt{-1}v0$ at once, then $\mathfrak{M}$ is called micro-hyperbolic at $x+\sqrt{-1}v0$. A (square matrix of) pseudo-differential operator P is said to be (partially) micro-hyperbolic at $x+\sqrt{-1}v0$, if so is the system of pseudo-differential equations $Pu=0$. In particular, if we set V the zeros of the principal symbol of P, and V satisfies the above condition, then P is (partially) micro-hyperbolic.

In order to describe the situation which we will encounter sometimes in this paper, we will give the following notation. Let M be a real analytic manifold and N be its submanifold, ${}^N\widetilde{M}$ be the real monoidal transform of M with center N and G be a closed subset in M. (See S-K-K [1] Chap. I § 1.2.) The intersection of S_NM and the closure of $G-N$ in ${}^N\widetilde{M}$ is called normal set of G along N and denoted by S_NG. The polar of S_NG is called the conormal set of G along N and denoted by S_N^*G. Therefore we have

$$S_N^*G=\{(x, \eta\infty)\in S_N^*M\,;\ \langle\xi, \eta\rangle\leqq 0 \qquad \text{for any} \quad x+\xi 0\in S_NG\}\,.$$

According to this terminology, $\mathfrak{M}$ is partially micro-hyperbolic at $x+\sqrt{-1}v0$ if the normal set of the support of $\mathfrak{M}$ along L does not contain $x+\sqrt{-1}v0$.

Let $f: M'\to M$ be a real analytic map and N' and N be submanifolds of M' and M respectively such that $f(N')\subset N$. Then there is a map $S_{N'}M'-S_{N'}f^{-1}(N)\to S_NM$, which is denoted by q. Let G' and G be closed subsets of M' and M respectively. If $G'\subset f^{-1}(G)$, then we have $S_{N'}G'\subset q^{-1}(S_NG)\cup S_{N'}f^{-1}(N)$. If f is smooth and $G'=f^{-1}(G)$, then $S_{N'}G'-S_{N'}f^{-1}(N)=q^{-1}(S_NG)$. These facts are easily obtained from the fact that there exists a canonical continuous map

$${}^{N'}\widetilde{M'}-(f^{-1}(N)-N')-S_{N'}f^{-1}(N)\longrightarrow {}^N\widetilde{M}\,.$$

In order to describe the results of our paper, we will give several explanations of purely imaginary contact manifolds.

$L=\sqrt{-1}S^*M$ has a purely imaginary contact structure. It means that there are given a principal $\boldsymbol{R}^+$-bundle $\hat{L}$ of L and a purely imaginary 1-form θ on $\hat{L}$ satisfying

(1.1) θ is homogeneous of degree 1, that is, $\theta(c\lambda)=c\theta(\lambda)$ for $(c, \lambda)\in\boldsymbol{R}^+\times\hat{L}$;

(1.2) θ vanishes nowhere;

(1.3) $(d\theta)^n$ vanishes nowhere, where $2n$ is the dimension of L.

We call $\hat{L}$ the associated purely imaginary symplectic manifold. A 1-form

ω_L on L is said to be a canonical 1-form if the pull back of ω_L to $\hat{L}$ is θ up to a non-vanishing function as multiple. In fact it suffices to take $T^*_M X - M$ as $\hat{L}$, and the canonical 1-form $\theta = \sqrt{-1}\langle \xi, dx\rangle = \sqrt{-1}\sum_{j=1}^{n} \xi_j dx_j$. In this notation $(x_1, \cdots, x_n)$ is a coordinate system of M and $(\xi_1, \cdots, \xi_n)$ the fiber coordinate system of T^*M.

There is a canonical isomorphism $\hat{H}$ from $T^*\hat{L}$ to $\sqrt{-1}T\hat{L}$ defined by

$$n\omega \wedge \omega' \wedge (d\theta)^{n-1} = \langle \hat{H}(\omega), \omega'\rangle (d\theta)^n .$$

Using the local coordinate system $(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ such that $\theta = \sqrt{-1}\langle \xi, dx\rangle$, this isomorphism is represented by

$$\hat{H}:\ dx_j \longmapsto \frac{-1}{\sqrt{-1}} \frac{\partial}{\partial \xi_j},$$

$$d\xi_j \longmapsto \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j} \qquad (j=1, 2, \cdots, n).$$

$\hat{H}$ induces an isomorphism $S^*\hat{L} \xrightarrow{\sim} \sqrt{-1}S\hat{L}$. We denote by $L \hat{\times} L$ the quotient of $\hat{L}\times\hat{L}$ by the group $\boldsymbol{R}^+$, where $\boldsymbol{R}^+$ operates on $\hat{L}\times\hat{L}$ under the law

$$(c, x_1, x_2) \longmapsto (cx_1, cx_2) \qquad \text{for } c \in \boldsymbol{R}^+ \text{ and } x_1, x_2 \in \hat{L}.$$

Since $L = \sqrt{-1}S^*M$, $L\hat{\times}L$ is isomorphic to $\sqrt{-1}S^*(M\times M) - M\times\sqrt{-1}S^*M - \sqrt{-1}S^*M\times M$. In this paper, we identify these two manifolds by

$$\hat{L}\times\hat{L} \ni (x_1, \sqrt{-1}\langle \xi_1, dx_1\rangle \infty)\times(x_2, \sqrt{-1}\langle \xi_2, dx_2\rangle\infty)$$
$$\longleftrightarrow (x_1, x_2\,;\ \sqrt{-1}\langle \xi_1, dx_1\rangle\infty - \langle \xi_2, dx_2\rangle\infty)) \in \sqrt{-1}T^*M\times\sqrt{-1}T^*M.$$

Let p_1 and p_2 be the first and the second projections from $L\hat{\times}L$ to L respectively. We will use the same letter p_ν to express the projection from $\hat{L}\times\hat{L}$ to $\hat{L}$. $L\hat{\times}L$ is also a purely imaginary contact manifold with the canonical form $\theta_{L\hat{\times}L} = p_1^*(\theta_L) - p_2^*(\theta_L)$. We identify L with a submanifold of $L\hat{\times}L$ by the diagonal embedding $x \mapsto (x, x)$. We identify $S^*\hat{L}$ with $\hat{L}\times_L S^*_L(L\hat{\times}L) = S^*_{\hat{L}}(\hat{L}\times\hat{L})$ by $T^*\hat{L} \ni \omega \mapsto p_1^*\omega - p_2^*\omega \in T^*_{\hat{L}}(\hat{L}\times\hat{L})$. Analogously we identify $\hat{L}\times_L S_L(L\hat{\times}L) = S_{\hat{L}}(\hat{L}\times\hat{L})$ with $S\hat{L}$ by $T(\hat{L}\times\hat{L}) \ni v \mapsto p_{1*}v - p_{2*}v \in T\hat{L}$. Under the map $\hat{L}\times_L S^*L \to S^*\hat{L} \cong \hat{L}\times_L S^*_L(L\hat{\times}L)$, we consider S^*L as a subbundle of $S^*_L(L\times L)$ of codimension 1. Let Θ be the subbundle in $S^*_L(L\hat{\times}L)$ defined by $\{\pm\sqrt{-1}\theta_{L\hat{\times}L}\}$. Θ is contained in S^*L, that is $\{\pm\sqrt{-1}\theta_L\}$. Then there is a canonical map $S^*_L(L\hat{\times}L) - \Theta \to \sqrt{-1}SL$ induced by $T^*_L(\hat{L}\times\hat{L}) \cong T^*\hat{L} \xrightarrow{\hat{H}} \sqrt{-1}T\hat{L} \to \sqrt{-1}TL$. We denote it by H. Note that the image of $S^*L - \Theta$ under H is the orthogonal bundle $\Theta^\perp$ of Θ, that is $\{x + iv0\,;\ \langle v, \theta\rangle = 0\}$. The isomorphism $\hat{H}: T^*\hat{L} \xrightarrow{\sim} \sqrt{-1}T\hat{L}$ induces an isomorphism $S^*_L(L\hat{\times}L) \xrightarrow{\sim} S_L(L\hat{\times}L)$, which is also denoted by $\hat{H}$.

DEFINITION 1.2. Let x be a point in L and Γ be a subset of $S_L^*(L\hat{\times}L)$ over x. We denote by $\mathcal{A}_\Gamma$ the set of all germs K of $C_{M\times M}^{(0,n)}$ at x such that the fiber of the conormal set of the support of K over x contains a neighborhood of the antipodal set of Γ. It is evident that $\mathcal{A}_\Gamma=\bigcap_{\vartheta\in\Gamma}\mathcal{A}_\vartheta$. If Δ is a subset of $\sqrt{-1}S_xL$, then $\mathcal{A}_{H^{-1}(\Delta)}$ is denoted by $\mathcal{H}_\Delta$.

The supports of the elements in $\mathcal{A}_\Gamma$ are so restricted that we can define the composition of two elements of $\mathcal{A}_\Gamma$ under which $\mathcal{A}_\Gamma$ becomes a ring. Since the following lemma which assures the composition of two kernel functions is an easy consequence of S-K-K [1] Chap. 1 § 2, we state it without proof.

LEMMA 1.3. *Let $L\hat{\times}L\hat{\times}L$ be the quotient of $\hat{L}\times\hat{L}\times\hat{L}$ by $\boldsymbol{R}^+$, p_{12}, p_{23} and p_{13} be the canonical projections from $L\hat{\times}L\hat{\times}L$ to $L\hat{\times}L$ defined by $(x_1, x_2, x_3)\mapsto(x_1, x_2), (x_2, x_3)$ and (x_1, x_3) respectively. Then there is a canonical bilinear homomorphism, which is a "product of operators":*

$$p_{13!}(p_{12}^{-1}C_{M\times M}^{(0,n)}\times p_{23}^{-1}C_{M\times M}^{(0,n)})\longrightarrow C_{M\times M}^{(0,n)}.$$

LEMMA 1.4. *Let K_ν $(\nu=1, 2)$ be a germ of $C_{M\times M}^{(0,n)}$ at x. Let G_ν be its support. If $S_LG_1\cap(S_LG_2)^a=\emptyset$ at x, then*

$$K(x, x'')=\int_{x'}K_1(x, x')K_2(x', x'')$$

makes sense as a germ of $C_{M\times M}^{(0,n)}$ at x, and the normal set of the support of K along L is in $\langle(S_LG_1)_x, (S_LG_2)_x\rangle$. (Here $\langle A, B\rangle$ denotes the union of A, B and arcs joining a point of A and a point in B.)

PROOF. It suffices to show that $p_{13}^{-1}(L)\cap p_{12}^{-1}(G_1)\cap p_{23}^{-1}(G_2)\subset L$ and that if we set $G=p_{13}(p_{12}^{-1}(G_1)\cap p_{23}^{-1}(G_2))$ then $(S_LG)_x\subset\langle(S_LG_1)_x, (S_LG_2)_x\rangle$. Let ξ_1 and ξ_2 be tangent vectors. Then we have $S_L(p_{12}^{-1}(G_1))\subset\{(x, x, x; (\xi_1, 0, \xi_3)0)\in S_L(L\hat{\times}L\hat{\times}L);\ \xi_1=0$ or $\xi_10\in S_L(G_1)\}$ and $S_L(p_{23}^{-1}(G_2))\subset\{(x, x, x; (\xi_1, 0, \xi_3)0\in S_L(L\hat{\times}L\hat{\times}L);\ \xi_3=0$ or $-\xi_30\in S_L(G_2)\}$. Therefore $S_L(p_{12}^{-1}(G_1)\cap p_{23}^{-1}(G_2))\subset\{(x, x, x; (\xi_1, 0, \xi_3)0)\in S_L(L\hat{\times}L\hat{\times}L);\ \xi_1=0$ or $\xi_3=0$, or $(\xi_10\in S_L(G_1)$ and $-\xi_30\in S_L(G_2))\}$. Therefore $S_L(p_{12}^{-1}(G_1)\cap p_{23}^{-1}(G_2)\cap p_{13}^{-1}(L))\subset\{(x, x, x; (\xi_1, 0, \xi_3)0)\in S_L(L\hat{\times}L\hat{\times}L);\ \xi_10\in S_L(G_1), -\xi_30\in S_L(G_2)$, and $\xi_1-\xi_3=0\}=\emptyset$, which implies the first statement.

Moreover $(S_L(p_{13}(p_{12}^{-1}(G_1)\cap p_{23}^{-1}(G_2))))$ is contained in $\{(x, x; (\xi, 0)0)\in S_L(L\hat{\times}L);\ \xi=\xi_1-\xi_3$ where $(\xi_1=0$ or $\xi_10\in S_L(G_1))$ and $(\xi_3=0$ or $-\xi_30\in S_L(G_2))\}$. Therefore the second statement follows. Q. E. D.

By the preceding lemma, we immediately obtain the following

PROPOSITION 1.5. *$\mathcal{A}_\Gamma$ is a ring by the operation $(K_1(x, x'), K_2(x, x'))\mapsto K(x, x'')=\int K_1(x, x')K_2(x', x'')$, if Γ is not empty.*

If the condition on the supports of the kernels and microfunctions is imposed suitably, we can define their operations on microfunctions.

DEFINITION 1.6. Let x be a point in L and Γ be a subset in S_x^*L. The set of all germs $u(x)$ of $\mathcal{C}_M$ at x satisfying the following conditions is denoted by $\mathcal{M}_\Gamma$: The normal set of the support of $u(x)$ along $\{x\}$ does not intersect the polar of Γ.

Using this terminology we have

PROPOSITION 1.7. *$\mathcal{M}_\Gamma$ is an $\mathcal{A}_\Gamma$-module by the operation*

$$(K(x, x')dx', u(x)) \longmapsto (Ku)(x) = \int K(x, x')u(x')dx' .$$

This is an easy consequence of the following lemma:

LEMMA 1.8. *Let $K(x, x')dx'$ and $u(x)$ be germs of $\mathcal{C}_{M\times M}^{(0,n)}$ and $\mathcal{C}_M$ at x respectively. Denote by G and Z the supports of K and u respectively. Suppose that $S_LG\cap\hat{H}(\Theta)=\emptyset$ and that the image E of $S_L(G)$ under the canonical projection $S_L(L\hat{\times}L)_x-\hat{H}(\Theta)_x\rightarrow S_xL$, does not intersect the antipodal of S_xZ. Then Ku can be defined and the normal set of the support of Ku along $\{x\}$ is contained in $\langle E, S_xZ\rangle$.*

Since this is proved in the same way as Lemma 1.4, we omit the proof. The (partial) micro-hyperbolicity is expressed by using the purely imaginary tangential sphere bundle in Definition 1.1. But it is more natural to use the cotangential one. Therefore, we use sometimes the following terminology.

DEFINITION 1.9. Let $\mathfrak{M}$ be a system of pseudo-differential equations. Let x be a point in L and let Γ be a connected set in $S_x^*L-\Theta$ (or more generally in $S_L^*(L\hat{\times}L)_x-\Theta$). We say that $\mathfrak{M}$ is partially micro-hyperbolic at x with respect to the direction in Γ if $\mathfrak{M}$ is partially micro-hyperbolic at any $x+\sqrt{-1}v0$ contained in $H(\Gamma)^a$.

§ 2. Formal elementary solution.

In this section, we construct an elementary solution as a formal series of pseudo-differential operators, and in the next two sections we will show that the elementary solution thus constructed formally can be realized as microfunction if the operator is microhyperbolic. (See Treves [**1**] for related topics.)

Let $P(x, D_x)$ be a (matrix of) pseudo-differential operator. We may assume without loss of generality that the surface $\{x_1=0\}$ is non characteristic with respect to $P(x, D_x)$. Then $P(x, D_x)$ can be represented as

$$S(x, D_x)\{D_1^m+A_1(x, D_x')D_1^{m-1}+\cdots+A_m(x, D_x')\}$$

where $S(x, D_x)$ is an invertible operator and $A_j(x, D_x')$ is an operator of order $\leqq j$ which commutes with x_1 (see S-K-K [**1**] Chap. II, § 2.2). In this way, the equation $Pu=0$ is seen to be equivalent to $(D_1^m+A_1(x, D_x')D_1^{m-1}+\cdots+A_m(x, D_x'))u$

$=0$. At the point under consideration we may assume some of D_j is invertible, say D_n. Then $Pu=0$ is equivalent to

$$\left[D_1-\begin{pmatrix} 0 & D_n & & & \\ & 0 & D_n & & \\ & & \ddots & \ddots & \\ & & & 0 & D_n \\ -A_m D_n^{-m+1} & \cdots & & -A_2 D_n^{-1} & -A_1 \end{pmatrix}\right]\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}=0$$

by the relation $u_j=(D_1/D_n)^{j-1}u$. Therefore, we may assume $P(x, D_x)$ is of the form

$$D_1-A(x, D'_x)$$

where $A(x, D'_x)$ is a matrix of pseudo-differential operators whose components are of order equal to or less than 1. In the sequel, we use the local coordinate system $(t, x_1, \cdots, x_n)$ instead of $(x_1, \cdots, x_n)$ so that $P=D_t-A(t, x, D_x)$. We seek for a formal solution $R(t, x, D_x)$ of the following equation

$$\text{(2.1)} \qquad \begin{cases} \dfrac{\partial}{\partial t} R(t, x, D_x)-A(t, x, D_x)R(t, x, D_x)=0\,, \\ R(t, x, D_x)|_{t=0}=1\,. \end{cases}$$

Before discussing this equation, we prepare several notions.

DEFINITION. Let $(t, x_0, \xi_0)=(0, x_0, \xi_0)$ be a point of cotangential vector bundle of the x-space, $R(t, x, D_x)$ be a formal series $\sum_{j=-\infty}^{\infty} R_j(t, x, D_x)$, where $R_j(t, x, \xi)$ is a holomorphic function homogeneous of degree j with respect to ξ satisfying the following conditions.

(2.2) There is a positive number δ such that $R_j(t, x, \xi)$ is holomorphic on

$$|t|<\delta, \quad |x-x_0|<\delta, \quad |\xi-\xi_0|<\delta\,.$$

(2.3) There is a positive constant v satisfying the following condition: For any $\varepsilon(0<\varepsilon<\delta)$, there is C_ε such that

$$|R_j(t, x, \xi)|\leqq \frac{C_\varepsilon}{j!}(v\varepsilon|\xi|)^j$$

holds for $j\geqq 0$, $|x-x_0|<\delta$, $|\xi-\xi_0|<\delta$ and $|t|\leqq\varepsilon$.

(2.4) There is a constant A such that

$$|R_j(t, x, \xi)|\leqq(-j)!A^{-j} \qquad \text{for } j<0,\ |t|<\delta,\ |x-x_0|<\delta,\ |\xi-\xi_0|<\delta\,.$$

If $R(t, x, D_x)$ satisfies the above conditions, then it is called an operator with finite velocity defined at $(0, x_0, \xi_0)$. The minimum of v which satisfies estimate (2.3) by replacing δ with a suitably small one, is called the velocity of $R(t, x, D_x)$ at $(0, x_0, \xi_0)$. The equation (2.1) obtains its substantial meaning by the following Lemma 2.1.

LEMMA 2.1. *Let $P(t, x, D_t, D_x)=\sum P_j(t, x, D_t, D_x)$ be a pseudo-differential operator defined on $\{(t, x, (\tau dt+\xi dx)\infty)\,;\, |t|<\delta, |x-x_0|<\delta, |\xi-\xi_0|<\delta, |\tau|<\mu|\xi|\}$ and $R(t, x, D_x)=\sum R_j(t, x, D_x)$ be an operator defined at $(0, x_0, \langle\xi_0, dx\rangle\infty)$ with velocity $\leqq v<\mu$. Then*

$$S_l(t, x, \xi)=\sum_{l=j+k-\nu-|\alpha|}\frac{1}{\nu!\alpha!}(D_\tau^\nu D_\xi^\alpha P_j(t, x, \tau, \xi)|_{\tau=0})D_t^\nu D_x^\alpha R_k(t, x, \xi)$$

converges absolutely and uniformly on a neighbourhood of $(0, x_0, \xi_0)$ independent of l, and $S(t, x, D_x)=\sum S_l(t, x, D_x)$ is an operator with velocity $\leqq(v^{-1}-\mu^{-1})^{-1}$. We say that S is a product of P and R and denote $S=PR$.

PROOF. Since $\sum_{k\leqq 0} R_k(t, x, D_x)$ is a pseudo-differential operator, $S_l^{(1)} = \sum_{\substack{l=j+k-\nu-|\alpha| \\ k\leqq 0}}\frac{1}{\nu!\alpha!}D_\tau^\nu D_\xi^\alpha P_j(t, x, 0, \xi)D_t^\nu D_x^\alpha R_k(t, x, \xi)$ converges normally and $\sum S_l^{(1)}(t, x, D_x)$ is also a pseudo-differential operator. Therefore, we can assume that $R_k(t, x, \xi)=0$ for $k<0$.

We may assume that R is with velocity $<v$. We replace δ by a sufficiently small one so that

$$|R_k(t, x, \xi)|\leqq\frac{C_\varepsilon}{k!}(v\varepsilon|\xi_0|)^k$$

holds for $|t|<\varepsilon$, $|x-x_0|$, $|\xi-\xi_0|<2\delta$.

Now set $\rho=(v_1-v)/v$ for $v_1>v$. Then, $v_1\varepsilon=v(1+\rho)\varepsilon$. Therefore, we have

$$|R_k(t, x, \xi)|\leqq\frac{C_\varepsilon}{k!}(v_1\varepsilon|\xi_0|)^k$$

for $|t|\leqq\varepsilon(1+\rho)$, $|x-x_0|$, $|\xi-\xi_0|<2\delta$.

It follows that

$$|D_t^\nu D_x^\alpha R_k(t, x, \xi)|\leqq\frac{C_\varepsilon}{k!}\nu!\alpha!(\varepsilon\rho)^{-\nu}A^{|\alpha|}(v_1\varepsilon|\xi_0|)^k$$

holds for sufficiently large A for $|t|\leqq\varepsilon$, $|x-x_0|<\delta$, $|\xi-\xi_0|<\delta$ by Cauchy's formula.

Moreover we may assume, without loss of generality, that we have the following estimate

$$|P_j(t, x, \tau, \xi)|\leqq\frac{C_h}{j!}h^j$$

for $j>0$, $|t|$, $|x-x_0|$, $|\xi-\xi_0|<2\delta$, $|\tau|\leqq\mu|\xi_0|$

by replacing δ by a sufficiently small one and μ by a suitable one. Hence we have

$$|D_\tau^\nu D_\xi^\alpha P_j(t, x, 0, \xi)|\leqq\frac{C_h}{j!}h^j\nu!\alpha!(\mu|\xi_0|)^{-\nu}A^{|\alpha|}$$

for $|t|$, $|x-x_0|$, $|\xi-\xi_0|<\delta$.

It follows that

$$S_l^{(2)}=\sum_{\substack{l=j+k-\nu-|\alpha|\\ j\geqq 0,\ k\geqq 0\\ \nu\geqq 0,\ \alpha\geqq 0}}\frac{1}{\alpha!\nu!}D_\tau^\nu D_\xi^\alpha P_j(t,x,0,\xi)D_t^\nu D_x^\alpha R_k(t,x,\xi)$$

is smaller than

$$(2.5)\qquad \sum_{\substack{l=j+k-\nu-|\alpha|\\ j\geqq 0,\ k\geqq 0}}\frac{1}{\nu!\alpha!}\Big(\frac{\nu!\alpha!}{j!}C_h h^j(\mu|\xi_0|)^{-\nu}A^{|\alpha|}\Big)\Big(\frac{\nu!\alpha!}{k!}C_\varepsilon(v_1\varepsilon|\xi_0|)^k(\varepsilon\rho)^{-\nu}A^{|\alpha|}\Big)$$

$$=C_\varepsilon C_h\sum_{l=j+k-\nu-|\alpha|}\frac{\nu!\alpha!}{j!k!}h^j(v_1\varepsilon|\xi_0|)^k(\varepsilon\rho\mu|\xi_0|)^{-\nu}A^{2|\alpha|}$$

$$=C_\varepsilon C_h\sum_{l=p-\nu-|\alpha|}\frac{\nu!\alpha!}{p!}(h+v_1\varepsilon|\xi_0|)^p(\varepsilon\rho\mu|\xi_0|)^{-\nu}A^{2|\alpha|}.$$

If $l\geqq 0$, $S_l^{(2)}$ is, therefore, smaller than

$$C_\varepsilon C_h\sum_{\nu,\alpha}\frac{\nu!\alpha!}{(\nu+|\alpha|+l)!}(h+v_1\varepsilon|\xi_0|)^{\nu+|\alpha|+l}(\varepsilon\rho\mu|\xi_0|)^{-\nu}A^{2|\alpha|}$$

$$=\frac{C_\varepsilon C_h}{l!}(h+v_1\varepsilon|\xi_0|)^l\sum_{\nu,\alpha}\frac{\nu!\alpha!l!}{(\nu+|\alpha|+l)!}\Big(\frac{h+v_1\varepsilon|\xi_0|}{\varepsilon\rho\mu|\xi_0|}\Big)^\nu(A^2(h+v_1\varepsilon|\xi_0|))^{|\alpha|}$$

$$=\frac{C_\varepsilon C_h}{l!}(h+v_1\varepsilon|\xi_0|)^l\Big\{\sum_\nu\Big(\frac{h+v_1\varepsilon|\xi_0|}{\varepsilon\rho\mu|\xi_0|}\Big)^\nu\Big\}\Big\{\sum_{s=0}^\infty(A^2(h+v_1\varepsilon|\xi_0|))^s\Big\}^n.$$

Hence, if $A^2(h+v_1\varepsilon|\xi_0|)<1$ and $(h+v_1\varepsilon|\xi_0|)/\varepsilon\rho\mu|\xi_0|<1$, then $S_l^{(2)}$ converges uniformly and satisfies condition (2.3) with velocity $\leqq v_1$ because we can take h as small as we please. Clearly $A^2(h+v_1\varepsilon|\xi_0|)<1$ by taking h and ε sufficiently small. Moreover $(h+v_1\varepsilon|\xi_0|)/\varepsilon\rho\mu|\xi_0|<1$ is verified if $v_1\varepsilon|\xi_0|/\varepsilon\rho\mu|\xi_0|=v_1/\rho\mu<1$ since we can take h as small as we please. The inequality $v_1/\rho\mu<1$ is verified if $v_1>(v^{-1}-\mu^{-1})^{-1}$. Therefore $S_l^{(2)}$ $(l\geqq 0)$ satisfies the condition (2.3) with velocity $(v^{-1}-\mu^{-1})^{-1}$.

Suppose $l<0$. Then $S_l^{(2)}$ is smaller than

$$C_\varepsilon C_h\sum_{l=p-\nu-|\alpha|}\frac{\nu!\alpha!}{p!}(h+v_1\varepsilon|\xi_0|)^p(\varepsilon\rho\mu|\xi_0|)^{-\nu}A^{2|\alpha|}$$

$$\leqq C_\varepsilon C_h\sum_{l=p-q}\frac{q!}{p!}(h+v_1\varepsilon|\xi_0|)^p\Big(\frac{1}{\varepsilon\rho\mu|\xi_0|}+nA^2\Big)^q$$

$$\leqq C_\varepsilon C_h\sum\frac{(p-l)!}{p!}(h+v_1\varepsilon|\xi_0|)^p\Big(\frac{1}{\varepsilon\rho\mu|\xi_0|}+nA^2\Big)^{p-l}$$

$$\leqq C_\varepsilon C_h(-l)!\Big\{2\Big(\frac{1}{\varepsilon\rho\mu|\xi_0|}+nA^2\Big)\Big\}^{-l}\sum_p\Big\{(h+v_1\varepsilon|\xi_0|)\Big(\frac{1}{\varepsilon\rho\mu|\xi_0|}+nA^2\Big)\Big\}^p.$$

Since we can take ε and h so that $(h+v_1\varepsilon|\xi_0|)(1/\varepsilon\rho\mu|\xi_0|+nA^2)<1$, $S_l^{(2)}$ $(l<0)$ satisfies condition (2.3).

In this way we can prove that $S_l^{(2)}$ converges absolutely and uniformly and satisfies conditions (2.3) and (2.4) for every v_1 such that $v_1^{-1}+\mu^{-1}<v^{-1}$. Lastly

we show that

$$S_l^{(3)}=\sum_{\substack{l=j+k-\nu-|\alpha| \\ j\geqq 0,\ k\geqq 0 \\ \nu=0,\ \alpha\geqq 0}}\frac{1}{\nu!\alpha!}D_\tau^\nu D_\xi^\alpha P_{-j}(t, x, 0, \xi)D_t^\nu D_x^\alpha R_k(t, x, \xi)$$

possesses the same property. We may assume $P_j(t, x, \tau, \xi)$ satisfies

$$|P_{-j}(t, x, \tau, \xi)|\leqq j!A^j \qquad \text{for } |t|<\mu|\xi_0|,\ |x-x_0|,\ |\xi-\xi_0|<2\delta,\ j>0.$$

Therefore we have

$$|D_\tau^\nu D_\xi^\alpha P_j(t, x, 0, \xi)|\leqq j!\nu!\alpha!(\mu|\xi_0|)^{-\nu}A^{j+|\alpha|}$$
$$\text{for } \ j>0,\ |x-x_0|,\ |\xi-\xi_0|<\delta.$$

Therefore $S_l^{(3)}$ is smaller than

$$\sum_{\substack{l=j+k-\nu-|\alpha| \\ j\geqq 0,\ k\geqq 0 \\ \nu\geqq 0,\ \alpha\geqq 0}}\frac{1}{\nu!\alpha!}j!\nu!\alpha!(\mu|\xi_0|)^{-\nu}A^{j+|\alpha|}\frac{C_\varepsilon}{k!}\nu!\alpha!(\varepsilon\rho)^{-\nu}A^{|\alpha|}(v_1\varepsilon|\xi_0|)^k$$
$$=C_\varepsilon\sum_{l=k-j-\nu-|\alpha|}\frac{j!\nu!\alpha!}{k!}A^j(\varepsilon\rho\mu|\xi_0|)^{-\nu}(v_1\varepsilon|\xi_0|)^kA^{2|\alpha|}$$
$$\leqq C_\varepsilon\sum_{l=k-p-|\alpha|}\frac{p!\alpha!}{k!}(A+(\varepsilon\rho\mu|\xi_0|)^{-1})^p(v_1\varepsilon|\xi_0|)^kA^{2|\alpha|}$$
$$\leqq C_\varepsilon\sum_{l=k-p-|\alpha|}(v_1\varepsilon|\xi_0|)^k(\varepsilon\rho\mu|\xi_0|/(1+A\varepsilon\rho\mu|\xi_0|))^{-p}A^{2|\alpha|}.$$

Since this series has the same form as series (2.5), we can conclude that $S_l^{(3)}$ converges uniformly and satisfied the estimations concerning the operator with velocity v_1. Q.E.D.

Since the meaning of the equation (2.1) is clarified by Lemma 2.1, we can give the unique existence theorem of the solution of the equation (2.1).

PROPOSITION 2.2. *Suppose that* $P(t, x, D_t, D_x)=D_t-A(t, x, D_x)$ *is defined at* $(0, x_0, \xi_0\infty)$. *Then, there exists a unique solution* $R(t, x, D_x)$ *of the equation* (2.1) *which is an operator defined at* $(0, x_0, \xi_0\infty)$ *with finite velocity.*

PROOF. In order to prove the uniqueness, it is sufficient to show

$$\begin{cases}(D_t-A(t, x, D_x))R(t, x, D_x)=0\\ R(0, x, D_x)=0\end{cases}$$

implies $R=0$. For that purpose if suffices to show that $\left(\frac{\partial}{\partial t}\right)^\nu R(t, x, D_x)|_{t=0}=0$ for every ν. We prove this by induction on ν. Since

$$\left(\frac{\partial}{\partial t}\right)^\nu R(t, x, D_x)=\left(\frac{\partial}{\partial t}\right)^{\nu-1}(AR)$$
$$=\sum_{\mu=0}^{\nu-1}\binom{\nu-1}{\mu}\left(\frac{\partial}{\partial t}\right)^{\nu-\mu-1}A\left(\frac{\partial}{\partial t}\right)^\mu R,$$

$$\left(\frac{\partial}{\partial t}\right)^{\nu} R|_{t=0}=\sum_{\mu=0}^{\nu-1}\binom{\nu-1}{\mu}\left(\frac{\partial}{\partial t}\right)^{\nu-\mu-1} A|_{t=0}\left(\frac{\partial}{\partial t}\right)^{\mu} R|_{t=0}$$
$$=0$$

by the induction hypothesis. Hence follows the uniqueness.

Now let us prove the existence of R. We define $R^{(j)}(t, x, D_x)$ inductively by

$$R^{(0)}(t, x, D_x)=1,$$
$$D_t(R^{(j)}(t, x, D_x))=A(t, x, D_x)R^{(j-1)}(t, x, D_x),$$
$$R^{(j)}(t, x, D_x)|_{t=0}=0 \quad \text{for} \quad j \geqq 1.$$

Then, if $R=\sum_j R^{(j)}$ converges (that is, each homogeneous part converges) then R is evidently a solution of the equation (2.1). Note that each $R^{(j)}$ is a pseudo-differential operator of order $\leqq j$. Set $R^{(j)}=\sum_{k=0}^{\infty} R_{j-k}^{(j)}(t, x, D_x)$, where $R_{j-k}^{(j)}(t, x, \xi)$ are functions homogeneous of degree $(j-k)$ with respect to ξ.

We use the norm N_j introduced in Boutet de Monvel and Krée [**1**]. This is a power series in λ defined by

$$N_j(Q)(\lambda, t)=\sum_{k,\alpha,\beta} \frac{2(2n)^{-k} k!}{(|\alpha|+k)!(|\beta|+k)!} \sup_{\substack{|x-x_0|<\delta \\ |\xi-\xi_0|<\delta}} |D_x^{\alpha} D_{\xi}^{\beta} Q_{j-k}(t, x, \xi)| \lambda^{2k+|\alpha+\beta|}$$

for a pseudo-differential operator $Q(t, x, D_x)=\sum_{k=0}^{\infty} Q_{j-k}(t, x, D_x)$ of order j.

Concerning the properties of this norm we refer to Boutet de Monvel-Krée [**1**]. Using this norm we have

$$N_j(D_t R^{(j)}) \ll N_1(A) N_{j-1}(R^{(j-1)}).$$

It follows that $N_j(R^{(j)}) \leqq 2N_1(A)^j |t|^j/j!$. In fact, it is true for $j=0$. If this is true for $j-1$, then

$$N_j(D_t R^{(j)}) \leqq 2N_1(A)^j |t|^{j-1}/(j-1)!.$$

Since

$$R^{(j)}=\int_0^t D_t R^{(j)} dt,$$

we have

$$N_j(R^{(j)}) \leqq \int_0^{|t|} 2N_1(A)^j |t|^{j-1}/(j-1)! |dt|$$
$$=2N_1(A)^j |t|^j/j!.$$

We may suppose that $N_1(A) \ll \dfrac{b}{1-a\lambda}$. Therefore

$$|R_{j-k}^{(j)}| \leqq \frac{k!}{(2n)^{-k}} \frac{|t|^j}{j!} b^j a^{2k}\binom{j+2k-1}{j}.$$

Consequently for $j \geqq l \geqq 0$

$$|R_l^{(j)}| \leqq \frac{(j-l)!(2n)^{j-l}}{j!} |t|^j b^j a^{2(j-l)} \binom{2j-2l-1}{j}$$

$$\leqq \frac{(2n)^{j-l} 2^{2j-2l-1}}{j!} b^j a^{2(j-l)} |t|^j,$$

which implies

$$\sum_{j=l}^{\infty} |R_l^{(j)}| \leqq \sum_{\nu=0}^{\infty} \frac{(2n)^\nu 2^{2\nu-1} b^{l+\nu} a^{2\nu}}{(\nu+l)!} |t|^{l+\nu}$$

$$= \frac{(b|t|)^l}{2l!} \sum_{\nu=0}^{\infty} \frac{l!}{(\nu+l)!} (8na^2 b|t|)^\nu$$

$$\leqq \frac{(b|t|)^l}{2l!} e^{8na^2 b|t|}.$$

Now suppose that $l > 0$. Then

$$|R_{-l}^{(j)}| \leqq \frac{(j+l)!(2n)^{j+l} 2^{2j+2l-1}}{j!} \frac{|t|^j}{j!} b^j a^{2(j+l)}.$$

Hence,

$$\sum_{j=0}^{\infty} |R_{-l}^{(j)}| \leqq \frac{(8na^2)^l}{2} \sum_{j=0}^{\infty} \frac{(j+l)!}{(j!)^2} (8nba^2|t|)^j$$

$$\leqq \frac{(8na^2)^l 2^l l!}{2} \sum_{j=0}^{\infty} \frac{1}{j!} (16nba^2|t|)^j$$

$$\leqq \frac{(16na^2)^l l!}{2} e^{16nba^2|t|}.$$

Therefore, the estimate (2.3) is verified, and the proof of Proposition 2.2 is established.

REMARK. The above estimate shows that the velocity of R is $b/|\xi_0|$, and we can take b sufficiently close to $|A_1(0, x_0, \xi_0)|$. Moreover, we can choose a norm $|\ |$ in order that $|A_1(0, x_0, \xi_0)|$ is sufficiently close to the maximum of the absolute value of eigenvalues of $A_1(0, x_0, \xi_0)$. It follows that the velocity of R is less than the maximum m of the absolute values of the eigenvalues of $A_1(0, x_0, \xi_0)/|\xi_0|$.

In fact, Weierstrass' preparation theorem for pseudo-differential operators (S-K-K [**1**] Chap. II. § 2.2) allows us to localize the problem with respect to τ (Cf. S-K-K [**1**] p. 409) and we easily find that (at least some component of) R has the velocity m exactly.

§ 3. Operation of pseudo-differential operators.

In the preceding section, we constructed a formal elementary solution. We want to say that the domain of definition of the formal elementary solution is so large that we can obtain its boundary value. In order to facilitate this program, we will give an explicit representation of the operation of pseudo-differential operators on microfunctions by using their defining functions. We begin by expressing the integration of microfunctions from the point of view of defining functions.

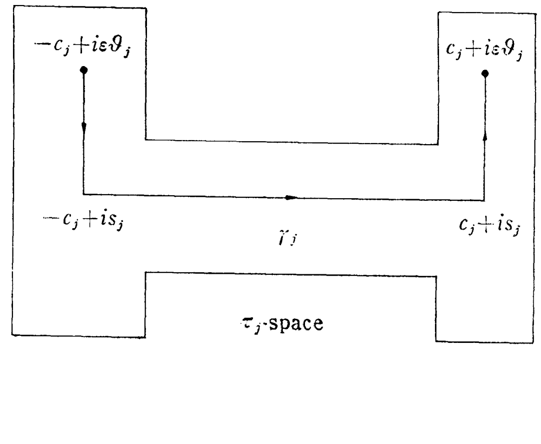
LEMMA 3.1. *Let $u(t, x)$ be a microfunction of variables $(t, x)=(t_1, \cdots, t_r, x_1, \cdots, x_n)\in G\times D\subset \boldsymbol{R}^r\times\boldsymbol{R}^n$, where G is an open set in t-space and D is an open set in x-space. Let K be a compact set in G, Γ be a non void open convex cone in (t, x)-space and $\tilde{\Gamma}$ be an open convex cone containing Γ and $(\vartheta, 0)$ for some vector $\vartheta=(\vartheta_1, \cdots, \vartheta_r)$. Suppose $u(t, x)$ is the spectrum of the boundary value of a holomorphic function $\varphi(\tau, z)$ defined on $\Omega=\Omega_1\cup\Omega_2$, where $\Omega_1=\{(\tau, z)\in \boldsymbol{C}^r\times\boldsymbol{C}^n; \operatorname{Re}\tau\in G, \operatorname{Re}z\in D, |\operatorname{Im}\tau|, |\operatorname{Im}z|<\delta$ and $(\operatorname{Im}\tau, \operatorname{Im}z)\in\Gamma\}$ and $\Omega_2=\{(\tau, z); \operatorname{Re}\tau\in G-K, |\operatorname{Re}z|<a, |\operatorname{Im}\tau|, |\operatorname{Im}z|<\delta, (\operatorname{Im}\tau, \operatorname{Im}z)\in\tilde{\Gamma}\}$. Then $v(x)=\int u(t, x)dt$ is a boundary value of a holomorphic function $\psi(z)$ defined in $\{z; |\operatorname{Re}z|<a, |\operatorname{Im}z|<\delta/2, (s, \operatorname{Im}z)\in\Gamma$ for some s with $|s|<\delta/2\}$, which is given by*

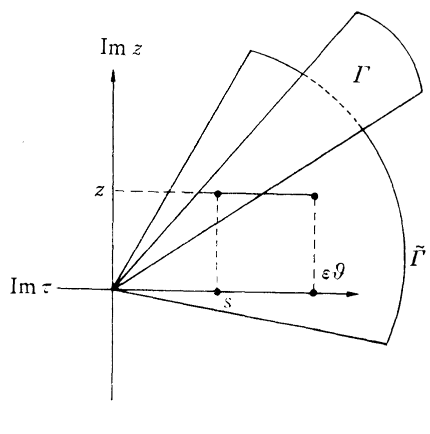
$$\psi(z)=\int_\sigma \varphi(\tau, z)d\tau$$

where σ is an r-dimensional chain (depending on z) such that (σ, z) is contained in Ω and that its coboundary $\beta=\partial\sigma$ is an $(r-1)$-dimensional cycle independent of z in $U=\{\tau; \operatorname{Re}\tau\in G-K, |\operatorname{Im}\tau|<\delta, (\operatorname{Im}\tau, 0)\in\tilde{\Gamma}\}$, whose homology class in $H_{r-1}(U: \boldsymbol{C})=H_{r-1}(G-K; \boldsymbol{C})$ is the image of $1\in\boldsymbol{C}$ by the homomorphism $\boldsymbol{C}\to H^0(K; \boldsymbol{C})\simeq H_r(G, G-K; \boldsymbol{C})\to H_{r-1}(G-K; \boldsymbol{C})$. If G and K are polydiscs, i.e., $G=\{t; |t_j|<b_j\}$ and $K=\{t; |t_j|\leqq a_j\}$, then we can take another chain $\gamma_1\times\cdots\times\gamma_n$ so that

$$\psi(z)=\int_{\gamma_1\times\cdots\times\gamma_n}\varphi(\tau, z)d\tau\,,$$

where γ_j is a path in τ_j-space constructed in the following way:





Let c_j *be a fixed number such that* $b_j < c_j < a_j$, *and let* ε *be such that* $|\varepsilon\vartheta| < \delta$. *If* $(\operatorname{Im} z, s) \in \Gamma$, *then* γ_j *is a path which starts from* $-c_j + i\varepsilon\vartheta_j$ *and ends at* $c_j + i\varepsilon\vartheta_j$ *through* $-c_j + is_j$ *and* $c_j + is_j$; *that is,* $\gamma_j = \{-c_j + i\lambda\,;\, \varepsilon\vartheta_j \geqq \lambda \geqq s_j\} \cup \{\lambda + is_j\,;\, -c_j \leqq \lambda \leqq c_j\} \cup \{c_j + i\lambda\,;\, s_j \leqq \lambda \leqq \varepsilon\vartheta_j\}$, *as seen in the above figures.*

Note that $\psi(z)$ is independent of the choice of σ. In fact, if σ' is another chain such that $\partial\sigma' = \beta$, then $\partial(\sigma - \sigma') = 0$. Moreover, since $\{\tau\,;\,(\tau, z) \in \Omega\}$ is homotopically equivalent to G, its r-th homology group is zero. It follows that there is an $(r+1)$-chain γ such that $\partial\gamma = \sigma - \sigma'$, which implies $\int_{\sigma-\sigma'} \varphi(\tau, z)d\tau = 0$ by Cauchy's integral formula. If we replace β with another β', then the obtained integral $\psi'(z) = \int_{\sigma'} \varphi(\tau, z)d\tau$ with $\partial\sigma' = \beta'$ is equal to $\psi(z)$ modulo real analytic function. In fact, since $\beta - \beta'$ is homologous to zero, there is an r-dimensional chain γ in U such that $\partial\gamma = \beta - \beta'$. Therefore

$$\int_{\sigma-\sigma'} \varphi(\tau, z)d\tau = \int_{\sigma-\sigma'-\gamma} \varphi(\tau, z)d\tau + \int_{\gamma} \varphi(\tau, z)d\tau = \int_{\gamma} \varphi(\tau, z)d\tau$$

because $\partial(\sigma - \sigma' - \gamma) = 0$. Since γ is in U, $\int_{\gamma} \varphi(\tau, z)d\tau$ is a real analytic function in z.

If G and K are polydiscs, the integral $\int_{\gamma_1 \times \cdots \times \gamma_n} \varphi(\tau, z)d\tau$ is equal to $\int_{\sigma} \varphi(\tau, z)d\tau$ where $\beta = \partial\sigma = \{t + \sqrt{-1}\varepsilon\vartheta\,;\, -c_j \leqq t \leqq c_j\}$ by Cauchy's integral formula.

Now let us prove the lemma. Note that the support of $u(t, x)$ is contained in $\sqrt{-1}\Gamma^\circ\infty$, where Γ° is the polar of Γ, and contained in $\sqrt{-1}\tilde{\Gamma}^\circ\infty$ on $(G-K) \times D$. By the flabbiness of $\mathcal{C}$, there is a microfunction $u'(t, x)$ defined on $\boldsymbol{R}^r \times D$ such that $u = u'$ in $K \times D$, its support is contained in $\sqrt{-1}\Gamma^\circ\infty$ on $\boldsymbol{R}^r \times D$ and in $\sqrt{-1}\tilde{\Gamma}^\circ\infty$ on $(G-K) \times D$, and that u' is zero on $(\boldsymbol{R}^r - G) \times D$. Therefore, if G' is a polydisc containing G, then we obtain a holomorphic function $\varphi'(\tau, z)$ defined on $\Omega \cup (G - {}'G) \times D$, such that u' is the boundary value of φ' by shrinking δ, D, Γ, $\tilde{\Gamma}$ a little and replacing K with a sufficiently large one.

Note that $\psi(\tau, z) = \varphi(\tau, z) - \varphi'(\tau, z)$ is defined on $\{(\tau, z) : \operatorname{Re}\tau \in G, \operatorname{Re} z \in D, |\operatorname{Im} z|, |\operatorname{Im}\tau| < \delta, (\operatorname{Im}\tau, \operatorname{Im} z) \in \tilde{\Gamma}\}$. It follows that $\int_{\sigma} \psi(\tau, z)d\tau$ is real analytic in z. Therefore $\int_{\sigma} \varphi(\tau, z)d\tau = \int_{\sigma} \varphi'(\tau, z)d\tau$ modulo real analytic function. Since $\int u(t, x)dt = \int u'(t, x)dt$, it is sufficient to prove in the case of φ' instead of φ. Therefore, we may assume, from the first, that G and K are polydiscs, i.e., $G = \{t\,;\, |t| < b_j\}$, $K = \{t\,;\, |t| \leqq a_j$ and that $\tilde{\Gamma} = \boldsymbol{R}^r$ and $\vartheta = 0$. Since $v(x) = \int dt_1 \cdots \int dt_r u(t, x)$ and $\psi(z) = \int_{\gamma_1} d\tau_1 \cdots \int_{\gamma_r} d\tau_r \varphi(\tau, z)$, we can assume $r = 1$ by the induction on r. Set $f(\tau, z)=\int_{-c}^{\tau}\varphi(\tau, z)d\tau$ with $a<c<b$. This is analytic on Ω. We have $\frac{\partial}{\partial\tau}f(\tau, z)=\varphi(\tau, z)$. If we set $w(t, x)$ the boundary value of the holomorphic function $f(\tau, z)$, then $\frac{\partial}{\partial t}w(t, x)=u(t, x)$. It follows that

$$\int u(t, x)dt=\int\frac{\partial}{\partial t}w(t, x)dt=w(c, x)-w(-c, x)=w(c, x)$$

$w(c, x)$ is a boundary value of $f(c, z)=\int_{-c}^{c}\varphi(\tau, z)d\tau$. This completes the proof of Lemma 3.1.

Now, we will explain the operation of a pseudo-differential operator on microfunctions. Let M be a real analytic manifold and X be its Stein complexification. We take a (local) coordinate system $z=(z_1, \cdots, z_n)$ of X. We do not assume that z_j is real valued on M. Let $P(z, D_z)$ be a pseudo-differential operator on X, with the defining function $L(z, w)dw$, where $w=(w_1, \cdots, w_n)$ are coordinates of a copy of X corresponding to z. Assume that $L(z, w)$ has the form $L(z, w)=L_0(z, w)+\frac{1}{2\pi\sqrt{-1}}L_1(z, w)\log(z_1-w_1)$, and $L_0(z, w)$ is defined on $\Omega_0=\{(z, w)\in X\times X;\ z_\nu\neq w_\nu$ for every $\nu\}$ and $L_1(z, w)$ is defined on $\Omega_1=\{(z, w)\in X\times X;\ |z-w|<\delta,\ |z_\nu-w_\nu|>a_\nu|z_1-w_1|$ for $\nu=2, \cdots, n\}$. Therefore $L(z, w)$ is a (multivalued) analytic function defined on $\Omega=\{(z, w)\in X\times X;$ $|z-w|<\delta,\ |z_\nu-w_\nu|>a_\nu|z_1-w_1|>0$ for $\nu=2, \cdots, n\}$.

Remember that if $P(z, D_z)=\sum a_\alpha(z)D_z^\alpha$ then we can take $L(z, w)=\sum a_\alpha(z)\Phi_\alpha(z-w)$. Here $\Phi_\alpha(z)=\Phi_{\alpha_1}(z_1)\cdots\Phi_{\alpha_n}(z_n)$, where

$$\Phi_j(z_k)=\begin{cases}\dfrac{1}{2\pi\sqrt{-1}}\dfrac{j\,!}{(-z_k)^{j+1}} & \text{for } j\geqq 0\\[2ex] -\dfrac{z_k^{-j-1}}{2\pi\sqrt{-1}(-j-1)\,!}(\log(-z_k)-\psi(-j)) & \text{for } j<0\,.\end{cases}$$

$$(\psi(\nu)=\sum_{k=1}^{\nu-1}\frac{1}{k}-\gamma \text{ denotes the di-gamma function.})$$

It is easy to see that P is defined on $Z=\{(z, \langle\zeta, dz\rangle\infty)\in P^*X;\ \sum_{\nu=2}^{n}a_\nu|\zeta_\nu|<|\zeta_1|\}$. Let γ be the canonical projection $S^*X\to P^*X$. Let $\tilde{U}$ be an open convex cone in $\sqrt{-1}SM$ such that $\tau(\tilde{U})=M$ and U be an open set in X such that $\tilde{U}\cup(U-M)$ is an open set in the real monoidal transform $\widetilde{^MX}$. Let $\varphi(z)$ be a holomorphic function defined on U, then φ defines a section of $\tilde{\mathfrak{A}}_M$ on $\tilde{U}$. Let u be the spectrum of the hyperfunction obtained as boundary value of $\varphi(z)$. u is a microfunction defined on $\sqrt{-1}\ S^*M$ whose support is contained in the polar $\tilde{U}^\circ$ of $\tilde{U}$. We assume that $\tilde{U}^\circ$ is contained in $\gamma^{-1}(Z)$. The example of such $\tilde{U}$ is

(3.1) $\left\{z+\mathrm{Re}\left\langle\tau, \frac{\partial}{\partial z}\right\rangle 0 \in S_M X;\ \mathrm{Re}\left\langle\tau, \frac{\partial}{\partial z}\right\rangle\right.$ is not parallel to M at z, $b_\nu \mathrm{Re}(\lambda\tau_1) > |\tau_\nu|$ for $\nu=2, \cdots, n$ and $\left.\mathrm{Im}(\lambda\tau_1) < c|\mathrm{Re}(\lambda\tau_1)|\right\}$ where c is a positive number, λ is a non zero complex number and b_ν is a positive number such that $b_\nu > \sqrt{1+c^2}\, a_\nu / c|\lambda|$,

and

$$U=\{z\in X;\ z=z'+\tau,\ z'\in M,\ |\tau|<\delta,\ b_\nu(\mathrm{Im}\,\lambda\tau_1)>|\tau_\nu|$$
$$\text{for } \nu=2, \cdots, n \text{ and } \mathrm{Im}(\lambda\tau_1)>c|\mathrm{Re}(\lambda\tau_1)|$$
$$\text{for some } \delta>0\}.$$

Let $z^0=(z_1^0, \cdots, z_n^0)$ be a point in M.

LEMMA 3.2. *There is* $\lambda\in \boldsymbol{C}^*$ *such that* $\tilde{U}^\circ \cap \pi^{-1}(z^0)$ *is contained in* $\{\mathrm{Re}\langle\zeta, dz\rangle\infty;\ |\lambda|\mathrm{Im}(\lambda^{-1}\zeta_1) > \sum_{\nu=0}^{n} a_\nu|\zeta_\nu|\}$.

PROOF. The set $C=\{(\alpha, t)\in \boldsymbol{C}\times\boldsymbol{R};$ there is $\zeta\in\tilde{U}^\circ\cap\pi^{-1}(z^0)$ such that $\alpha=\zeta_1,\ t=\sum a_\nu|\zeta_\nu|\}$ is a convex closed cone in $\boldsymbol{C}\times\boldsymbol{R}$ contained in $|\alpha|>t\geqq 0$. Therefore, there is λ such that $\mathrm{Im}(\lambda^{-1}\alpha)>|\lambda^{-1}|t$ on C. Q. E. D.

By this lemma, $\tilde{U}^\circ$ is contained in $\{(z, \mathrm{Re}\langle\zeta, dz\rangle\infty)\in S_M^*X;\ |\lambda|^{-1}\sum_{\nu=0}^{n} b_\nu|\zeta_\nu| < \mathrm{Im}(\lambda^{-1}\zeta_1) - c|\mathrm{Re}(\lambda^{-1}\zeta_1)|\}$ with suitable $b_\nu > \sqrt{1+c^2}\, a_\nu$. It follows that $\tilde{U}$ contains $\left\{z+\mathrm{Re}\langle\tau, \frac{\partial}{\partial z}\rangle 0\in S_M X;\ \mathrm{Im}(\lambda\tau_1)\geqq c^{-1}|\mathrm{Re}(\lambda\tau_1)|,\ \mathrm{Im}(\lambda\tau_1)\geqq \frac{|\lambda|}{b_\nu}|\tau_\nu|\ (\nu=2, \cdots, n)\right\}$ and therefore $\{\mathrm{Im}(\lambda\tau_1)\geqq c^{-1}|\mathrm{Re}(\lambda\tau_1)|,\ c_\nu|\tau_1|\geqq|\tau_\nu|\ (\nu=2, \cdots, n)\}$, if we take $a_\nu < c_\nu < b_\nu/\sqrt{1+c^2}$.

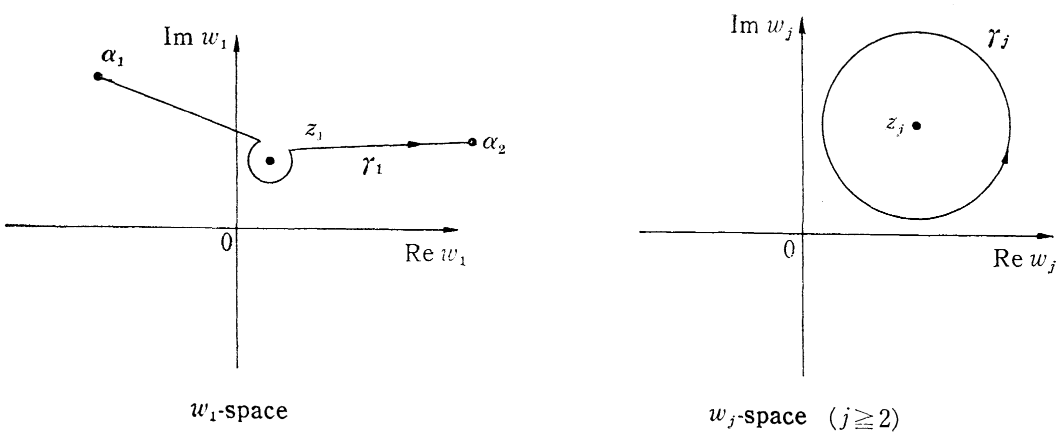
THEOREM 3.3. *Let* α_1, α_2 *be two points sufficiently near* z_1^0 *such that* $c\,\mathrm{Im}((\alpha_k - z_1^0)\lambda^{-1}) > |\mathrm{Re}((\alpha_k - z_1^0)\lambda^{-1})|$ $(k=1, 2)$.

Set

$$\psi(z)=\int_{\gamma_1\times\cdots\times\gamma_n} L(z, w)\varphi(w)dw$$

where γ_j *are paths determined as follows:*

γ_1 *is a path starting from* α_1 *and ending at* α_2 *around* z_1 *counterclockwise, and* γ_j $(j\geqq 2)$ *is a cycle rounding* z_j *counterclockwise* (*with radius* $> a_j|z_1 - w_1|$).



Then $\phi(z)$ can be defined in an open set U' in $X-M$ such that $U' \cup \tilde{U}$ is open at $\pi^{-1}(z^0)$, and Pu coincides with the spectrum of the boundary value of $\phi(z)$.

Since the proof of the theorem is rather long and tedious, we decompose the proposition into several lemmas. First, we will prove

SUBLEMMA 3.4. *Let Δ be a compact set in $S^{2n-1}=(\boldsymbol{C}^n-0)/\boldsymbol{R}^+$ such that $z^0+\mathrm{Re}\langle\delta, \frac{\partial}{\partial z}\rangle 0$ is contained in $\tilde{U}$ for every $\delta\in\Delta$. Then there exists a sufficiently small $\varepsilon>0$ satisfying the following condition: If $z^0+\mathrm{Re}\langle\tau, \frac{\partial}{\partial z}\rangle 0$ is contained in $\tilde{U}$, then U contains $z+t\tau'+\delta$ for any $z\in M$, $t\in\boldsymbol{R}^+$, $\delta\in\boldsymbol{C}^n-\{0\}$ such that $|z-z^0|\ll 1$, $0<t\ll 1$, $|\delta|<\varepsilon$, $|\tau-\tau'|\ll 1$ and $\delta 0\in\Delta$.*

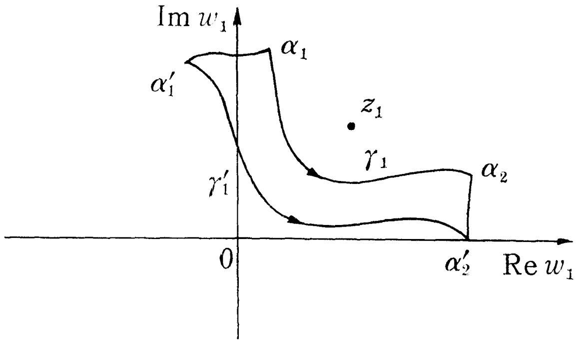
PROOF. If this is false, then there are sequences $\delta^{j,\mu}\in\boldsymbol{C}^n-\{0\}$, $\tau^{j,\mu}\in\boldsymbol{C}^n$, $t^{j,\mu}\in\boldsymbol{R}^+$, $z^{j,\mu}\in M$ such that $U\not\ni z^{j,\mu}+t^{j,\mu}\tau^{j,\mu}+\delta^{j,\mu}$ and $\delta^{j,\mu}0\in\Delta$, $|\delta^{j,\mu}|<1/j$ $t^{j,\mu}\to 0$, $|z^{j,\mu}-z^0|$, $|\tau^{j,\mu}-\tau|\to 0$ if $\mu\to\infty$. If $(t^{j,\mu}\tau^{j,\mu}+\delta^{j,\mu})0$ has a cluster point ρ when $\mu\to\infty$, $j\to\infty$, then $\tilde{U}\not\ni z^0+\mathrm{Re}\langle\rho, \frac{\partial}{\partial z}\rangle 0$. Since ρ is in the convex hull of τ and Δ, this is a contradiction.

LEMMA 3.5. *$\phi(z)$ is holomorphic in some U' chosen as in Theorem 3.3 and independent of α_k modulo real analytic function.*

PROOF. We can take $\lambda=1$ without loss of generality. Let $z^0+\mathrm{Re}\langle\tau^1, \frac{\partial}{\partial z}\rangle 0$ be in $\tilde{U}$. Then $z+t\tau^2+\sigma$ is in U, if $|z-z^0|\ll 1$, $z\in M$, $0<t\ll 1$, $|\tau^1-\tau^2|\ll 1$, $|\sigma|<\varepsilon$, $\mathrm{Im}\,\sigma_1\geqq c^{-1}|\mathrm{Re}\,\sigma_1|$, $c_\nu|\sigma_1|\geqq|\sigma_\nu|$, $\nu=2,\cdots,n$ for some ε independent of τ^1. We choose convex U and α_k such that $|\alpha_k|-z_1^0<\varepsilon$. It suffices to show that ϕ can be defined at $z=\tilde{z}+t\tau$ with $\tilde{z}\in M$, $t\in\boldsymbol{R}$, $\tau\in\boldsymbol{C}^n$ such that $|\tilde{z}-z|\ll 1$, $0<t\ll 1$, $|\tau-\tau^1|\ll 1$. Set $W=\{(z,w);(z,w)\in\Omega, w\in U\}$. It suffices to show that $\gamma_1\times\cdots\times\gamma_n$ is contained in W. We take as γ_j the cycle $|w_j-z_j|=c_j|w_1-z_1|$ $(j=2,\cdots,n)$. Since U contains a neighbourhood of z, it suffices to show that $\{w;|w_j-z_j|=c_j|z_1-w_1|, w_1$ is in a segment jointing z_1 and $\alpha_k\}$ is contained in U if $z=\tilde{z}+t\tau$. We have $\mathrm{Im}(w_1-z_1)\geqq c^{-1}|\mathrm{Re}(w_1-z_1)|$ because $|z_1-z_1^0|\ll 1$. Therefore $w=z+(w-z)=\tilde{z}+t\tau+(w-z)$ is contained in U. If we

obtain function $\psi'(z)$ by changing α_k by other α'_k $(k=1, 2)$ we have

$$\psi(z)-\psi'(z)=\int_{\alpha_1}^{\alpha'_1} dw_1\int_{\gamma_2\times\cdots\times\gamma_n} L(z, w)\varphi(w)dw$$
$$-\int_{\alpha_2}^{\alpha'_2} dw_1\int_{\gamma_2\times\cdots\times\gamma_n} L(z, w)\varphi(w)dw\,.$$



It is obvious that $\int_{\alpha_k}^{\alpha'_k} dw_1\int L(z, w)\varphi(w)dw$ is defined in a neighbourhood of z^0. Q. E. D.

By virtue of the preceding lemma, we can localize Theorem 3.3 in $\sqrt{-1}S^*M$, that is, we may assume that $\tilde{U}^\circ$ is sufficiently small.

In order to prove Theorem 3.3, we will construct a micro-local operator corresponding to P, which is given in S-K-K [1] Chap. III. Prop. 1.2.1. There we constructed a homomorphism $\gamma^{-1}\mathscr{P}_X \to \mathscr{L}_M$ as the composition of $\gamma^{-1}\mathscr{P}_X \to \mathscr{P}^R_X \to \mathscr{L}_M$, where $\mathscr{P}^R_X = C^{R(0,n)}_{X|X\times X} = C^R_{X|X\times X} \underset{p_2^{-1}\mathcal{O}_X}{\otimes} p_2^{-1}\Omega^n_X$ ($p_2: P^*_X(X\times X)\to X\times X\to X$ the second projection). Therefore, we prove the theorem corresponding to Theorem 3.3 by using $\mathscr{P}^R$ instead of $\mathscr{P}$. Let $L(z, w)$ be a holomorphic function defined on $\Omega=\{(z, w)\in X\times X;\ |z_\nu-w_\nu|>a_\nu|z_1-w_1|$ (for $\nu=2, \cdots, n$), $c\,\mathrm{Im}\,(\lambda(z_1-w_1))> -|\mathrm{Re}(\lambda(z_1-w_1))|\}$. Then $V_1=\{(z, w)\in X\times X;\ c\,\mathrm{Im}(\lambda(z_1-w_1))>-|\mathrm{Re}(\lambda(z_1-w_1))|\}$, $V_\nu=\{(z, w)\in X\times X;\ |z_\nu-w_\nu|>a_\nu|z_1-w_1|\}$ $(\nu=2, \cdots, n)$, constitute a Stein covering of $X\times X-Z$ where $Z=\{(z, w)\in X\times X;\ |z_\nu-w_\nu|\leqq a_\nu|z_1-w_1|$ $(\nu=2, \cdots, n)$, $c\,\mathrm{Im}\,(\lambda(z_1-w_1))\leqq-|\mathrm{Re}\,(\lambda(z_1-w_1))|\}$, and $\Omega=\bigcap_{j=1}^{n} V_j$. Therefore, we have a mapping $H^0(\Omega;\mathcal{O}^{(0,n)}_{X\times X})\to H^n_Z(X\times X;\mathcal{O}^{(0,n)}_{X\times X})\to\Gamma(\mathscr{Z};\mathscr{P}^R_X)$ (see S-K-K [1] Chap. I. Prop. 1.2.4.), where

$$\mathscr{Z}=\{(z,\ \mathrm{Re}\langle\zeta,\ dz\rangle\infty)\in S^*X;$$
$$\mathrm{Im}\,\lambda^{-1}\zeta_1>c|\mathrm{Re}\,\lambda^{-1}\zeta_1|+|\lambda^{-1}|\sqrt{1+c^2}\sum_{\nu=2}^{n}a_\nu|\zeta_\nu|\}\subset S^*X\,,$$

the antipodal of the polar of the normal of Z along the diagonal. The operator in $\Gamma(\mathscr{Z}, \mathscr{P}^R_X)$ corresponding to $L(z, w)dw$ is denoted by P. Let $\tilde{U}$ be a convex open set in S_MX such that $\pi(\tilde{U})=M$ and the polar $\tilde{U}^\circ=\{(z, \mathrm{Re}\langle\zeta,\ dz\rangle\infty)\in S^*_MX\subset S^*X;\ \mathrm{Re}\langle\zeta,\ \tau\rangle\leqq 0$ for every τ such that $z+\mathrm{Re}\langle\tau, \frac{\partial}{\partial z}\rangle 0\in\tilde{U}\}$ of

$\tilde{U}$ is contained in $\mathcal{Z}$. Therefore we may assume that $\tilde{U}^\circ$ is contained in $\{(z, \mathrm{Re}\langle\zeta, dz\rangle\infty); \mathrm{Im}(\lambda^{-1}\zeta_1)-c_1|\mathrm{Re}(\lambda^{-1}\zeta_1)|>|\lambda|^{-1}\sum_{\nu=2}^{n} b_\nu|\zeta_\nu|\}$ for suitable c_1 and b_ν with $c<a_1$, $\sqrt{1+c_1^2}\,a_\nu<b_\nu$. Then $\tilde{U}$ contains $z+\mathrm{Re}\left(\left\langle\tau, \frac{\partial}{\partial z}\right\rangle\right)\in S_M X$ if $\mathrm{Im}(\lambda\tau_1)\geqq c_1^{-1}|\mathrm{Re}(\lambda\tau_1)|$, $\mathrm{Im}(\lambda\tau_1)\geqq\frac{|\lambda|}{b_\nu}|\tau_\nu|$ $(\nu=2, \cdots, n)$.

Then the theorem corresponding to Theorem 3.3 is the following

THEOREM 3.6. *Let α_1, α_2 be two points sufficiently near z_1^0 such that* $0<c\,\mathrm{Im}(\lambda(\alpha_1-z_1^0))<-\mathrm{Re}(\lambda(\alpha_1-z_1^0))<c_1\mathrm{Im}(\lambda(\alpha_1-z_1^0))$ *and that* $0<c\,\mathrm{Im}(\lambda(\alpha_2-z_1^0))<\mathrm{Re}(\lambda(\alpha_2-z_1^0))<c_1\mathrm{Im}(\lambda(\alpha_2-z_1^0))$.

Set

$$\phi(z)=\int_{\gamma_1\times\cdots\times\gamma_n} L(z, w)\varphi(w)dw$$

where γ_j are chains given in Theorem 3.3. *Then $\phi(z)$ can be defined in an open set U' in $X-M$ such that $U'\cup\tilde{U}$ is open at $\pi^{-1}(z^0)$ and Pu coincides with the spectrum of the boundary value of $\phi(z)$.*

We can prove the lemma corresponding to Lemma 3.5. Since the proof, however, goes on in the same way, we omit its proof. It follows that this theorem has also a micro-local nature.

$\phi(z)$ depends only on the cohomology class of L and is independent of the choice of L. In fact, it is obvious that ϕ is real analytic, if L is a coboundary because of Cauchy's integral formula.

Theorem 3.3 is an obvious consequence of Theorem 3.6.

We will prove Theorem 3.6 in several steps. Firstly, we will show that Theorem 3.6 is independent of the change of coordinates and secondly show Theorem 3.6 in real coordidate case.

In order to make smooth the discussion of change of coordinates we consider the following statement $A_{(p,z,G)}$: Let p be a point in S_M^*X, $z=(z_1, \cdots, z_n)$ be a local coordinate system of X around $x=\pi(p)$ and G be a closed subset in S_xX of the type $G=\left\{x+\mathrm{Re}\left\langle\tau, \frac{\partial}{\partial z}\right\rangle 0; c\,\mathrm{Im}(\lambda\tau_1)\leqq-|\mathrm{Re}(\lambda\tau_1)|, |\tau_\nu|\leqq a_\nu|\tau_1|\right\}$, such that the antipodal of the polar of G contains p. (This is equivalent to say that $\mathrm{Im}(\lambda^{-1}\zeta_1)>c|\mathrm{Re}(\lambda^{-1}\zeta_1)|+\sum_{\nu=2}^{n}\sqrt{1+c^2}\,a_\nu|\zeta_\nu|$ if $p=\mathrm{Re}\langle\zeta, dz\rangle\infty$.) For any positive numbers c', a'_ν, δ such that $c>c'$, $a_\nu>a'_\nu$, $L(z, w)$ defined on $\{(z, w)\in X\times X; c'\,\mathrm{Im}(\lambda(z_1-w_1))>-|\mathrm{Re}\,\lambda(z_1-w_1)|, |z_\nu-w_\nu|>a'_\nu|z_1-w_1|, |z-z(x)|, |w-z(x)|<\delta\}$ and any $\varphi(z)$ whose boundary value has spectrum u with a support in a sufficiently small neighbourhood of p, the spectrum of the boundary value of the function $\phi(z)$ defined in Theorem 3.6 coincides with Pu at p. It is evident that Theorem 3.6 is equivalent to $(A)_{(p,z,G)}$.

LEMMA 3.7. *$V=\{z\in \boldsymbol{C}^n; c\,\mathrm{Im}\,z_1+|\mathrm{Re}\,z_1|>\sum_{\nu=2}^{n}|z_\nu|\}$ is Stein.*

PROOF. Since $\{z\in \boldsymbol{C}^n;\ c\operatorname{Im} z_1\pm\operatorname{Re} z_1>\sum_{\nu=2}^{n}|z_\nu|\}$ are convex sets, they are pseudo-convex. Therefore V is pseudo-convex in $\Omega=\{z\in \boldsymbol{C}^n;\ \operatorname{Re} z_1\neq 0\}$. Since Ω is Stein so is V. Q.E.D.

In order to prove Theorem 3.6, we reduce, firstly, to the case where z_j is real valued on M. The following lemma is a preparation for this step.

LEMMA 3.8. *Let* $p\in S_M^*X$, $z(t)=(z_1(t),\cdots,z_n(t))$ $(0\leqq t\leqq 1)$ *be a continuous family of coordinates of* X, $a(t)=(a_2(t),\cdots,a_n(t))$ *be a continuous family of positive sequences and* $c(t)>0$, $\lambda(t)\in \boldsymbol{C}^*$ *depends continuously on* t. *Determine* $\zeta(t)$ *by* $p=\operatorname{Re}\langle\zeta(t), dz(t)\rangle\infty$. *Suppose that* $G_t=\Big\{\pi(p)+\operatorname{Re}\langle\tau(t), \frac{\partial}{\partial z(t)}\rangle 0\in S_{\pi(p)}X;\ c(t)\operatorname{Im}\lambda(t)\tau_1(t)\leqq-|\operatorname{Re}\lambda(t)\tau_1(t)|,\ |\tau_\nu(t)|\leqq a_\nu|\tau_1(t)|$ *for* $\nu=2,\cdots,n\Big\}$ *contains* G_0 *in* $S_{\pi(p)}X$, *and that* $\operatorname{Im}(\lambda(t)^{-1}\zeta_1(t))>c(t)|\operatorname{Re}\lambda(t)^{-1}\zeta_1(t)|+|\lambda(t)^{-1}|\sqrt{1+c(t)^2}\sum_{\nu=2}^{n}a_\nu(t)|\zeta_\nu(t)|$. *Then* $A_{\{p,z(1),G_1\}}$ *implies* $A_{\{p,z(0),G_0\}}$.

PROOF. We assume that $z_\nu(t)=0$ at $\pi(p)$. $Z_t=\{(x,y)\in X\times X;\ |z_\nu(t)(x)-z_\nu(t)(y)|\leqq a_\nu|z_1(t)(x)-z_1(t)(y)|$ $(\nu=2,\cdots,n)$, $c(t)\operatorname{Im}(\lambda(t)(z_1(t)(x)-z_1(t)(y))\leqq-|\operatorname{Re}(\lambda(t)(z_1(t)(x)-z_1(t)(y)))|\}$, and assume that $Z_t\supset Z_0$. Let P belong to $H^n_{Z_0}(X\times X;\ \mathcal{O}^{(0,n)}_{X\times X})$. Set $V_\nu(t)=\{(x,y)\in X\times X;\ |z_\nu(t)(x)-z_\nu(t)(y)|>a_\nu|z_1(t)(x)-z_1(t)(y)|\}$ $(\nu=2,\cdots,n)$ and $V_1(t)=\{(x,y)\in X\times X;\ c(t)\operatorname{Im}(\lambda(t)(z_1(t)(x)-z_1(t)(y)))>-|\operatorname{Re}\lambda(t)(z_1(t)(x)-z_1(t)(y))|\}$. This constitutes a Stein covering of $X\times X-Z_\nu(t)$. Therefore, the image of P under the homomorphism $H^n_{Z_0}(X\times X;\ \mathcal{O}^{(0,n)}_{X\times X})\to H^n_{Z_t}(X\times X;\ \mathcal{O}^{(0,n)}_{X\times X})$ is represented by $L_t(x,y)dz_1(t)(y)\wedge\cdots\wedge dz_n(t)(y)$ where $L_t(x,y)$ is a holomorphic function defined on $\Omega(t)=\bigcap_{\nu=1}^{n}V_\nu(t)$. It suffices to show the following statement: If $\varphi(\lambda)$ is a holomorphic function whose singular support is contained in a sufficiently small neighbourhood of p, and $\psi_t(x)$ is a holomorphic function defined in Theorem 3.6 using $L_t(x,y)$ with the coordinate system $z(t)$, then $\psi_t(x)-\psi_{t'}(x)$ is holomorphic in a neighbourhood of $\pi(p)$ when t and t' are sufficiently close to each other. We may assume $\lambda(t)=1$ without loss of generality. For the sake of simplicity, we set $V_j=V_j(t)$, $V'_j=V_j(t')$, $Z=Z_t$, $Z'=Z_{t'}$, $L=L_t$, $L'=L_{t'}$, $z_j=z_j(t)(x)$, $w_j=z_j(t)(y)$, $z'_j=z_j(t')(x)$, $w'_j=z_j(t')(y)$, $a_\nu=a_\nu(t)$, $a'_\nu=a_\nu(t')$, $c=c(t)$, $c'=c(t')$ and $\zeta_j=\zeta_j(t)$. We put $\tilde V_\nu=\{(z,w)\in X\times X;\ |z_\nu-w_\nu|>\tilde a_\nu|z_1-w_1|+\varepsilon\sum_{j=2}^{n}|z_j-w_j|\}$ $(\nu=2,\cdots,n)$ and $\tilde V_1=\{(z,w)\in X\times X;\ \tilde c\operatorname{Im}(z_1-w_1)>-|\operatorname{Re}(z_1-w_1)|+\varepsilon\sum_{j=2}^{n}|z_j-w_j|\}$, $\tilde Z=X\times X-\bigcup_{j=1}^{n}\tilde V_j$, $\tilde\Omega=\bigcap_{j=1}^{n}\tilde V_j$ and $\tilde{\tilde\Omega}=\{(z,w)\in X\times X;\ |z_\nu-w_\nu|>\tilde{\tilde a}_\nu|z_1-w_1|,\ \tilde{\tilde c}\operatorname{Im}(z_1-w_1)>-|\operatorname{Re}(z_1-w_1)|\}$ for some $\tilde{\tilde a}_\nu>\tilde a_\nu>a_\nu$, $\tilde c>c$ and $1\gg\varepsilon>0$ such that we have $\tilde V_j\subset V_j$, $\tilde V_j\subset V'_j$, $\tilde\Omega\supset\tilde{\tilde\Omega}$ and $\operatorname{Im}\zeta_1>\tilde c|\operatorname{Re}\zeta_1|+\sqrt{1+\tilde c^2}\sum_{\nu=2}^{n}\tilde a_\nu|\zeta_\nu|$ by shrinking X. This is possible if t and t' are sufficiently close to each other. Let $\mathcal{V}$, $\mathcal{V}'$, $\tilde{\mathcal{V}}$, be coverings given by $\{V_j\}$, $\{V'_j\}$ and $\{\tilde V_j\}$. We have $\mathcal{V}\supset\tilde{\mathcal{V}}$, $\mathcal{V}'\supset\tilde{\mathcal{V}}$.

Since $L\,dw \in H^{n-1}(\mathcal{V}, \mathcal{O}^{(0,n)}_{X\times X})$ and $L'dw' \in H^{n-1}(\mathcal{V}', \mathcal{O}^{(0,n)}_{X\times X})$ are the images of the same P, these coincide in $H^{n-1}(\tilde{\mathcal{V}}, \mathcal{O}^{(0,n)}_{X\times X})$. It means that $L\,dw - L'dw'$ is in the coboundary with respect to the covering $\tilde{\mathcal{V}}$. Therefore there is F_j defined on $\hat{V}_j = \bigcap_{k\neq j} \tilde{V}_k$ such that $L\,dw - L'dw' = \sum_{j=1}^{n} F_j dw$.

Let $\sigma = \gamma_1 \times \cdots \times \gamma_n$ (resp. $\sigma' = \gamma_1' \times \cdots \times \gamma_n'$) be a chain given in Theorem 3.6 with respect to the coordinate system z (resp. z'). Then we have

$$\int_\sigma L(x, y)\varphi(y)dw - \int_\sigma L'(x, y)\varphi(y)dw' = \sum_j \int_\sigma F_j(x, y)\varphi(y)dy\,.$$

We will show that $\int_\sigma F_j(x, y)\varphi(y)dy$ and $\int_{\sigma-\sigma'} L'(x, y)\varphi(y)dw'$ are real analytic, which implies the desired result. The first fact is obvious. We will prove the second fact. Set δ be an $(n+1)$-chain defined by

$$\{\sigma_s\,;\ t \leqq s \leqq t'\}\,,$$

where σ_s is an n-chain defined in Theorem 3.6 with the boundary $\{w_1(s) = \alpha_2,\ |z_\nu(s) - w_\nu(s)| = \tilde{a}_\nu |\alpha_2 - z_1(s)|\} - \{w_1(s) = \alpha_1,\ |z_\nu(s) - w_\nu(s)| = \tilde{a}_\nu |\alpha_1 - z_1(s)|\}$. Therefore

$$\partial\delta = \sigma' - \sigma + \rho_2 - \rho_1$$

with

$$\rho_k = \{x \in X\,;\ w_1(s) = \alpha_2,\ |z_\nu(s) - w_\nu(s)| = \tilde{a}_\nu |\alpha_k - z_1(s)| \text{ at } x \text{ and } t \leqq s \leqq t'\}\,.$$

Therefore, we have

$$\int_{\sigma-\sigma'} L'(x, y)\varphi(y)dw' = \int_{\rho_2} L'(x, y)\varphi(y)dw' - \int_{\rho_1} L'(x, y)\varphi(y)dw'\,.$$

The right hand side is clearly real analytic. This proves Lemma 3.8. Q.E.D.

LEMMA 3.9. *$A_{(p,z,G)}$ is true if all z_j are real valued on M, $\lambda = 1$ and $p = \sqrt{-1}(dz_1 + \varepsilon \sum_{\nu=2}^{n} dz_\nu)\infty$ for $0 < \varepsilon \ll 1$.*

PROOF. We may assume that $\pi(p) = 0$, $X = \{z \in \boldsymbol{C}^n\,;\ |z| < 1\}$ and $L(z, w)$ is defined on $\Omega = \bigcap_{j=1}^{n} V_j$. We will construct a micro-local operator corresponding to $L(z, w)dw$ according to S-K-K [**1**] Chap. III. Prop. 1.2.4. We may assume $\sum a_\nu \sqrt{1+c^2} < 1$. Then Z is contained in

$$\begin{aligned} Z' = \{(z, w) \in X \times X\,;&\\ &\operatorname{Im}(w_1 - z_1) \geqq \operatorname{Im}(w_\nu - z_\nu) + \operatorname{Re}\sum_\mu (z_\mu - w_\mu)^2 \ (\nu = 2, \cdots, n),\\ &\operatorname{Im}(w_1 - z_1) \geqq -\sum_{\mu=2}^{n} \operatorname{Im}(w_\mu - z_\mu) + \operatorname{Re}\sum_\mu (z_\mu - w_\mu)^2,\ |z|,\ |w| < \delta\}\,. \end{aligned}$$

Set

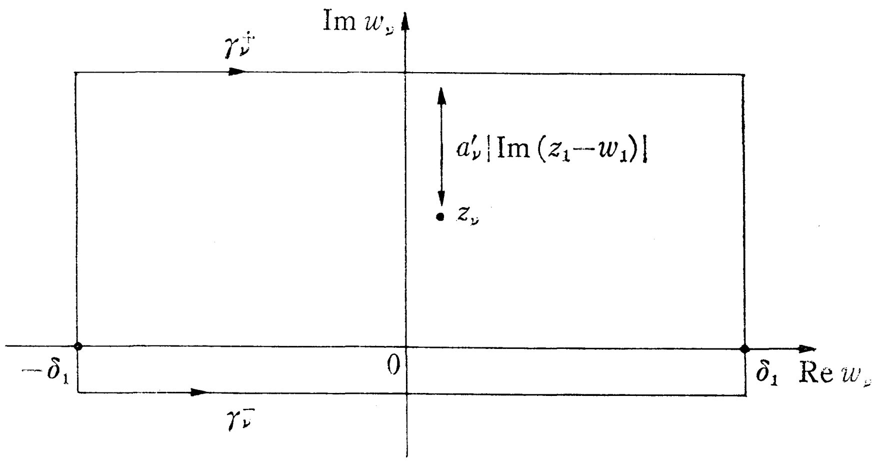
$$W=\{(z, w)\,;\ |z|,\ |w|<\delta\}\,,$$
$$U_1=\{(z, w)\in W\,;\ \mathrm{Im}\,(w_1-z_1)<-\sum_{\mu=2}^{n}\mathrm{Im}\,(w_\mu-z_\mu)+\mathrm{Re}\sum_{\mu}(z_\mu-w_\mu)^2\}\,,$$
$$U_\nu=\{(z, w)\in W\,;\ \mathrm{Im}\,(w_1-z_1)<\mathrm{Im}\,(w_\nu-z_\nu)+\mathrm{Re}\sum_{\mu}(z_\mu-w_\mu)^2\},\ \nu=2, \cdots, n\,,$$
$$\Omega'=\bigcap_{\nu=1}^{n}U_\nu\,.$$

Then $\mathcal{U}=\{U_1, \cdots, U_n\}$ is a Stein covering of $W-Z'$. Therefore, $H^n_{Z'}(W\,;\mathcal{O}_W)=\mathcal{O}(\Omega')$ modulo coboundary. Let $T(z, w)\in\mathcal{O}(\Omega')$ be a representative of the image of $L(z, w)$ by the homomorphism $H^n_{Z\cap W}(W\,;\mathcal{O}_W)\to H^n_{Z'\cap W}(W\,;\mathcal{O}_W)$. Then $T(z, w)dw$ defines a micro-local operator corresponding to $L(z, w)dw$. $L(z, w)$ is represented as an $(n-1)$-cocycle with respect to Leray covering $\mathcal{V}=\{V_j\}$. Therefore $L(z, w)$ and $T(z, w)$ are zero in $H^{n-1}(\mathcal{U}\cap\mathcal{V}\,;\mathcal{O}_W)$, this is, $L(z, w)-T(z, w)$ is a coboundary in a complex of Leray covering $\mathcal{U}\cap\mathcal{V}$. Especially, there are holomorphic functions $S_j(z, w)$ defined on $\bigcap_{k\neq j}(U_k\cap V_k)$ such that $L-T=\sum_{j=1}^{n}S_j$ in $\Omega\cap\Omega'$.

We may assume that u is a boundary value of holomorphic function $\varphi(z)$ defined on

$$U=\{z\in X\,;\ |z|<\varepsilon,\ |\mathrm{Im}\,z_\nu|<n\,\mathrm{Im}\,z_1\ (\nu=2, \cdots, n)\}\,.$$

We choose α_ν $(\nu=1, 2)$ according to Theorem 3.6 so that $\mathrm{Im}\,\alpha_1=\mathrm{Im}\,\alpha_2>0$. γ_1 be a path starting from α_1, ending at α_2 and encircling z_1 and $\gamma_\nu^{\pm}$ be a path in w_ν-space consisting of three segments starting from $-\delta_1$ and ending at δ_1 through $-\delta_1+\sqrt{-1}(\mathrm{Im}\,z_\nu\pm a'_\nu|\mathrm{Im}\,(z_1-w_1)|)$ and $\delta_1+\sqrt{-1}\,\mathrm{Im}\,z_\nu\pm\sqrt{-1}\,a'_\nu|\mathrm{Im}(z_1-w_1)|$, where $a'_\nu>1$ and $\sum a'_\nu<n$. (See the figure in the below.) Then a chain $\gamma_1\times\gamma_2^{\pm}\times\cdots\times\gamma_n^{\pm}=\sigma_{\pm,\cdots,\pm}$ is contained in $\{(z, w)\in\Omega\cap\Omega'\,;\ |z|\ll 1\}$ if $0<\delta_1\ll 1$.



w_ν-space $(\nu=2, \cdots, n)$

We have

$$\psi(z)=(-)^{n-1}\sum_{\pm}\pm_2\cdots\pm_n\int_{\gamma_1\times\gamma_2\times\cdots\times\gamma_n}L(z,w)\varphi(w)dw\,.$$

Set $\psi_{\pm_2,\cdots,\pm_n}=\pm_2\cdots\pm_n\int_{\gamma_1\times\gamma_2\times\cdots\times\gamma_n}L(z,w)\varphi(w)dw$. Suppose that one of $\pm_j$ is of a positive sign. If $\operatorname{Im}z_j<0$, then we may take a chain γ'_j instead of γ_j^{+}, where γ'_j is a path starting from $-\delta_1$ and ending at δ_1 through $-\delta_1+\sqrt{-1}\,a'_j\operatorname{Im}(w_1-z_1)$ and $\delta_1+\sqrt{-1}\,a'_j\operatorname{Im}(w_1-z_1)$. Therefore $\psi_{\pm_1,\cdots,\pm_n}$ is defined on $\{z\,;\,|z|\ll 1\,;\,|\operatorname{Im}z_\nu|<n\operatorname{Im}z_1\ (\nu\neq 1,j)$ and $\operatorname{Im}z_j<n\operatorname{Im}z_1\}$, which implies that the singular support of $\psi_{\pm_1,\cdots,\pm_n}$ does not contain p if any of the sign is positive. Therefore

$$\psi(z)=\int_{\gamma_1\times\gamma_2^-\times\cdots\times\gamma_n^-}L(z,w)\varphi(w)dw$$

modulo holomorphic function whose boundary value is real analytic at p. Moreover, we have

$$\int_{\gamma_1\times\gamma_2^-\times\cdots\times\gamma_n^-}L(z,w)\varphi(w)dw=\int_{\gamma_1\times\gamma_2^-\times\cdots\times\gamma_n^-}T(z,w)\varphi(w)dw$$
$$+\sum_{j=1}^{n}\int_{\gamma_1\times\gamma_2^-\times\cdots\times\gamma_n^-}S_j(z,w)\varphi(w)dw\,.$$

Consider the integral $\int S_1(z,w)\varphi(w)dw$. Since $S_1(z,w)$ is defined on $\{(z,w)\in X\,;\,|z_\nu-w_\nu|>a_\nu|z_1-w_1|,\ \operatorname{Im}(w_1-z_1)<\operatorname{Im}(w_\nu-z_\nu)+\operatorname{Re}\sum_\mu(z_\mu-w_\mu)^2\ (\nu=2,\cdots,n)\}$, $\gamma_1\times\gamma_2^-\times\cdots\times\gamma_n^-$ can be deformed to $\gamma'_1\times\gamma'_2\times\cdots\times\gamma'_n$, where γ'_1 is a segment from α_1 to α_2 and γ'_ν is a path from $-\delta_1$ to δ_1 through $-\delta_1+\sqrt{-1}(\operatorname{Im}z_\nu+k|z_1-w_1|)$ and $\delta_1+\sqrt{-1}(\operatorname{Im}z_\nu+k|z_1-w_1|)$ such that $n\operatorname{Im}\alpha_\nu>k|\alpha_\nu|>\operatorname{Im}\alpha_\nu$. Then $\int_{\gamma_1\times\gamma_2^-\times\cdots\times\gamma_n^-}S_1(z,w)\varphi(w)dw$ is real analytic.

Suppose $j\neq 1$. Then $S_j(z,w)$ is defined on

$$\begin{aligned}W_j=\{(z,w)\in X\,;\ &\operatorname{Im}(w_1-z_1)<c|\operatorname{Re}(w_1-z_1)|,\\ &|w_\nu-z_\nu|>a_\nu|z_1-w_1|,\\ &\operatorname{Im}(w_\nu-z_\nu)>\operatorname{Im}(w_1-z_1)-\operatorname{Re}\big(\textstyle\sum_\mu(z_\mu-w_\mu)^2\big)\qquad\text{for }\nu\neq 1,j\\ &\text{and}\\ &\operatorname{Im}(w_1-z_1)<-\textstyle\sum_{\mu=2}^{n}\operatorname{Im}(w_\mu-z_\mu)+\operatorname{Re}\sum_{\mu=1}^{n}(w_\mu-z_\mu)^2\}\,.\end{aligned}$$

Therefore the integration $\int S_j(z,w)\varphi(w)dw$ can be performed over the cycle $\gamma_1\times\gamma_2^-\times\cdots\times\gamma'_j\times\cdots\times\gamma_n^-$, where γ'_j is a segment from $-\delta_1$ to δ_1. Then $\int S_j(z,w)\varphi(w)dw$ is defined on $\{z\in X\,;\,|z|\ll 1,\ |\operatorname{Im}z_\nu|<n\operatorname{Im}z_1$ for $\nu\neq j\}$. Therefore $\int S_j(z,w)\varphi(w)dw$ is real analytic at p for every j. It follows that $\psi(z)$ is

equal to $\int_{\gamma_1\times\gamma_2^-\times\cdots\times\gamma_n^-} T(z, w)\varphi(w)dw$ modulo holomorphic function which is real analytic at p. By Lemma 3.1, $\int T(z, w)\varphi(w)dw$ is nothing but Pu, which completes the proof of the lemma. Q. E. D.

Now return to the original situation. Set $T=\boldsymbol{C}$. We consider T as a two dimensional real analytic manifold. We denote by T_C the complexification of T. We can take T_C as $\boldsymbol{C}\times\bar{\boldsymbol{C}}$ where $\bar{\boldsymbol{C}}$ is a complex conjugate of T. Let $(t, \tilde{t})$ be a coordinate system on T_C such that $\bar{t}=\tilde{t}$ on T. $u(x)$ can be considered as a microfunction on $M'=M\times T$. $u(x)$ is also a boundary value of a holomorphic function $\varphi(z)$ defined on $X'=X\times T_C$. Set $p=(0, \mathrm{Re}\langle\zeta, dz\rangle\infty)$ and assume that the support of u is contained in a sufficiently small neighbourhood of p. In order to prove $A_{(p,z,G)}$, we may assume that $\lambda=1$. We introduce a new coordinate system $(\tilde{z}, t, \tilde{t})$ on X' given by $\tilde{z}_1=z_1+t\langle\zeta, z\rangle/\zeta_1$, $\tilde{z}_\nu=z_\nu$ $(\nu=2, \cdots, n)$. We may assume that $L(z, w)$ is defined on

$$\Omega=\{(z, w)\in X\,;\ c(\mathrm{Re}\, z_1-w_1)<|\mathrm{Im}\,(z_1-w_1)|,$$
$$|z_\nu-w_\nu|<a_\nu|z_1-w_1|(\nu=2, \cdots, n)\}\,.$$

Let $\tilde{L}(\tilde{z}, t, \tilde{t}\,;\tilde{w}, s, \tilde{s})$ be a holomorphic function $L(\tilde{z}, \tilde{w})\Phi_0(t-s)\Phi_0(\tilde{t}-\tilde{s})$, which is defined on $\Omega'=\{(\tilde{z}, t, \tilde{t}\,;\tilde{w}, s, \tilde{s})\in X'\times X'\,;\, c\,\mathrm{Re}\,(\tilde{z}_1-\tilde{w}_1)<|\mathrm{Im}\,(z_1-w_1)|, t\neq z, \tilde{t}\neq\tilde{s}, |\tilde{z}_\nu-\tilde{w}_\nu|<a_\nu|\tilde{z}_1-\tilde{w}_1|\}$. Let $\tilde{P}$ be a corresponding section of $\mathscr{P}_{X'}^{R}$. $\tilde{P}$ is defined on $\tilde{Z}=\{(\tilde{z}, t, \tilde{t}\,;\mathrm{Re}\,(\langle\zeta, d\tilde{z}\rangle+kdt+\tilde{k}d\tilde{t})\infty\,;\mathrm{Re}\,\zeta_1>c|\mathrm{Im}\,\zeta_1|+\sqrt{1+c^2}\sum_{\nu=2}^{n}a_\nu|\zeta_\nu|\}$. Then the support of $u(x)$ is contained in a sufficiently small neighborhood of $\{(\tilde{z}, t, \bar{t}\,;\mathrm{Re}\,(\zeta_1 d\tilde{z}_1+(1-t)\sum_{\nu=2}^{n}\zeta_\nu d\tilde{z}_\nu-\langle\zeta, z\rangle dt)\infty)\}$, which is contained in $\tilde{Z}$ if t belongs to a neighbourhood of $[0, 1]$, since Z is convex. Consider the integral

$$\tilde{\varphi}(\tilde{z}, t, \tilde{t})=\int L(\tilde{z}, t, \tilde{t}, \tilde{w}, s, \tilde{s})\varphi(w)d\tilde{w}dsd\tilde{s}\,.$$

Note that $\tilde{\varphi}(\tilde{z}, t, \tilde{t})$ satisfies $\frac{\partial}{\partial t}\tilde{\varphi}=0$. It means that the spectrum $\tilde{v}$ of the boundary value of $\tilde{\varphi}$ satisfies $\frac{\partial}{\partial\bar{t}}\tilde{v}=0$; that is, $\tilde{v}$ depends holomorphically on t. It is evident that $\tilde{\varphi}(\tilde{z}, t, \tilde{t})|_{t=\tilde{t}=0}=\varphi(z)$. It is also obvious that $\tilde{P}u$ depends holomorphically on t, and that $\tilde{P}u|_{t=\tilde{t}=0}=Pu$. Therefore, in order to prove that $Pu=\tilde{v}|_{t=\tilde{t}=0}$, it suffices to show that $\tilde{P}u=\tilde{v}$ in a neighbourhood of $t=0$. Since both sides depend holomorphically on t, it suffices to show that $\tilde{P}u=\tilde{v}$ in a neighbourhood of $t=1$ because of the unique continuation property of microfunctions with holomorphic parameter. (See S-K-K [**1**] Chap. III. Theorem 2.2.9.) Therefore, by replacing P and X by $\tilde{P}$ and X', we can assume from the beginning that p is sufficiently near $\mathrm{Re}\,(\zeta_1 dz_1)\infty$. By using Lemma 3.8, we may assume that $p=\mathrm{Re}\,(\alpha dz_1)\infty$ for some $\alpha\in\boldsymbol{C}^*$. By a coordinate transformation, we may set $\alpha=\sqrt{-1}$. Because of the connectivity of $GL(n-1;\boldsymbol{C})$, there is a continuous family $(p, z(t), G_t)$ such that $z(0)=z$ and $z(1)$ is real valued on M and that $p=\mathrm{Re}(\sqrt{-1}dz_1(t))\infty$, and that G_t is sufficiently large $(t>0)$. By virtue of Lemma 3.8, we can reduce the problem to the case where $p=\mathrm{Re}(\sqrt{-1}dz_1)\infty$ and that z_ν are all real valued on M. By a small perturbation of a coordinate system, $A_{\{p,z,G\}}$ is reduced to Lemma 3.9. This is the end of the long proof of Theorem 3.6 and Theorem 3.3.

We end this section by giving a generalization of Theorem 3.3 to the case with parameters. Let X, M and z be the same as before, and T be a real analytic manifold and T_C be its complexification with a coordinate system t. Let $L(t, z, w, D_t)$ be a (multivalued) differential operator defined on $\Omega=\{(t, s, w)\in T_C\times X\times X;\ |z-w|<\delta,\ |z_\nu-w_\nu|>a_\nu|z_1-w_1|>0$ for $\nu=2, \cdots, n\}$ of the type

$$L(t, z, w, D_t)=L_0(t, z, w, D_t)+\frac{1}{2\pi\sqrt{-1}}L_1(t, z, w, D_t)\,,$$

where $L_0(t, w, D_t)$ is (single-valued and) defined on $\Omega_0=\{(t, z, w)\in T_C\times X\times X;\ z_\nu\neq w_\nu\ (\nu=1, \cdots, n)\}$ and $L_1(t, z, w, D_t)$ is defined on

$$\Omega_1=\{(t, z, w)\in T_C\times X\times X;\ |z-w|<\delta,\ |z_\nu-w_\nu|>a_\nu|z_1-w_1| \quad \text{for}\quad \nu=2, \cdots, n\}\,.$$

Then $L(t, z, w, D_t)$ determines a pseudo-differental operator $P(t, z, D_t, D_z)$ defined on $Z=\{(t, z; (kdt+\langle\zeta, dz\rangle)\infty)\in P^*(T_C\times X);\ \sum_{\nu=2}^{n}a_\nu|\zeta_\nu|<|\zeta_1|\}$. Let $\tilde{U}$ be an open convex cone in $T\times\sqrt{-1}SM$ such that $\tau(\tilde{U})=T\times M$ and U be an open set in X such that $\tilde{U}\cup T\times(U-M)$ is an open set in $T\times\widetilde{^{M}X}$. Let $\varphi(t, z)$ be a hyperfunction on U depending holomorphically on z. Then the boundary value of $\varphi(t, z)$ is a hyperfunction on $T\times M$ with singular support contained in $\{(t, z; \mathrm{Re}(\sqrt{-1}kdt+\langle\zeta, dz\rangle)\infty)\in\sqrt{-1}S^*(T\times M);\ (t, z, \mathrm{Re}\sqrt{-1}\langle\zeta, dz\rangle\infty)$ is in the polar $\tilde{U}^\circ$ or $\zeta=0\}$ (see S-K-K [1] Chap. I. §3.2). Let u be a restriction of the spectrum of φ to $\sqrt{-1}S^*(T\times M)-\sqrt{-1}S^*T\times M$. Then the support of u is contained in $\gamma^{-1}(Z)$, where γ is the projection $\sqrt{-1}S^*(T\times M)\to P^*(T_C\times X)$.

THEOREM 3.10. *Let $(t^0, z^0)\in T\times M$, and let λ, α_1, α_2 and c be the same as in Theorem 3.3. Set*

$$\psi(t, z)=\int_{\gamma_1\times\cdots\times\gamma_n}L(t, z, w, D_t)\varphi(t, w)dw\,,$$

where γ_j are paths determined in Theorem 3.3. Then $\psi(t, z)$ is defined on an open set U' in $T\times X$ such that $\tilde{U}\cup(U'-M)$ is open in $T\times\widetilde{^{M}X}$, and Pu coincides with the spectrum of $\psi(t, z)$ on $\sqrt{-1}S^*(T\times M)-\sqrt{-1}S^*T\times M$.

PROOF. The lemma corresponding to Lemma 3.5 is verified in the same way and we omit its proof. We may express $\varphi(t, z)$ as the boundary value of a holomorphic function $\varphi_j(t, z)$. All $\varphi_j(t, z)$ can be considered also as sections of $\widetilde{\mathfrak{A}}_{T\times M}$ (see S-K-K [1] Chap. I. Theorem 3.1.1). Let u_j be a boundary value of $\varphi_j(t, z)$, and set $\psi_j(t, z)=\int L(t, z, w, D_t)\varphi_j(t, z)dz$. Then Pu_j is the spectrum of ψ_j by virtue of Theorem 3.3. Therefore $Pu=\sum Pu_j$ is the spectrum of $\psi=\sum\psi_j$. Q.E.D.

§4. The analytic continuation of the formal elementary solution.

In §2 we constructed a formal elementary solution $R(t, x, D_x)$, which is an operator with finite velocity. If there is no assumption on P, the singularity of $R(t, x, D_x)$ spreads into the complex domain. Therefore, R cannot be endowed with the meaning of a microfunction. However, if P is micro-hyperbolic, the singularity of R propagates only along the real domain so that R can be considered as a microfunction. In order to analyze this situation, we assume that $P(t, x, D_t, D_x)=D_t-A(t, x, D_x)$ is partially micro-hyperbolic at $(t, x, i(\tau, \xi)\infty)=(0, x_0, i(\tau, \xi_0)\infty)$ for every real τ with respect to the direction t. This means, if we set $A_1(t, x, i\xi)$ the matrix which is obtained by taking the first order part of each component, and $g(t, x, \tau, \xi)=\det(i\tau-A_1(t, x, i\xi))$, then $g(t, x, \tau+ik, \xi)\neq 0$ for every real t, x, τ, ξ, k such that $|t|\ll 1$, $|x-x_0|\ll 1$, $0<k\ll 1$, $|\xi-\xi_0|\ll 1$. In this case, we say that P is partially hyperbolic at $(0, x_0, i\xi_0\infty)$ with respect to the direction t.

In this section we will show that the function

$$\sum_{j=-\infty}^{\infty} R_j(t, x, \xi)\Phi_j(\langle x, \xi\rangle-p) \tag{4.1}$$

converges in a conical domain so that it can be considered as a microfunction. For the sake of simplicity, we assume $x_0=0$, $\xi_0=(1, 0, \cdots, 0)$, and we set $x'=(x_2, \cdots, x_n)$. We define the multivalued analytic function $G(t, x)$ by

$$\begin{aligned} G(t, x) &= \sum_{j=-\infty}^{\infty} R_j(t, x, \xi_0)\Phi_j(x_1) \\ &= G_0(t, x)+\frac{1}{2\pi\sqrt{-1}}G_1(t, x)\log x_1 . \end{aligned}$$

Recall that

$$\begin{aligned} \Phi_j(x) &= \frac{1}{2\pi\sqrt{-1}}\frac{j!}{(-x)^{j+1}} \qquad \text{for } j\geqq 0 \\ &= -\frac{x^{-j-1}}{2\pi\sqrt{-1}(-j-1)!}(\log(-x)-\psi(-j)) \qquad \text{for } j<0, \end{aligned}$$

where $\psi(\nu)=\sum_{k=1}^{\nu-1}\frac{1}{k}-\gamma$ is the di-gamma function. By estimates (2.3) and (2.4), $G(t, x)$ converges on $\Omega=\{(t, x)\in \boldsymbol{C}\times\boldsymbol{C}^n ; |t|<\delta, |x_1|<\delta, |x_1|>v|t|\}$. More precisely, G_0 converges on Ω and G_1 converges on $V_\delta=\{(t, x)\in\boldsymbol{C}\times\boldsymbol{C}^n ; |t|<\delta, |x|<\delta\}$. Under the assumption of partial micro-hyperbolicity of P, we can extend the domain of definition of G.

In order to perform this program, we reformulate the meaning of $PR=0$ by using the defining functions.

Developing $A(t, x, D_x)$ into a series of D_x^α, i.e.,

$$A(t, x, D_x)=\sum_{\alpha\in \boldsymbol{Z}\times\boldsymbol{Z}_+^{n-1}} A_\alpha(t, x)D_x^\alpha ,$$

we set

$$\begin{aligned} L(t, x, y)&=\sum A_\alpha(t, x)\Phi_\alpha(x-y)\\ &=L_0(t, x, y)+L_1(t, x, y)\log(x_1-y_1),\end{aligned}$$

where $L_0(t, x, y)$ is the kernel of a differential operator, that is, $L_0(t, x, y)$ is holomorphic on the domain

$$\begin{aligned} V_\delta=\{(t, x, y) ; |t|<\delta, |x|<\delta, |y|<\delta,\\ x_j\neq y_j\ (j=1, \cdots, n)\} ,\end{aligned}$$

and $L_1(t, x, y)$ is holomorphic on the domain

$$\begin{aligned} \tilde{V}_\delta=\{(t, x, y) ; |t|<\delta, |x|<\delta, |y|<\delta,\\ |x_1-y_1|<\delta|x_j-y_j| \quad \text{for} \quad j=2, \cdots, n\}\end{aligned}$$

for a sufficiently small δ.

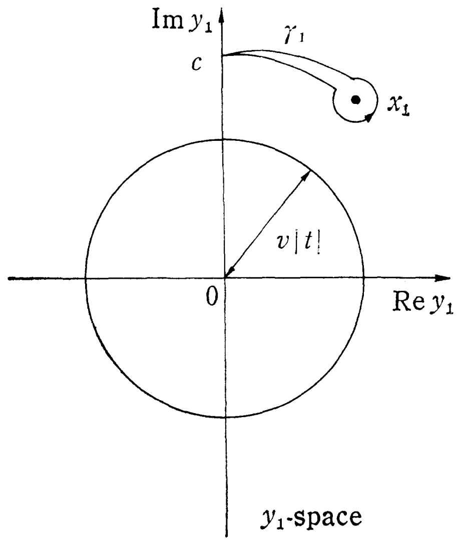
It follows that $A(t, x, D_x)$ is defined on $\{(t, x, i\langle\xi, dx\rangle\infty) ; |t|<\delta, |x|<\delta, |\xi_j|<\delta|\xi_1|, j=2, \cdots, n\}$. Then, we have the following interpretation of the relation $PR=0$.

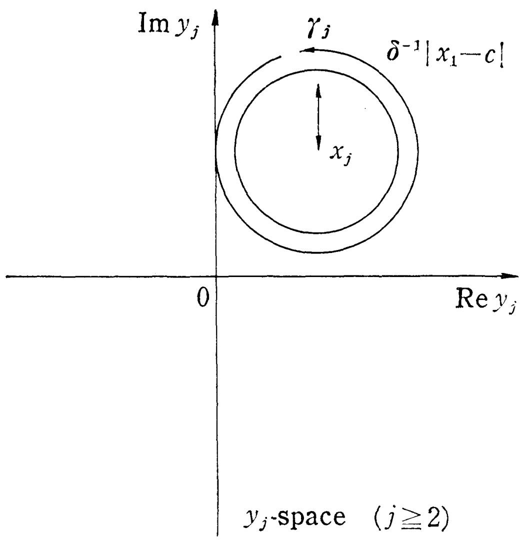
LEMMA 4.1. *The function*

$$\frac{\partial}{\partial t}G(t, x)-\oint L(t, x, y)G(t, y)dy$$

is holomorphic at the origin $(t, x)=(0, 0)$.

In this notation, the integral is taken along the path $\gamma_1\times\cdots\times\gamma_n$ *where* γ_j $(2\leqq j\leqq n)$ *is a path around* x_j *and* γ_1 *is a path around* x_1 *which starts from some sufficiently small fixed* $c\in\sqrt{-1}\boldsymbol{R}^+$ *and ends at the same point c as shown in the following figures.*





PROOF. We have

$$\oint L(t, x, y)G(t, y)dy$$
$$=\sum\oint A_\alpha(t, x)\Phi_\alpha(x-y)R_j(t, y, \xi_0)\Phi_j(y_1)dy\,,$$

because L and G converge uniformly. Moreover we have

$$\oint \Phi_\alpha(x-y)R_j(t, y, \xi_0)\Phi_j(y_1)dy$$
$$=\int_{\gamma_1}\Phi_{\alpha_1}(x_1-y_1)\Phi_j(y_1)dy_1\oint\Phi_{\alpha'}(x'-y')R_j(t, y, \xi_0)dy'$$
$$=\int_{\gamma_1}\Phi_{\alpha_1}(x_1-y_1)\Phi_j(y_1)D_{x'}^{\alpha'}R_j(t, y_1, x', \xi_0)dy_1$$
$$=\sum_k\frac{1}{k!}\int_{\gamma_1}(y_1-x_1)^k\Phi_{\alpha_1}(x_1-y_1)\Phi_j(y_1)D_{x_1}^kD_{x'}^{\alpha'}R_j(t, x, \xi_0)dy_1$$

and

$$\int_{\gamma_1}\Phi_{\alpha_1}(x_1-y_1)\Phi_j(y_1)(y_1-x_1)^kdy_1$$
$$=\begin{cases} 0 & k>\alpha_1\geqq 0\\[2mm] \dfrac{\alpha_1!}{(\alpha_1-k)!}\Phi_{j+\alpha_1-k}(x_1) & \alpha_1\geqq k\\[2mm] (-)^k\dfrac{(k-\alpha_1-1)!}{(-\alpha_1-1)!}\Phi_{j+\alpha_1-k}(x_1) & \\ \qquad +(-)^{k+1}\dfrac{(k-\alpha_1-1)!}{(-\alpha_1-1)!}\displaystyle\sum_{\nu=0}^{k-\alpha_1-1}\frac{1}{\nu!}\Phi_{j+\alpha_1-k+\nu}(c)(x_1-c)^\nu & \alpha_1<0\,. \end{cases}$$

It follows that

$$\oint \Phi_\alpha(x-y)R_j(t, y, \xi_0)\Phi_j(y_1)dy$$

$$=\begin{cases} \sum_{k=0}^{\alpha_1} \frac{\alpha_1 !}{k !(\alpha_1-k) !} D_{x_1}^k D_{x'}^{\alpha'} R_j(t, x, \xi_0)\Phi_{j+\alpha_1-k}(x_1) & \text{for } \alpha_1 \geqq 0 \\ \sum_{k=0}^{\infty} \frac{(-)^k(k-\alpha_1-1)!}{k!(-\alpha_1-1)!} D_{x_1}^k D_{x'}^{\alpha'} R_j(t, x, \xi_0)\Phi_{j+\alpha_1-k}(x_1) & \\ + \sum_{\substack{0\leqq\nu\leqq k-\alpha_1-1 \\ 0\leqq k}} \frac{(-)^{k+1}(k-\alpha_1-1)!}{\nu !k!(-\alpha_1-1)!} D_{x_1}^k D_{x'}^{\alpha'} R_j(t, x, \xi_0)\Phi_{j+\alpha_1-k+\nu}(c) & \\ \qquad \times(x_1-c)^\nu & \text{for } \alpha_1<0 . \end{cases}$$

Therefore

$$\oint L(t, x, y)G(t, y)dy$$

$$= \sum_{\alpha_1\geqq k\geqq 0} \frac{\alpha_1 !}{k !(\alpha_1-k)!} A_\alpha(t, x)D_{x_1}^k D_{x'}^{\alpha'} R_j(t, x, \xi_0)\Phi_{j+\alpha_1-k}(x_1)$$

$$+ \sum_{\substack{\alpha_1<0 \\ k\geqq 0}} \frac{(-)^k(k-\alpha_1-1)!}{k!(-\alpha_1-1)!} A_\alpha(t, x)D_{x_1}^k D_{x'}^{\alpha'} R_j(t, x, \xi_0)\Phi_{j+\alpha_1-k}(x_1)$$

$$+ \sum_{\substack{0\leqq\nu\leqq k-\alpha_1-1 \\ 0\leqq k, 0>\alpha_1}} \frac{(-)^{k+1}(k-\alpha_1-1)!}{k!(-\alpha_1-1)!\nu !} A_\alpha(t, x)D_{x_1}^k D_{x'}^{\alpha'} R_j(t, x, \xi_0)\times$$
$$\times\Phi_{j+\alpha_1-k+\nu}(c)(x_1-c)^\nu$$

$$= \sum_{\substack{\alpha_1\geqq 0 \\ \alpha\geqq\beta\geqq 0}} \frac{1}{\beta !}[D_\xi^\beta(A_\alpha(t, x)\xi^\alpha)]|_{\xi=\xi_0} D_x^\beta R_j(t, x, \xi_0)\Phi_{j+|\alpha|-|\beta|}(x_1)$$

$$+ \sum_{\substack{\alpha_1<0 \\ \beta\geqq 0}} \frac{1}{\beta !}[D_\xi^\beta(A_\alpha(t, x)\xi^\alpha)]|_{\xi=\xi_0} D_x^\beta R_j(t, x, \xi_0)\Phi_{j+|\alpha|-|\beta|}(x_1)$$

$$+ \sum_{\substack{0\leqq\nu\leqq k-\alpha_1-1 \\ 0\leqq k, 0>\alpha_1}} \frac{(-)^{k+1}(k-\alpha_1-1)!}{k!(-\alpha_1-1)!\nu !} A_\alpha(t, x)D_{x_1}^k D_{x'}^{\alpha'} R_j(t, x, \xi_0)\times$$
$$\times\Phi_{j+\alpha_1-k+\nu}(c)(x_1-c)^\nu$$

$$= \sum \frac{1}{\beta !}[D_\xi^\beta(A_\alpha(t, x)\xi^\alpha)D_x^\beta R_j(t, x, \xi)]|_{\xi=\xi_0}\Phi_{j+|\alpha|-|\beta|}(x_1)$$

$$+ \sum_{\substack{0\leqq\nu\leqq k-\alpha_1-1 \\ 0\leqq k, 0>\alpha_1}} \frac{(-)^{k+1}(k-\alpha_1-1)!}{k!(-\alpha_1-1)!\nu !} A_\alpha(t, x)D_{x_1}^k D_{x'}^{\alpha'} R_j(t, x, \xi_0)\times$$
$$\times\Phi_{j+\alpha_1-k+\nu}(c)(x_1-c)^\nu .$$

The first term of the above sum is equal to

$$\sum \frac{1}{\beta !}[D_\xi^\beta A_k(t, x, \xi)D_x^\beta R_j(t, x, \xi)]|_{\xi=\xi_0}\Phi_{j+k-|\beta|}(x_1) .$$

By the assumption, this is equal to $\frac{\partial}{\partial t}G(t, x)$. Therefore, we obtain

(4.2) $$\frac{\partial}{\partial t}G(t, x)-\oint L(t, x, y)G(t, y)dy$$

$$=\sum_{\substack{0\leqq\nu\leqq k-\alpha_1-1\\ 0\leqq k,0>\alpha_1}}\frac{(-)^k(k-\alpha_1-1)!}{k!(-\alpha_1-1)!\nu!}A_\alpha(t,x)D_{x_1}^kD_{x'}^{\alpha'}R_j(t,x,\xi_0)\Phi_{j+\alpha_1-k+\nu}(c)(x_1-c)^\nu.$$

It is sufficient to show that the series in the right hand side converges absolutely in a neighbourhood of $t=x=0$. Remember that $A_\alpha(t,x)$ satisfies the following estimate

$$|A_\alpha(t,x)|\leqq C_{|\alpha|}B^{-\alpha_1},$$

where B is a sufficiently large constant and C_j is a sequence satisfying

$$\lim_{j\to\infty} j\sqrt[j]{C_j}=0,\qquad \overline{\lim_{j\to\infty}}\frac{\sqrt[j]{C_{-j}}}{j}<\infty, \tag{4.3}$$

and that $D_x^\beta R_j(t,x,\xi_0)$ satisfies the estimate

$$|D_x^\beta R_j(t,x,\xi_0)|\leqq\begin{cases}\beta!C_\varepsilon\dfrac{1}{j!}(\nu\varepsilon)^j & \text{if}\quad |t|<\varepsilon,\ j\geqq 0\\ \beta!(-j)!B^{-j} & \text{if}\quad j<0.\end{cases}$$

Therefore, the right hand side of formula (4.2) is estimated by

$$\sum_{\substack{0\leqq\nu\leqq k-\alpha_1-1\\ 0\leqq k,0>\alpha_1\\ j\geqq 0}}\frac{(k-\alpha_1-1)!}{k!(-\alpha_1-1)!\nu!}C_{|\alpha|}B^{-\alpha_1}\frac{k!\alpha'!}{j!}C_\varepsilon(\nu\varepsilon)^j|\Phi_{j+\alpha_1-k+\nu}(c)||x_1-c|^\nu$$

$$+\sum_{\substack{0\leqq\nu\leqq k-\alpha_1-1\\ 0\leqq k,0>\alpha_1\\ j<0}}\frac{(k-\alpha_1-1)!}{k!(-\alpha_1-1)!\nu!}C_{|\alpha|}B^{-\alpha_1}k!\alpha'!|(-j)!|B^{-j}|\Phi_{j+\alpha_1-k+\nu}(c)||x_1-c|^\nu.$$

Since we can easily see that

$$\sum_{\alpha'}C_{j+|\alpha'|}\alpha'!$$

satisfies the same estimate as (4.3), it suffices to show that the following series converge uniformly at $t=x=0$:

$$\sum_{\substack{0\leqq\nu\leqq k+\mu\\ 0\leqq k,\mu\geqq 0\\ j\geqq 0}}\frac{(k+\mu)!}{\mu!\nu!j!}C_{-\mu}B^\mu(\nu\varepsilon)^j|\Phi_{j-k-\mu+\nu-1}(c)(x_1-c)^\nu| \tag{4.4}$$

and

$$\sum_{\substack{0\leqq\nu\leqq k+\mu\\ 0\leqq k,\mu,j}}\frac{(k+\mu)!j!}{\mu!\nu!}C_{-\mu}B^{\mu+j}|\Phi_{-j-k-\mu+\nu-1}(c)(x_1-c)^\nu|. \tag{4.5}$$

Since $C_{-\mu}\leqq\mu!B^\mu$, the first one (4.4) is estimated by

$$\sum\frac{(k+\mu)!}{\nu!j!}B^{2\mu}(\nu\varepsilon)^j|\Phi_{j-k-\mu+\nu-1}(c)(x_1-c)^\nu|$$

$$\leqq\frac{1}{1-B^{-2}}\sum_{\substack{0\leqq\nu\leqq k\\ 0\leqq j}}\frac{k!}{\nu!j!}B^{2k}(\nu\varepsilon)^j|\Phi_{j-k+\nu-1}(c)(x_1-c)^\nu|$$

$$\leqq (1-B^{-2})^{-1} \sum_{0\leqq \nu,\mu,j} \frac{(\nu+\mu)!}{\nu!j!} B^{2(\nu+\mu)}(\nu\varepsilon)^j |\Phi_{j-\mu-1}(c)(x_1-c)^\nu|$$

$$\leqq (1-B^{-2})^{-1} \sum_{0\leqq \nu,\mu,j} \frac{\mu!2^{\nu+\mu}}{j!} B^{2(\nu+\mu)}(\nu\varepsilon)^j |\Phi_{j-\mu-1}(c)(x_1-c)^\nu|$$

$$\leqq (1-B^{-2})^{-1}(1-2B^2|x_1-c|)^{-1} \sum_{0\leqq \mu,j} \frac{\mu!}{j!} B^{2\mu}(\nu\varepsilon)^j |\Phi_{j-\mu-1}(c)| .$$

It is easy to see that

$$\sum_{0\leqq k,j} \frac{k!}{j!} A^k \varepsilon^j |\Phi_{j-k-n}(c)| \qquad (n\in \boldsymbol{Z})$$

converges if $\varepsilon(A+|c|^{-1})<1$, $A(|c|+\varepsilon)<1$. Therefore (4.4) converges at $t=x=0$ if $|c|\ll 1$.

Finally, the second term (4.5) is estimated by

$$\sum_{0\leqq \nu\leqq k+\mu} \frac{(k+\mu)!j!}{\nu!} B^{2\mu+j} |\Phi_{-j-k-\mu+\nu-1}(c)(x_1-c)^\nu|$$

$$\leqq (1-B^{-2})^2 \sum_{0\leqq \nu\leqq k} \frac{k!j!}{\nu!} B^{2k+j} |\Phi_{-j-k+\nu-1}(c)(x_1-c)^\nu|$$

$$\leqq (1-B^{-2})^2 \sum_{0\leqq \nu,\mu,j} \frac{(\nu+\mu)!j!}{\nu!} B^{2(\nu+\mu)+j} |\Phi_{-j-\mu-1}(c)(x_1-c)^\nu|$$

$$\leqq (1-B^{-2})^2 \sum \mu!j!2(\nu+\mu) B^{2(\nu+\mu)+j} |\Phi_{-j-\mu-1}(c)(x_1-c)^\nu|$$

$$\leqq (1-B^{-2})^2(1-2B^2|x_1-c|)^{-1} \sum_{l=0}^{\infty} l!(2B^2+B)^l |\Phi_{-l-1}(c)| .$$

This converges if $|c|\ll 1$, $2|x_1-c|B^2<1$. Therefore, if $|c|\ll 1$, then the series (4.5) converges uniformly at $t=x=0$. Q. E. D.

In the sequel, we consider t as a real parameter and z as complex parameters. We denote by x and y the real parts and complex parts of z respectively. We use the notation $z'=(z_2, \cdots, z_n)$, $y'=(y_2, \cdots, y_n)$, etc. We will assume that

(4.6) $G(t, z)$ is analytic on

$$\{(t, z)\in \boldsymbol{R}\times \boldsymbol{C}^n ;\ |t|<\delta,\ |z|<\delta,\ |z_1|>vt\},$$

(4.7) $L(t, z, w)$ is analytic on

$$\{(t, z, w)\in \boldsymbol{R}\times \boldsymbol{C}^n\times \boldsymbol{C}^n ;\ |t|<\delta,\ |z|,\ |w|<\delta,$$
$$0<|z_1-w_1|<\delta|z_j-w_j| ,\quad j=2, \cdots, n\} ,$$

(4.8) $P(t, z, D_t, D_z)$ is therefore, defined on

$$\{(t, z ; (\tau dt+\zeta dz)\infty) ;\ |t|<\delta,\ |z|<\delta,\ \sum_{\nu=2}^{n} |\zeta_\nu|<\delta|\zeta_1|\}$$

and that

$$H(t,z)=\frac{\partial}{\partial t}G(t,z)-\oint_{\gamma_1\times\cdots\times\gamma_n}L(t,z,w)G(t,w)dw \tag{4.9}$$

is holomorphic on

$$\Omega_0=\{(t,z);\ |t|<\delta_0<\delta,\ |z|<\delta_0<\delta\}$$

for a sufficiently small δ_0. Here γ_j is a chain given in Lemma 4.1. We assume $\delta_0+(n-1)\delta^{-1}(|c|+\delta_0)<\delta$.

Moreover we assume that

$$g(t,z,\tau,\zeta)=\det(\tau-A_1(t,z,\zeta)) \tag{4.10}$$

never vanishes on

$$\{(t,z,\tau,\zeta)\in\boldsymbol{R}\times\boldsymbol{C}^n\times\boldsymbol{C}\times\boldsymbol{C}^n;\ 0\leqq t<\delta,\ |z|<\delta,$$

$$|\zeta'|<\delta|\zeta_1|,\ -\mathrm{Im}(\tau/\zeta_1)>M(|y|+\sum_{\nu=2}^n|\mathrm{Im}(\zeta_\nu/\zeta_1)|\}.$$

Our next step is to prove the following theorem under the above assumptions.

THEOREM 4.2. *There is $\delta_1<\delta_0$ and $M_1\gg 1$ such that $G(t,z)$ is analytic on*

$$\{(t,z)\in\boldsymbol{R}\times\boldsymbol{C}^n;\ \delta_1>t>0,\ |z|<\delta_1,\ \mathrm{Im}\,z_1>M_1t(\sum_{j=2}^n|\mathrm{Im}\,z_j|)\}.$$

The essential part of the proof of this theorem is concentrated to the following lemma. Once this lemma is established, the rest of the proof is a routine.

LEMMA 4.3. *Suppose that $\varphi(t,x',y')$ (or we will sometimes denote it by $\varphi(t,z',\bar z')$) is a positive valued real analytic function defined on $U=\{(t,z');\ 0<t<\delta_1<\delta_0,\ |x'|<\delta_2,\ |y'|<\delta_3\}(\delta_2^2+\delta_3^2<\delta_0^2)$ satisfying the following:*

1) $\dfrac{\partial\varphi}{\partial t}>M\Big(|y'|+\varphi+\sum_{\nu=2}^n\Big|\dfrac{\partial\varphi}{\partial x_\nu}\Big|\Big)$,

2) $\sum_{\nu=2}^n\Big|\dfrac{\partial\varphi}{\partial z_\nu}\Big|<\dfrac{\delta}{2}$ *on* U.

Moreover, we assume that the domain of definition of $G(t,z)$ can be extended to

$$V=\{(t,z);\ 0<t<\delta_1,\ |z|<\delta_2,\ y_1>\varphi(t,x',y')\}.$$

Then $G(t,z)$ can be extended to a holomorphic function defined on an open set V' which contains

$$\{(t,z);\ 0<t<\delta_1,\ |z|<\delta_2,\ y_1\geqq\varphi(t,x',y')\}.$$

PROOF. Set $N=\{(t,z);\ 0<t<\delta_1,\ |z|<\delta_2,\ y_1=\varphi(t,x',y')\}$. We regard N as a real analytic manifold of dimension $2n$. Let Y be a complex neighborhood of N and $Y'\subset Y$ be a subset of Y such that t is real. We take the coordinates $(t, z_1, z', \tilde{z}')$ of Y' with $t|_N=t$, $z_1|_N=z_1$, $z'_\nu|_N=z'_\nu$, $\tilde{z}'_\nu|_N=\bar{z}'_\nu$ ($\nu=2, \cdots, n$). Therefore N is defined by $\tilde{z}'_\nu=\bar{z}'_\nu$. Consider an analytic function $G(t, z)$ defined on the intersection of a neighbourhood of N and $\{(t, z, \tilde{z})\in Y'; 0<t<\delta_1, |z|<\delta_2, \operatorname{Im} z_1>\varphi(t, z', \bar{z}')$. Therefore $G(t, z)$ defines a section of $\tilde{\mathfrak{A}}_N$ on $\Big\{(t, z)+\Big(ic\frac{\partial}{\partial z_1}+ic\frac{\partial}{\partial \bar{z}_1}+\Big\langle\zeta, \frac{\partial}{\partial z'}+2i\frac{\partial\varphi}{\partial z'}\frac{\partial}{\partial z_1}\Big\rangle-\Big\langle\bar{\zeta}, \frac{\partial}{\partial \bar{z}'}-2i\frac{\partial\varphi}{\partial \bar{z}'}\frac{\partial}{\partial \bar{z}'_1}\Big\rangle\Big)0\in \sqrt{-1}\,SN; c>0\Big\}$. In fact, $\operatorname{Im}\Big(z_1-\varepsilon\Big(ic+2i\Big\langle\zeta, \frac{\partial\varphi}{\partial z'}\Big\rangle-\varphi(t, z'+\varepsilon\zeta, \bar{z}'+\varepsilon\bar{\zeta})\Big)\Big)=\operatorname{Im} z_1-\varphi(t, z', \bar{z}')+\varepsilon\Big(c+2\operatorname{Re}\Big\langle\zeta, \frac{\partial\varphi}{\partial z'}\Big\rangle-\Big\langle\zeta, \frac{\partial\varphi}{\partial z'}\Big\rangle-\Big\langle\bar{\zeta}, \frac{\partial\varphi}{\partial \bar{z}'}\Big\rangle\Big)+O(\varepsilon^2)=\varepsilon c+O(\varepsilon^2)>0$ if $0<\varepsilon\ll 1$ and $c>0$. Let $\tilde{G}$ be the boundary value of G. Then the singular support of $\tilde{G}$ is contained in the polar set Z of the domain of definition of G, where $Z=\Big\{\Big((t, z); \Big(\frac{\partial\varphi}{\partial t}+ia\Big)dt+idz_1+2\Big\langle\frac{\partial\varphi}{\partial z'}, dz'\Big\rangle\Big)\infty\Big)\in\sqrt{-1}S^*N; a\in\boldsymbol{R}\Big\}$. $\tilde{G}$ satisfies the tangential Cauchy-Riemann equation: $\frac{\partial}{\partial\bar{z}_\nu}\tilde{G}=0$ ($\nu=2, \cdots, n$). P is defined on Z and invertible there, because we have

$$-\operatorname{Im}\Big(\frac{\tau}{\zeta_1}\Big)=\frac{\partial\varphi}{\partial t}>M\Big(|\operatorname{Im} z|+\Big|\operatorname{Im}\Big(\frac{\zeta'}{\zeta_1}\Big)\Big|\Big)$$

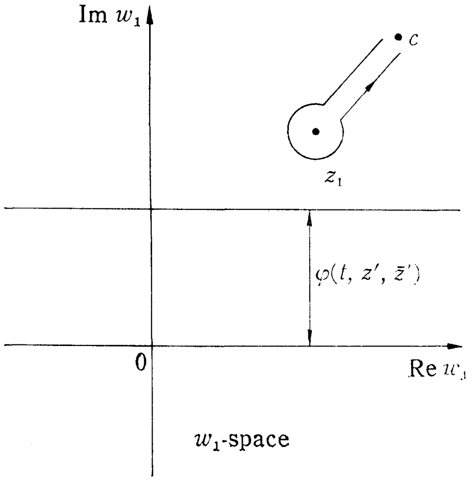
$$=M\Big(\sqrt{\varphi^2+|\operatorname{Im} z'|^2}+2\Big|\frac{\partial\varphi}{\partial z'}\Big|\Big).$$

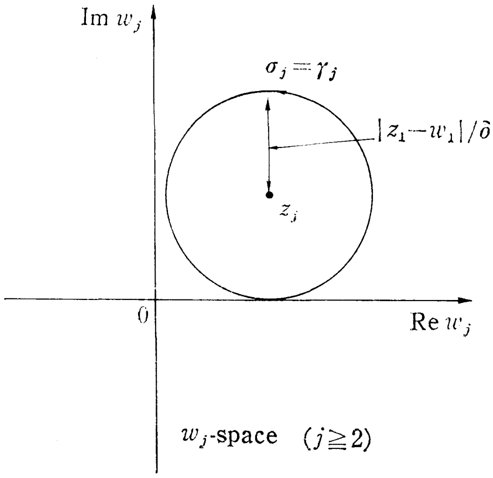
Therefore, in order to prove this lemma, it suffices to show that $P\tilde{G}=0$.

We will express $P\tilde{G}$ by making use of defining functions. $\frac{\partial\tilde{G}}{\partial t}$ is a boundary value of $\frac{\partial G}{\partial t}$ and $A\tilde{G}$ is a boundary value of the following $K(t,z)$ in a neighbourhood of $(t^0, z^0)\in N$ by;

$$K(t, z)=\int_{\sigma_1\times\cdots\times\sigma_n} L(t, z, w)G(t, w)dw$$

where σ_j is a chain defined by the following: σ_1 is a path starting from a fixed point c and ends at the same point c around z_1 counterclockwise in w_1-space, where c is a point sufficiently near to $z_1^0=x_1^0+i\varphi(t^0, z'^0, \bar{z}'^0)$, and $\operatorname{Im} c>\operatorname{Im} z_1^0$. σ_j ($j\geqq 2$) is a cycle around z_j counterclockwise with radius greater than $|z_1-w_1|/\delta$. (See the following figures.)





Then, clearly, $K(t, z)$ is defined in the intersection of V and a neighbourhood of (t^0, z^0). $P\tilde{G}$ is the boundary value of $\frac{\partial}{\partial t}G(t, z)-K(t, z)=H(t, z)$. By virtue of (4.9), this is holomorphic in Ω_0, which implies $P\tilde{G}=0$. Q. E. D.

Now we will prove Theorem 4.2 by using Lemma 4.3. Let $\varphi_1(t, x')$ be a positive valued real analytic function defined on $\{(t, x')\,;\, 0<t\leqq\delta_1,\ |x'|\leqq\delta_2\}$ satisfying

a) $\displaystyle\sum_{\nu=2}^{n}\left|\frac{\partial}{\partial x_\nu}\varphi_1(t, x')\right|<\frac{\delta}{2}$,

b) $\displaystyle\frac{\partial}{\partial t}\varphi_1(t, x')>M\sum_{\nu=0}^{n}\left|\frac{\partial}{\partial x_\nu}\varphi_1(t, x')\right|\geqq 0$,

c) $\varphi_1(t, x')>vt$ if $|x'|=\delta_2$ or $t=\delta_1$,

d) $\varphi_1(t, x')>\rho$ for some $\rho>0$.

Set $\varphi_2(t, y')=at\sqrt{|y'|^2+\varepsilon^2}$ with $\varepsilon>0$. Then $\frac{\partial\varphi_2}{\partial t}>M|y'|$, $\sum_{\nu=0}^{n}\left|\frac{\partial\varphi_2}{\partial y_\nu}\right|<\frac{\delta}{2}$ if $t<\delta/2na$ with $a>M$. Define $\varphi(t, x', y')$ by $e^{Mt}(\varphi_1(t, x')+\varphi_2(t, y'))$. This satisfies the condition in Lemma 4.3 if $\delta_1<\delta/2nMa$. Moreover $Z_\varepsilon=\{(t, z)\,;\, y_1=\varphi(t, x', y'),\ 0<t<\delta_1,\ |x'|<\delta_2,\ |y'|<\delta_3,\ |z_1|\leqq vt\}$ is compact if $v<a\delta_3$. If $\varepsilon>va^{-1}$, then Z_ε is an empty set. By using the preceding lemma, we can say that G is holomorphic on Z_ε for every $\varepsilon>0$ by the well-known method of Holmgren. Therefore G is holomorphic on $\{(t, z)\,;\, y_1-ae^{M\delta_1 t}|y'|>e^{M\delta_1}\varphi_1(t, x'),\ 0<t<\delta_1,\ |x'|<\delta_2,\ |y'|<\delta_3\}$. In order to prove Theorem 4.2, it suffices to show that for every $\varepsilon>0$, there is $\varphi_1(t, x')$ satisfying a), b), c) and d) and $\varphi_1(t, x')<\varepsilon$ for $0<t<\delta_4$ and $|x'|<\delta_4$ for some δ_4 independent of ε. This is possible by taking $\varphi_1(t, x')=f(at+|x'|^2)$ with $a\gg 0$ for a suitable choice of f. This completes the proof of Theorem 4.2.

§ 5. Construction of elementary solutions.

In the preceding section, we proved that the formal solution is analytic in an imaginary conical neighbourhood under the assumption of partial micro-hyperbolicity. By using this fact, we will construct an elementary solution of partially micro-hyperbolic operators.

LEMMA 5.1. *Suppose that* $P(t, x, D_t, D_x)=D_t-A(t, x, D_x)$ *is partially micro-hyperbolic with respect to the direction* t *at any point in the set* $F=\{(0, 0, \sqrt{-1}(kdt+dx_1)\infty)\,;\ -\infty<k<\infty\}$, *that is, all the eigenvalues of* $A_1(t, x, \sqrt{-1}\xi)$ *have non positive real part for any* (t, x, ξ) *sufficiently close to* $(0, 0, (1, 0, \cdots, 0))$. *Then, we have a microfunction* $u(t, x)$ *defined in a neighbourhood of* F *satisfying*

i) $Pu=\delta(t)\delta(x_1)$,

ii) *there are positive numbers* v *and* M *such that*

$$\operatorname{supp} u\subset\{(t, x\,;\ \sqrt{-1}(kdt+\langle\xi, dx\rangle)\infty)\,;\ t\geqq 0,\ |x_1|\leqq vt,\ |\xi'|\leqq Mt\,\xi_1\}\,.$$

PROOF. We have obtained the analytic function $G(t, z)$ defined on $\{(t, z)\in\boldsymbol{R}\times\boldsymbol{C}^n\,;\ 0<t<\delta,\ |z|<\delta,\ \operatorname{Im} z_1>Mt(\sum_{\nu=2}^{n}|\operatorname{Im} z_\nu|)\}\cup\{(t, z)\in\boldsymbol{R}\times\boldsymbol{C}^n\,;\ |t|,\ |z|<\delta,\ |z_1|>v|t|\}$, which satisfies the statement of Lemma 4.1. Then $G^+(t, z)=Y(t)G(t, z)$ is a hyperfunction in (t, z) with complex parameters z defined on

$$\{(t, z)\,;\ |z|,\ |t|<\delta,\ \operatorname{Im} z_1>M|t|(\sum_{\nu=2}^{n}|\operatorname{Im} z_\nu|)\}\,.$$

Let $v(t, x)$ be the boundary value of $G^+(t, z)$ and $u(t, x)$ be the restriction of the spectrum of $v(t, x)$ to $\{(t, x\,;\ \sqrt{-1}(kdt+\langle\xi, dx\rangle)\infty)\,;\ |t|,\ |x|<\delta,\ \xi\neq 0\}$. Since the singular support of v is contained in $\{(t, x\,;\ \sqrt{-1}(kdt+\langle\xi, dx\rangle\infty)\,;\ t\geqq 0,\ |\xi_\nu|\leqq Mt|\xi_1|,\ |x_1|\leqq vt\}$, the support of u is contained in the same set. Therefore, it suffices to show that $Pu=\delta(t)\delta(x_1)$. $A(t, x, D_x)u(t, x)$ is the boundary value of

$$K^+(t, z)=\int_{\gamma_1\times\cdots\times\gamma_n} L(t, z, w)G^+(t, w)dw\,,$$

where γ_j are paths given in Lemma 4.1 by virtue of Theorem 3.3.

Hence we have $K^+(t, z)=K(t, z)Y(t)$, where $K(t, z)=\int_{\gamma_1\times\cdots\times\gamma_n} L(t, z, w)G(t, w)dw$. Thus, we see that Pu is the boundary value of $\frac{\partial}{\partial t}G^+(t, z)-K(t, z)Y(t)=\frac{\partial}{\partial t}(G(t, z)Y(t))-K(t, z)Y(t)=\left(\frac{\partial}{\partial t}G(t, z)-K(t, z)\right)Y(t)+G(0, z)\delta(t)$. As noted in Lemma 4.1, $\frac{\partial}{\partial t}G(t, z)-K(t, z)$ is real analytic and $G(0, z)=\Phi_0(z_1)$, hence $Pu=\delta(t)\delta(x_1)$. Q. E. D.

This lemma yields the following main theorem.

THEOREM 5.2. *Let P be a pseudo-differential operator defined at $x^* \in L = \sqrt{-1}\, S^*M$ which is partially micro-hyperbolic with respect to the direction $\theta \in S_L^*(L \hat{\times} L)$. Then P is invertible in $\mathcal{A}_\theta$.*

PROOF. Firstly, we will prove this theorem in the case where $\theta \in S^*L$. By using the quantized contact transformation, we may assume without loss of generality that there is a coordinate system $(t, x_1, \cdots, x_n)$ on M so that $x^* = (0, \sqrt{-1}\, dx_1 \infty)$ and that $\theta = dt$. We may assume that $P_m(x^*) = 0$ where P_m denote the principal symbol of P. By Weierstrass' preparation theorem for pseudo-differential operators (S-K-K [**1**] Chapter II Theorem 2.2.1), we can reduce the problem to the case when $P = D_t - A(t, x, D_x)$, where A is a matrix of pseudo-differential operators of order $\leqq 1$ and that P is invertible at $(0, \sqrt{-1}(kdt + dx_1)\infty)$ if $k \neq 0$. By the preceding lemma, we have a microfunction $u(t, s, x, y, \xi)$ defined in a neighbourhood of $t = s = x = y = 0$, $\xi = (1, 0, \cdots, 0)$ such that

i) $P_{t,x} u(t, s, x, y, \xi) = \delta(t-s) \Phi_{n-1}(\langle x-y, \xi \rangle + i0)$.

ii) $\operatorname{supp} u(t, s, x, y, \xi)$ is contained in

$$
\begin{aligned}
Z = \{ (t, s, x, y, \xi\, ;\ & \sqrt{-1}(\tau dt + \sigma ds + \langle \zeta, dx \rangle + \langle \eta, dy \rangle + \langle \rho, d\xi \rangle)\infty)\, ; \\
& t \geqq s,\ |\langle x-y, \xi \rangle| \leqq v(t-s),\ |\sigma + \tau| \leqq M(t-s)|\zeta_1|, \\
& |\rho - \xi_1^{-1}\zeta_1(x-y)| \leqq M(t-s)|\zeta_1|,\ |\zeta' - \zeta_1 \xi_1^{-1} \xi'| \leqq M(t-s)|\zeta_1| \text{ and} \\
& |\eta + \xi_1^{-1}\zeta_1 \xi| \leqq M(t-s)|\zeta_1| \}
\end{aligned}
$$

because $\tau dt + \sigma ds + \langle \zeta, dx \rangle + \langle \eta, dy \rangle + \langle \rho, d\xi \rangle = \tau d(t-s) + (\tau + \sigma) ds + \zeta_1 \xi_1^{-1} d\langle x-y, \xi \rangle + \langle \zeta' - \zeta_1 \xi_1^{-1} \xi', dx' \rangle + \langle \eta + \xi_1^{-1} \zeta_1 \xi', dy \rangle + \langle \rho - \xi_1^{-1} \zeta_1 (x-y), d\xi \rangle$. Therefore the intersection of the support of u and the set defined by $\rho = 0$ is contained in

$$
\begin{aligned}
Z' = \{ (t, s, x, y, \xi\, ;\ & \sqrt{-1}(\tau dt + \sigma ds + \langle \zeta, dx \rangle + \langle \eta, dy \rangle)\infty\, ; \\
& t \geqq s,\ |x-y| \leqq M(t-s),\ |\sigma + \tau| \leqq M(t-s)|\zeta_1|, \\
& |\zeta' + \eta'| \leqq M(t-s)|\zeta_1|,\ |\eta_1 + \zeta_1| \leqq M(t-s)|\zeta_1| \\
& |\zeta' - \zeta_1 \xi_1^{-1} \xi'| \leqq M(t-s)(1 + |\zeta'|/|\zeta_1|) \}\, .
\end{aligned}
$$

It follows that

$$
E(t, x, s, y) = \frac{1}{(2\pi i)^{n-1}} \int u(t, s, x, y, \xi) \omega(\xi)
$$

can be defined and its support is contained in $\{(t, x, s, y\, ;\ \sqrt{-1}(\tau dt + \langle \zeta, dx \rangle + \sigma ds + \langle \eta, dy \rangle)\infty)\, ;\ t \geqq s,\ |x-y| \leqq M(t-s),\ |\zeta + \eta| \leqq M(t-s)|\zeta_1|,\ |\sigma + \tau| \leqq M(t-s)|\zeta_1|\}$. Therefore $Edsdy$ belongs in $\mathcal{A}_{dt}$. $P_{t,x} E(t, x, s, y) = \frac{1}{(2\pi i)^{n-1}} \int P_{t,x} u(t, s, x, y, \xi) \omega(\xi) = \frac{1}{(2\pi i)^{n-1}} \int \delta(t-s) \Phi_{n-1}(\langle x-y, \xi \rangle) \omega(\xi) = \delta(t-s) \delta(x-y)$. This implies that $Edsdy$

is a right inverse of P. By starting from the adjoint operator of P, we can construct the left inverse of P in $\mathcal{A}_{-dt}$ in the same way as above. This completes the proof in the case when $\theta\in S^*_{x^*}L$.

Now, we consider the general case. Set $M''=M\times\boldsymbol{R}$ and $L''=\sqrt{-1}S^*M''-\sqrt{-1}S^*M\times\boldsymbol{R}-M\times\sqrt{-1}S^*\boldsymbol{R}=L\hat{\times}L'$, where $L'=\sqrt{-1}S^*\boldsymbol{R}$. Let p (resp. $\hat{p}$) be the projection $L''\to L$ (resp. $\hat{L}''\to\hat{L}$) and let q (resp. $\hat{q}$) be the projection $L''\to L'$ (resp. $\hat{L}''\to\hat{L}'$). Let (x, ξ, t, τ) be a coordinate system of $\hat{L}''$ such that

$$\theta_{L''}=\hat{p}^*\theta_L+\hat{q}^*\theta_{L'}=\sqrt{-1}\,(\langle\xi, dx\rangle+\tau dt)\,.$$

Then P can be considered as a section of $\mathcal{P}^f_{M''}$. Set $\theta=\langle a, dx\rangle+\langle b, d\xi\rangle$. Then P is partially micro-hyperbolic at $p^{-1}(x^*)$ with respect to the direction $\theta''=\theta+ld\tau$ for every l. $\theta''\in S^*L''$ is equivalent to the relation $l=-\tau^{-1}\langle b, \xi\rangle$. Therefore, for every point $\alpha\in p^{-1}(x^*)$ we have the unique section K_α of $\mathcal{C}^{(0,n+1)}_{M''\times M'',\alpha}$ such that $PK_\alpha=K_\alpha P=1$ and $K''_\alpha\in\mathcal{A}_{(\alpha,\theta'')}$. By the uniqueness of K''_α, we can patch K_α and obtain a section K'' defined in a neighbourhood of $p^{-1}(x^*)$ such that the germ of K'' coincides K''_α at any α. (See the following lemma). Moreover, since we have $[t, P]=[\partial/\partial t, P]=0$, $[t, K'']=[\partial/\partial t, K'']=0$. Therefore, there is a $K\in\mathcal{C}^{(0,n)}_{M\times M}$ such that K'' is a pull back of K. Therefore K must be contained in $\mathcal{A}_\theta$ and $PK=KP=1$. Q.E.D.

The following lemma is obvious by the preceding theorem and we omit its proof.

LEMMA 5.3. *Let F be a subset in $L=\sqrt{-1}S^*M$, and $\theta: F\to S^*_L(L\hat{\times}L)$ be a section of $S^*_L(L\hat{\times}L)$ over F and P be a pseudo-differential operator defined in a neighbourhood of F and partially micro-hyperbolic at any $x^*\in F$ with respect to the direction θ_{x^*}. Then there is a section K of $\mathcal{C}^{(0,n)}_{M\times M}$ defined in a neighbourhood of F such that K belongs to $\mathcal{A}_{\theta_{x^*}}$ at any point x^* in F.*

Now, consider the pseudo-differential operator P defined near x^*. Then, the maximal subset of $S^*_L(L\hat{\times}L)_{x^*}$ where P is partially micro-hyperbolic is an open set. Moreover, its connected component is convex. This is an easy consequence of S-K-K [**1**] Chapter I Proposition 1.5.4. The following proposition is, therefore, useful in application.

PROPOSITION 5.4. *Let Γ be a nonvoid open convex cone in $S^*_L(L\hat{\times}L)_{x^*}$ for $x^*\in L$, and P be a pseudo-differential operator which is partially micro-hyperbolic at x^* with respect to the direction Γ. Then P is invertible in $\mathcal{A}_\Gamma$.*

PROOF. Let θ_1, θ_2 be two points in Γ. There are $E_\nu\in\mathcal{A}_{\theta_\nu}$ such that $PE_\nu=E_\nu P=1$ $(\nu=1, 2)$. It suffices to show that $E_1=E_2$ because $\mathcal{A}_\Gamma=\bigcap_{\theta\in\Gamma}\mathcal{A}_\theta$. We employ the argument used in Theorem 5.2. Set $L'=\sqrt{-1}S^*\boldsymbol{R}$, $L''=L\hat{\times}L'$ and $p: L''\to L$. P is partially micro-hyperbolic at any point in $p^{-1}(x^*)$ with respect to direction $\theta''=(1-t)\hat{p}^*\theta_1+t\hat{p}^*\theta_2$ with $-\varepsilon<t<1+\varepsilon$ for sufficiently small $\varepsilon>0$. Therefore there is a section K'' of $\mathcal{C}^{(0,n+1)}_{M''\times M''}$ such that $PK''=K''P$

and K'' belongs to $\mathcal{A}_{(\alpha,\theta')}$ for every α. Moreover, since K'' commutes with t and $\partial/\partial t$, K'' is a pull back of the unique $K \in C^{(0,n)}_{M\times M}$ such that $PK = KP = 1$. Therefore K must belong to $\mathcal{A}_{\theta'}$ for every t such that $-\varepsilon < t < 1+\varepsilon$. Therefore $K \in \mathcal{A}_{\theta_\nu}$ $(\nu = 1, 2)$. It follows that $K = K_\nu$ $(\nu = 1, 2)$. Q. E. D.

When P is a micro-hyperbolic operator, we can obtain an elementary solution for the Cauchy problem.

THEOREM 5.5. *Let $P = D_t - A(t, x, D_x)$ be a pseudo-differential operator defined in a neighbourhood of $(t_0, x_0, \sqrt{-1}\langle\xi_0, dx_0\rangle\infty)$, where $A(t, x, D_x)$ is a matrix of pseudo-differential operators of order $\leqq 1$. Suppose that all roots τ of the equation $g(t, x, \tau, \xi) = \det(\tau - A_1(t, x, \xi))$ are pure imaginary for any (t, x, ξ) in a sufficiently small neighbourhood of $(t_0, x_0, \sqrt{-1}\xi_0)$. Then there is a microfunction $E(t, x, y)$ defined in a neighbourhood of $\{(t, x, y\,;\ \sqrt{-1}(\tau dt + \langle\xi, dx\rangle + \langle\eta, dy\rangle)\infty)\,;\ t = t_0,\ x = y = x_0,\ \xi = -\eta = \xi_0\}$ satisfying the following conditions.*

1) $PE = 0$;

2) $E(t, x, y)|_{t=t_0} = \delta(x-y)$;

3) *The support of E is contained in*

$$\{(t, x, y\,;\ \sqrt{-1}(\tau dt + \langle\xi, dx\rangle + \langle\eta, dy\rangle)\infty\,;\ |x-y| \leqq M|t-t_0|,$$
$$|\xi+\eta| \leqq M|t-t_0|\,|\xi|,\ |\tau| \leqq M|\xi|\}\,.$$

PROOF. If $\xi_0 = (1, 0, \cdots, 0)$ and $t_0 = 0$, then $G(t, z)$ constructed in the preceding section is holomorphic in $\{(t, z) \in \boldsymbol{R}\times\boldsymbol{C}^n\,;\ \operatorname{Im} z_1 > M|t|(\sum_{\nu=2}^{n}|\operatorname{Im} z_\nu|),\ |t|,\ |z| \ll 1\}$. Then, the spectrum $u(t, x)$ of the boundary value of $G(t, z)$ satisfies

a) $Pu = 0$,

b) $u|_{t=0} = \delta(x_1)$,

c) $\operatorname{supp} u \subset \{(t, x,\ \sqrt{-1}(\tau dt + \langle\xi, dx\rangle))\,;\ x_1 < v|t|\,,$

$$|\tau| \leqq M|\xi|,\ |\xi_\nu| \leqq M|t|\xi_1 \qquad (\nu = 1, \cdots, n)\}$$

in the same way as in the proof of Lemma 5.1. The argument employed in Theorem 5.2 easily gives us the elementary solution $E(t, x, y)$. The detailed proof is left to the reader. Q. E. D.

We will give several examples of partially micro-hyperbolic operators.

EXAMPLE 1. $M = \boldsymbol{R}^2 = \{(x_1, x_2)\}$, $P = D_1 - \alpha x_2^{2m} D_2$ $(\alpha \in \boldsymbol{C}^+,\ m \geqq 1)$. In this case, the set of real characteristics $V_{\boldsymbol{R}}$ of P is $\{(x,\ \sqrt{-1}\langle\xi, dx\rangle\infty) \in \sqrt{-1}S^*M\,;\ \xi_1 = 0,\ x_2 = 0\} = V_+ \cup V_-$ where $V_\pm = \{\xi_i = 0,\ \xi_2 \gtrless 0,\ x_2 = 0\}$. $V_\pm$ is one dimensional. We use x_1 as the parameter on $V_\pm$. P is partially micro-hyperbolic with respect to the direction

$$\theta = \pm(dx_1 + a dx_2 + b d\xi_1 + c d\xi_2) \text{ on } V_\pm \text{ for any } a,\ b,\ c \in \boldsymbol{R}$$

(see S-K-K [1] Chapter I Lemma 3.1.5).

If $c=0$, θ belongs to S^*L. Therefore P is solvable. The good elementary solution $E(x, y)$ is given explicitly as follows:

Set

$$f_\pm(x_1, y_1, z_2, w_2)=$$

$$-\frac{1}{2\pi\sqrt{-1}}\frac{Y(\pm(x_1-y_1))}{z_2/2m-1\sqrt{1-(2m-1)\alpha x_1 z_2^{2m-1}}-w_2/2m-1\sqrt{1-(2m-1)\alpha y_1 w_2^{2m-1}}}\,.$$

Then $f_\pm$ is a hyperfunction defined on $\operatorname{Im} z_2 \gtrless k \operatorname{Im} w_2$ and $|z_2|$, $|w_2| \ll 1$ for $k>1$. We set

$$E(x, y)=f_+(x_1, y_1, x_2+\sqrt{-1}0, y_2-\sqrt{-1}0)$$
$$-f_-(x_1, y_1, x_2-\sqrt{-1}0, y_2+\sqrt{-1}0)\,.$$

Then clearly, $P_x E=P_y^* E=\delta(x-y)$. In the sequel, we restrict ourselves to the analysis only on V_+. Let u be a section of the solution sheaf $\mathcal{C}^P$ defined on $\{x_1\in V_+;\ x_1>a\}$. Then there is a holomorphic function $\varphi(\tau)$ defined on $\tau\in U\subset \boldsymbol{C}$ satisfying; for every $b>a$, there is $\varepsilon>0$ such that U contains $\{\tau\in \boldsymbol{C};\ |\tau|<\varepsilon,\ \operatorname{Im}\tau>(\operatorname{Im}\alpha)b(\operatorname{Re}\tau)^{2m}\}$, and u is a boundary value of $\varphi(z_2/2m-1\sqrt{1-(2m-1)\alpha x_1 z_2^{2m-1}})$ from $\operatorname{Im} z_2>0$. Moreover $\varphi(\tau)$ is determined uniquely by u modulo holomorphic function defined in a neighbourhood of $\tau=0$. It follows that there is a solution of P defined on $\{x_1\in V_+;\ x_1>a\}$ that cannot be continued to a solution defined in a neighbourhood of $x_1=a$. In this case, the uniqueness of continuation holds in both directions.

EXAMPLE 2.

$$M=\boldsymbol{R}^2, \qquad P=x_1+\sqrt{-1}\,x_2^2\,.$$

$$V_{\boldsymbol{R}}=\{(x, \sqrt{-1}\langle\xi, dx\rangle\infty);\ x_1=x_2=0\}\,.$$

In this case, P is partially micro-hyperbolic with respect to the direction $\theta= d\xi_1+a dx_1+b dx_2+c d\xi_2$ for any $a, b, c\in\boldsymbol{R}$. θ belongs to S^*L if and only if $\xi_1=-c\xi_2$. Therefore, if $\xi_2\neq 0$, P is solvable. Furthermore, since it is known that $P:\mathcal{B}\to\mathcal{B}$ is surjective, P is micro-locally solvable everywhere. The authors conjecture that pseudo-differential operator is solvable even if it is partially micro-hyperbolic with respect to the direction $\theta\notin S^*L$. In the original case, the singularity propagates to the direction $\xi_1/|\xi_2|$. The elementary solution $E(x, y)$ of P is

$$E(x, y)=\frac{1}{x_1+\sqrt{-1}\,x_2^2}\,\delta(x-y)\,.$$

Remark that operator $x_1+\sqrt{-1}\,x_2^2$ considered in $\xi_2\neq 0$ is equivalent to that of Example 1 by means of a quantized contact transformation.

§ 6. Existence and uniqueness theorems.

In this section we list up the existence and uniqueness theorems which follow from the existence of the "good" elementary solution of (partially) micro-hyperbolic operators (Theorem 6.1).

In this section N denotes a hypersurface of a real analytic manifold M. We use the local coordinate system $x=(x_1, x')$ on M so that $N=\{x\in M | x_1=0\}$.

To begin with we note the following theorem which is a trivial corollary of Theorem 5.2.

THEOREM 6.1. *Let $P(x, D_x)$ be a single pseudo-differential operator of finite order which is partially micro-hyperbolic at $x^*+\sqrt{-1}\vartheta 0$, then there exists an elementary solution E of P in $\mathcal{H}_{x^*+\sqrt{-1}\vartheta 0}$, i.e. there exists an element E in $\mathcal{H}_{x^*+\sqrt{-1}\vartheta 0}$ satisfying*

$$PE=EP=1.$$

The proof immediately follows from Theorem 5.2. (Cf. the arguments at the beginning of § 2.)

The first consequence of Theorem 6.1 is the following existence theorem.

THEOREM 6.2. *Let $P(x, D_x)$ be partially micro-hyperbolic at $x^*+\sqrt{-1}\vartheta 0$ with $\langle\vartheta, \theta\rangle=0$ where x^* is a point in $\sqrt{-1}S^*M=L$ and θ is the canonical 1-form of L. Then*

$$\mathcal{C}_{x^*} \xrightarrow{P} \mathcal{C}_{x^*}$$

is surjective.

PROOF. For any microfunction f defined in a neighbourhood of x^* we can find $\tilde{f}$ which coincides f in a neighbourhood U of x^* and has its support in the closure of U. Taking U sufficiently small, we may assume that the elementary solution E of P defined in a neighbourhood of $(x^*, -x^*)\in L\subset L\hat{\times} L$ operates on $\tilde{f}$. Clearly $u=E\tilde{f}$ gives the required solution of the equation $Pu=f$ near x^*. This completes the proof of the theorem.

THEOREM 6.3. *Let x^* be a point in $L=\sqrt{-1}S^*M$ and Γ be a non empty open convex set in $S^*_{x^*}L$. Denote by $\mathcal{G}$ the set of all closed sets whose normal set at x^* is disjoint from the polar of Γ, and $\mathfrak{U}$ be the set of all open neighbourhood of x^*. Assume that a pseudo-differential operator $P(x, D_x)$ is partially microhyperbolic at x^* with respect to the direction Γ, then we have the following isomorphism:*

$$\mathcal{C}^P_{x^*}=\varinjlim_{U\in\mathfrak{U}} \mathcal{C}^P(U) \overset{r}{\underset{\sim}{\longrightarrow}} \varinjlim_{\substack{U\in\mathfrak{U}\\ G\in\mathcal{G}}} \mathcal{C}^P(U-G).$$

Here $\mathcal{C}^P$ denotes the microfunction solution sheaf of the pseudo-differential equa-

tion $P(x, D_x)u=0$. *In other words,* $\lim_{G\in\mathcal{G}} \mathcal{H}_G^k(C^P)_{x^*}=0$ *holds for any* k.

PROOF. We first show that the natural restriction map r is injective. Let u be in $C^P(U)$ with support in $G\cap U$ for some G in $\mathcal{G}$ and some $U\in\mathfrak{U}$. Clearly u belongs to $\mathcal{M}_\Gamma$. Therefore E operates on u by Proposition 1.7. Hence we have

$$u=PEu=EPu=0$$

in a neighbourhood of x^*. This implies the injectivity of the map r.

Next we prove the surjectivity of r. Let u be an element in $C^P(U-U\cap G)$ for some G in $\mathcal{G}$. Then the flabbiness of the sheaf of microfunctions allows us to find an extension $\tilde{u}$ of u to U. Clearly $\operatorname{Supp} P\tilde{u}\subset U\cap G$. Hence $\mu=P\tilde{u}$ belongs to $\mathcal{M}_\Gamma$. Therefore $E\mu$ is well-defined. Moreover there exists a closed set $\tilde{G}$ containing G which satisfies

$$S_{x^*}\tilde{G}\cap\Gamma^\circ=\emptyset$$

and

$$\operatorname{Supp} E\mu\cap U'\subset\tilde{G}\cap U'$$

for some neighbourhood $U'\subset U$ of x^* by the support property of E. It is also clear that $P(\tilde{u}-E\mu)=0$ holds. Clearly $u=\tilde{u}-E\mu$ coincides with u in $U'-\tilde{G}$. Thus u defines an extension of $u|U'-\tilde{G}$ to a neighbourhood of x^* satisfying the equation $Pu=0$. This implies the surjectivity of the map r.

As a trivial corollary of the above theorem, we have the following result about the propagation of analyticity of solutions.

COROLLARY. *Let* ρ *be the canonical map from* $\sqrt{-1}S^*M\underset{M}{\times}N-\sqrt{-1}S^*_NM$ *to* $\sqrt{-1}S^*N$. *Assume that* N *is non-characteristic with respect to a linear differential operator* $P(x, D_x)$ *and that* P *is partially micro-hyperbolic with respect to* x_1*-direction at any point in* $\rho^{-1}(x'^*)$ *for any* x'^* *in* $\sqrt{-1}S^*N$. *Then there exists a neighbourhood* U *of* N *such that any hyperfunction solution* $u(x)$ *of the equation* $P(x, D_x)u(x)=0$ *that is defined on* U *and real analytic in* $\{x\in U\,|\,x_1<0\}$ *is necessarily real analytic in* U.

Moreover partially micro-hyperbolic differential operators enjoy the following existence theorem.

THEOREM 6.4. *Let* $P(x, D_x)$ *be a linear differential operator which is partially micro-hyperbolic in some* $\langle\xi_{x^*}, dx\rangle$ *direction at any point* $x^*=(x^0, \sqrt{-1}\eta)$ *in* $\pi^{-1}(x^0)\subset\sqrt{-1}S^*M$ *for* x^0 *in* M. *Then we can find a neighbourhood* U *of* x^0 *such that*

$$(\mathcal{B}/\mathfrak{A})(U)\xrightarrow{P(x, D_x)}(\mathcal{B}/\mathfrak{A})(U)$$

is surjective.

PROOF. First we take a finite open covering $\{W_j\}_{j=1}^k$ of $\pi^{-1}(U)$ in $\sqrt{-1}S^*M$

for a sufficiently small neighbourhood U of x_0 so that it satisfies the following condition: There exists an open set $\widetilde{W}_j$ containing the closure of W_j where the equation $P(x, D_x)u=f$ has a microfunction solution u on $\pi^{-1}(U)$ whose support is in F_j for any microfunction f with compact support in the closure of W_j so that $\pi|_{F_j}$ is a proper map and that $\pi(W_j)$ contains a neighbourhood U of x^0.

Such a choice of $\{W_j\}$ is possible by the assumption of the partial micro-hyperbolicity of $P(x, D_x)$ in the $\langle \xi_{x^*}, dx\rangle$ direction. In fact P has an elementary solution in $\mathscr{H}_{x^*+\sqrt{-1}\vartheta 0}$ ($\vartheta \in H^{-1}(\xi_{x^*})$), hence it is possible to define F_j as the union of some cones with their vertexes in W_j if W_j is sufficiently small. The properness of F_j over U_j follows from the assumption that P is partially micro-hyperbolic in $\langle \xi_{x^*}, dx\rangle$-direction.

For any $f(x)$ in $(\mathscr{B}/\mathfrak{A})(U)$ we can find an extension $\tilde{f}(x)$ of $f(x)$ in $(\mathscr{B}/\mathfrak{A})(M)$ with its support in the closure of U by the flabbiness of the sheaf $\mathscr{B}/\mathfrak{A}$. Then using the flabbiness of the sheaf of microfunctions we can find microfunctions $\{f_j\}_{j=1}^k$ so that $\operatorname{Supp} f_j$ is contained in the closure of W_j and that $\sum_{j=1}^k f_j=\tilde{f}$. After this decomposition of $\tilde{f}$ we can find u_j so that $P(x, D_x)u_j=f_j$ in $\widetilde{W}_j$ and that $\operatorname{Supp} u_j$ is contained in F_j. Since $\pi|_{F_j}$ is a proper map, $u=\sum_{j=1}^k u_j$ makes sense. Clearly $Pu=\tilde{f}$ holds as an equation for microfunctions. Moreover u belongs to $C(\pi^{-1}(U))$. Therefore $P(x, D_x)u(x)=\tilde{f}(x)$ holds in $(\mathscr{B}/\mathfrak{A})(U)$. This proves the surjectivity of $P(x, D_x)$. Q. E. D.

Moreover we can prove the following

TEEOREM 6.5. *Let $P(x, D_x)$ satisfy the conditions in Theorem 6.4. Then we can find a neighborhood U of x^0 on which*

$$P(x, D_x)\colon \mathscr{B}(U) \longrightarrow \mathscr{B}(U)$$

is surjective.

PROOF. The method of the proof of the preceding theorem shows the existence of a hyperfunction $E(x, y)$ defined in $U_0\times U_0$ for a neighbourhood U_0 of x^0 which satisfies

$$P(x, D_x)E(x, y)=\delta(x-y)+f(x, y) \qquad (x, y\in U_0)$$

for a real analytic function $f(x, y)$ and which depends real analytically on y. Note that we can find microfunctions f_j so that $\operatorname{Supp} f_j$ is contained in $\overline{W_j}\cap \Delta^a$ and $\sum f_j=\delta(x-y)$ for any locally finite open covering $\{W_j\}$ of Δ^a, the antidiagonal set in $\sqrt{-1}S^*(M\times M)$, since the sheaf of microfunctions is flabby. The assumption of partial micro-hyperbolicity in $\langle \xi_{x^*}, dx\rangle$ direction implies that $p_m(x^0, \sqrt{-1}\eta)\neq 0$ for some η (maybe complex), the Cauchy-Kovalevsky theorem asserts the existence of a real analytic function $v(x, y)$ defined in a neighbourhood W of (x_0, x_0) in $M\times M$ which satisfies $P(x, D_x)v(x, y)=f(x, y)$ there.

Therefore we can find a neighbourhood U_1 of x_0 and a hyperfunction $F(x, y)$ defined on $U_1 \times U_1$ such that

$$P(x, D_x)F(x, y) = \delta(x-y)$$

holds for $(x, y) \in U_1 \times U_1$. This implies the surjectivity of

$$P(x, D_x): \mathcal{B}(U) \longrightarrow \mathcal{B}(U)$$

for any open set $U \Subset U_1$ by the aid of the flabbiness of the sheaf of hyperfunctions. Q. E. D.

As explained thus far, partially micro-hyperbolic operators enjoy the good existence and unique continuation theorems. Moreover some partially micro-hyperbolic operators have their inverse in $\mathcal{L}_M$, not merely in $\mathcal{H}_{x^*+\sqrt{-1}\vartheta_0}$. The typical example of such an operator is $D_1 + \sqrt{-1} x_1^{2k} D_2$ considered at $x_1 = 0$, $\eta_1 = 0$, $\eta_2 \neq 0$.

Generally we have the following result concerning pseudo-differential operators which have their inverse in $\mathcal{L}_M$.

THEOREM 6.6. *Let Γ be a convex cone in $S^*_{x_0^*} L$ for x_0^* in $L = \sqrt{-1} S^* M$. Assume that a pseudo-differential operator $P(x, D_x)$ is partially micro-hyperbolic in Γ at x_0^*. Define Z by $\{x^* \in L \hat{\times} L\,;\, p_m(p_1(x^*)) = p_m(p_2(x^*)) = 0\}$ and assume that the closure $S_L Z$ of $Z - L$ in $\widetilde{{}^L L \hat{\times} L}$ is disjoint from $\Gamma^{\circ a}$. Then $P(x, D_x)$ has a two-sided inverse in $\mathcal{L}_M$.*

PROOF. The assumption of partial micro-hyperbolicity assures the existence of the elementary solution E of P. Moreover, if we denote $\operatorname{Supp} E$ by G, then the closure $S_L G$ of $G - L$ in $\widetilde{{}^L L \hat{\times} L}$ is contained in $\Gamma^{\circ a}$ near x_0^*. On the other hand G is contained in $L \hat{\times} L \cap (Z \cup L)$, since a pseudo-differential operator is invertible where its principal symbol does not vanish. This implies that $S_L G$ is contained in $S_L Z$. Therefore the assumption that $S_L Z$ is disjoint from $\Gamma^{\circ a}$ implies that $S_L G$ is disjoint from $\Gamma^{\circ a}$. Since $S_L G$ is contained in $\Gamma^{\circ a}$, $S_L G$ is void. This implies that G is contained in the anti-diagonal set Δ^a in $L \hat{\times} L$, that is, G defines a kernel function of a micro-local operator by the definition. This completes the proof of the theorem. Q. E. D.

Moreover, if we assume that the characteristic variety V of P is regular in the complex domain, then we have the following result. (Cf. Egorov [**2**], Treves [**2**].)

THEOREM 6.7. *Assume that V is defined by $a(x, \eta) + \sqrt{-1} b(x, \eta) = 0$ where $(\sqrt{-1})^{-m} a(x, \sqrt{-1}\eta)$ and $(\sqrt{-1})^{-m} b(x, \sqrt{-1}\eta)$ are real for $(x, \sqrt{-1}\eta)$ in $\sqrt{-1} S^* M$ near $x_0^* = (x_0, \sqrt{-1}\eta_0)$ and that* $\operatorname{grad}_{(x,\eta)} a(x, \eta)$ *and ω are linearly independent there. Assume further that $(\sqrt{-1})^{-m} b(x, \sqrt{-1}\eta)$ is positive (or negative) on each real bicharacteristic strip of $(\sqrt{-1})^{-m} a(x, \sqrt{-1}\eta)$ and not identically zero there. Then $P(x, D_x)$ is invertible in $\mathcal{L}_{M, x_0^*}$.*

PROOF. By a suitable "quantized" contact transformation, we may assume from the beginning that $a(x, \eta)=\eta_1$. While, the assumption on b implies that b has the zero of (finite) even order with positive (or negative) coefficients on each bicharacteristics of $a(x, \eta)=\eta_1$. It is clear that we have an elementary solution $E(x, y)$ of P such that $G=\operatorname{Supp} E$ is contained in $\{x_1 \geqq y_1\}$ by the partial micro-hyperbolicity of P. On the other hand since $a(x, \eta)=\eta_1$, G is contained in $\{(x^*, y^*) \in L \hat{\times} L;\ x_1 \geqq y_1$ and x^* and y^* lie on the same bicharacteristic strip of $a\}$. Then $G \subset V \cup L$ by the invertibility of elliptic operators. Moreover Theorem 6.6 asserts that $G \subset L \subset L \hat{\times} L$. Thus P is seen to have a right inverse in $\mathcal{L}_{M, x_0^*}$. Applying the same argument to P^*, we see that P is invertible in $\mathcal{L}_{M, x_0^*}$. Q. E. D.

REMARK. Only the assumption of positivity (or negativity) of $(\sqrt{-1})^{-m} b(x \sqrt{-1} \eta)$ implies the solvability of the equation $P(x, D_x)u=f$, since P is still partially micro-hyperbolic under this weak assumption. (Cf. Egorov [1], Nirenberg and Treves [1]).

Thus far we have discussed the existence and unique extension theorems for partially micro-hyperbolic operators. If we assume the micro-hyperbolicity of the operator, then we can further show the well-posedness of the Cauchy problems for such an operator. Here "well-posedness" means the unique existence of the solutions.

For the sake of simplicity of the terminology, we introduce the following definition. We always assume that N is non-characteristic with respect to P.

DEFINITION 6.8. A pseudo-differential operator $P(x, D_x)$ defined in a neighbourhood of $dx_1=0$ is said to be micro-hyperbolic in x_1-direction at x'^* in $\sqrt{-1} S^* N \underset{N}{\times} M$, if P is micro-hyperbolic with respect to x_1-direction at any point in $\rho^{-1}(x'^*)$. We also say that P is micro-hyperbolic in x_1-direction in $U \subset \sqrt{-1} S^* N \underset{N}{\times} M$, if P is so at any point in U. (Cf. Kawai [1]).

Now we have the following theorem.

THEOREM 6.9. *Let $P(x, D_x)$ be micro-hyperbolic in U, V be the characteristic variety of P. Denote by k the number of the points in $V \cap \rho^{-1}(x'^*)$, counting their multiplicities for $x'^* \in U$. Then there exist a neighbourhood W of $V \cap \rho^{-1}(U)$ and a unique microfunction solution of the following Cauchy problem in W:*

$$\begin{cases} P(x, D_x) u=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j u\big|_{x_1=0}=\mu_j(x'), \qquad j=0, \cdots, k-1, \end{cases}$$

where $\mu_j(x')$ is a microfunction on U.

PROOF. Since Theorem 5.5 asserts the existence of the elementary solution for Cauchy problems, the proof is immediate.

Moreover, if the operator P is a hyperbolic differential operator, then the Cauchy problem in the space of hyperfunctions is obviously well-posed. In fact we have the following theorem. This is the case treated by Bony-Schapira [1] and [2].

THEOREM 6.10. *Let $P(x, D_x)$ be a hyperbolic linear differential operator of order m. Then there exist a neighbourhood W of N and a unique hyperfunction solution of the following Cauchy problem in W:*

$$\begin{cases} P(x, D_x)u(x)=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j u|_{x_1=0}=\mu_j(x'), \qquad j=0, \cdots, m-1, \end{cases}$$

where $\mu_j(x')$ is a hyperfunction in x' defined on $W \cap N$.

PROOF. Applying the Cauchy-Kovalevsky theorem we can find $E_k(x, y')$ which satisfies in a neighbourhood of N the following:

$$\begin{cases} PE_k=0 \\ \left(\dfrac{\partial}{\partial t}\right)^j E_k|_{x_1=0}=\delta_{jk}\delta(x'-y'), \qquad 0 \leqq j,\ k \leqq m-1. \end{cases}$$

Then the Holmgren's uniqueness theorem asserts that supp $E_k(x, y')$ is contained in $K_1 \cup (-K_1)$, where K_1 is a proper cone in $\{x_1 \geqq 0\}$, because E_k is real analytic outside the cone. Therefore the assertion follows immediately. Q. E. D.

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