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# A remark on ordered structures with unary predicates

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## Introduction.

Structures for the first order theory of linear order have a simple character even with unary predicates. So, many results have been obtained in this subject. We will add some pretty results to them in this paper.

Fraïssé method and the idea in Ehrenfeucht [**2**] are useful to study those structures as seen in Läuchli and Leonard [**6**]. Our main tools are also them.

Our main results are the following:

(a) An ordinal $\alpha$ has a cofinality larger than $\omega$, if and only if for any subset $U$ of $\alpha$, the structure $\langle \alpha, <, U\rangle$ has a proper elementary extension of the form $\langle \beta, <, V\rangle$.

(b) Any injective map from $\kappa^2$ into $\kappa$ is not definable in $\langle \kappa, <, S\rangle_{S\subseteq\kappa}$.

## §1. Preliminaries.

A structure $A$ is called an ordered structure with unary predicates if it has the form of $\langle |A|, <_A, P_\xi^A\rangle_{\xi<\alpha}$ where $<_A$ is a linear ordering of $|A|$ and $P_\xi^A$ is a subset of $|A|$ for each $\xi<\alpha$. The similarity type of the structure $A$ will be denoted by $\tau_\alpha$.

Let $A\cong\langle Q, <_1, S_\xi\rangle_{\xi<\alpha}$ and $B\cong\langle R, <_2, T_\xi\rangle_{\xi<\alpha}$ where $Q\cap R=\emptyset$. Then $A+B$ means the structure $\langle Q\cup R, <_1\cup(Q\times R)\cup <_2, S_\xi\cup T_\xi\rangle_{\xi<\alpha}$. In the same sense, when $A_i$ is an ordered structure with unary predicates for each element $i$ of an ordered set $I$, we denote by $\sum_{i\in I} A_i$ the structure which is obtained by concatenating all $A_i$ $(i\in I)$ following the ordering of $I$.

Given $a, b\in|A|$, $(a, b)_A$ will denote the substructure of $A$ whose field is the open interval of $A$ determined by $a$ and $b$. In the similar manner, we

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define $(-\infty, a)_A$, $(a, \infty)_A$, $[a, b)_A$, $[a, \infty)_A$, etc.

The following definition and Lemmas 1, 2 and 3 are copies of Definition and Lemmas 1, 2 and 3 of Läuchli and Leonard [6].

DEFINITION. We define recursively the $n$-equivalence ($\equiv_n^\alpha$) between two ordered structures of the similarity type $\tau_\alpha$ as follows:

(a) $A \equiv_0^\alpha B$ always

(b) $A \equiv_{n+1}^\alpha B$ if and only if for every $a \in |A|$ there exists $b \in |B|$, and for every $b \in |B|$ there exists $a \in |A|$, such that $(-\infty, a)_A \equiv_n^\alpha (-\infty, b)_B$, $(a, \infty)_A \equiv_n^\alpha (b, \infty)_B$ and $a \in P_\xi^A \Leftrightarrow b \in P_\xi^B$ for every $\xi < \alpha$.

Throughout this section, capital letters $A$, $B$, $C$, $D$ (possibly with index) will denote ordered structures of the similarity type $\tau_\alpha$.

LEMMA 1. (i) *$A \equiv B$ if and only if $A \equiv_n^\alpha B$ for every $n$, if $\alpha$ is finite.*

(ii) *$A \equiv_n^\alpha B$ implies "$A \models \Phi \Leftrightarrow B \models \Phi$ for every sentence $\Phi$ containing at most $n$ quantifiers (The language is of type $\tau_\alpha$)."*

LEMMA 2. *If $\alpha$ is finite, then there are only finite number of equivalence classes of $\equiv_n^\alpha$ for each $n$.*

LEMMA 3. *If $A_i \equiv_n^\alpha B_i$ for every element $i$ of an ordered set $I$, then $\sum_{i \in I} A_i \equiv_n^\alpha \sum_{i \in I} B_i$.*

COROLLARY 3.1. *If $A_i \equiv B_i$ for every element $i$ of an ordered set $I$, then $\sum_{i \in I} A_i \equiv \sum_{i \in I} B_i$.*

LEMMA 4. (i) *$A \prec B$ implies $A+C \prec B+C$.*

(ii) *If a substructure $A$ of $B$ satisfies that for any $a \in |A|$ there exists $a' > a$ in $|A|$ such that $(-\infty, a')_A \prec (-\infty, a')_B$ and $[a', \infty)_A \equiv [a', \infty)_B$, then $A \prec B$.*

PROOF. (i) Let $a_1, \cdots, a_m \in |A|$ and $c_1, \cdots, c_n \in |C|$. Then $(A, \{a_1, \cdots, a_m\}) \equiv (B, \{a_1, \cdots, a_m\})$ and $(C, \{c_1, \cdots, c_n\}) \equiv (C, \{c_1, \cdots, c_n\})$. So, by Corollary 3.1, $(A+C, \{a_1, \cdots, c_n\}) \equiv (B+C, \{a_1, \cdots, c_n\})$.

(ii) Similar to (i).

LEMMA 5. *If $A \equiv C$, $B \equiv D$ and $A \prec A+B$, then $C \prec C+D$.*

PROOF. From the assumption and Corollary 3.1, $(A, |A|)+(B, \emptyset) \equiv (C, |C|) +(D, \emptyset)$. That is $(A+B, |A|) \equiv (C+D, |C|)$. Hence $A \prec A+B$ implies $C \prec C+D$.

COROLLARY 5.1. *$A \prec A+B$ and $A \prec A+C$ implies $A+B \prec A+B+C$.*

PROOF. $A \equiv A+B$, $C \equiv C$ and Lemma 5.

LEMMA 6. *If $A \prec B_i$ for every element $i$ of an ordered set $I$, then $A+\sum_{i \in J} B_i \prec A+\sum_{i \in I} B_i$ for any subset $J$ of $I$.*

PROOF. If $I$ is a finite set, it can be obtained by iterative uses of Lemma 4 and Corollary 5.1. Let $I$ be infinite. Then, by the above, $A+\sum_{i \in K} B_i \prec A+\sum_{i \in J} B_i$ for any $J$, $K$ such that $J$ is finite and $K \subseteqq J \subseteqq I$. The desired conclusion can be obtained from this, using, for example, the following fact:

If $K$ is a non-empty family of systems such that any two systems in $K$ have a common arithmetical extension which is also in $K$, then the union of $K$ is a common arithmetical extension of all members of $K$ (Theorem 1.9 of Tarski and Vaught [8]).

LEMMA 7. *$B \prec A$ and $C = \bigcup_{b \in |B|} (-\infty, b]_A$ implies $C \prec A$.*

PROOF. It suffices to show it when the similarity type $\tau_\alpha$ of $A$, $B$, $C$ is finite. We may further assume $|A|$ is countable. (Otherwise, consider a countable elementary substructure of $(A, |B|, |C|)$.) Let $\langle b_i \rangle_{i \in \omega}$ be an unbounded ascending sequence in $B$. Then, for any $c \in |C|$, there exists $b_i$ such that $b_i > c$. By Corollary 3.1 and the fact $B \prec A$, $[b_i, \infty)_A \equiv [b_i, \infty)_B = \sum_{j \geqq i} [b_j, b_{j+1})_B \equiv \sum_{j \geqq i} [b_j, b_{j+1})_A = [b_i, \infty)_C$. Clearly $(-\infty, b_i)_A = (-\infty, b_i)_C$. So, by Lemma 4 (ii), $C \prec A$.

## § 2. Theorems.

THEOREM 1. *If $\mu$ is an ordinal whose cofinality is larger than $\omega$ and $\alpha$ is countable, then for any subsets $U_\xi$ $(\xi < \alpha)$ of $\mu$, there exist an ordinal $\nu > \mu$ and subsets $V_\xi$ $(\xi < \alpha)$ of $\nu$ such that $\langle \mu, <, U_\xi \rangle_{\xi < \alpha} \prec \langle \nu, <, V_\xi \rangle_{\xi < \alpha}$.*

PROOF. Let $B$ be a countable elementary substructure of $\langle \mu, <, U_\xi \rangle_{\xi < \alpha}$. Put $C = \bigcup_{b \in |B|} (-\infty, b]_A$. Then $C \prec \langle \mu, <, U_\xi \rangle_{\xi < \alpha}$ by Lemma 7 and $|C|$ is a proper initial segment of $\mu$ since $\omega < cf(\mu)$. Define the structure $D$ by $C + D = \langle \mu, <, U_\xi \rangle_{\xi < \alpha}$. Then, by Corollary 5.1, $\langle \mu, <, U_\xi \rangle_{\xi < \alpha} = C + D \prec C + D + D$. Put $\langle \nu, <, V_\xi \rangle_{\xi < \alpha} \cong C + D + D$.

COROLLARY 1.a. *If $\alpha$ is countable and $\lambda$ is an ordinal such that $\omega < cf(\lambda)$, then the following two conditions on an ordinal $\mu$ are equivalent:*

(i) *For any subsets $U_\xi$ $(\xi < \alpha)$ of $\lambda$, there exist subsets $V_\xi$ $(\xi < \alpha)$ of $\mu$ such that $\langle \lambda, <, U_\xi \rangle_{\xi < \alpha} \prec \langle \lambda, <, U_\xi \rangle_{\xi < \alpha} + \langle \mu, <, V_\xi \rangle_{\xi < \alpha}$.*

(ii) *$\mu = \omega_1 \cdot \nu$ for some ordinal $\nu$.*

PROOF. (ii) $\Rightarrow$ (i): By Theorem 1 and Lemma 5, there is a countable structure $\langle \mu', <, V'_\xi \rangle_{\xi < \alpha}$ such that $\langle \lambda, <, U_\xi \rangle_{\xi < \alpha} \prec \langle \lambda, <, U_\xi \rangle_{\xi < \alpha} + \langle \mu', <, V'_\xi \rangle_{\xi < \alpha}$. Since $\mu'$ is countable, $\sum_{\zeta < \omega_1 \cdot \nu} \langle \mu', <, V'_\xi \rangle_{\xi < \alpha}$ (the concatenation of $\omega_1 \cdot \nu$ copies of $\langle \mu', <, V'_\xi \rangle_{\xi < \alpha}$) has the form of $\langle \omega_1 \cdot \nu, <, V_\xi \rangle_{\xi < \alpha}$. These $V_\xi (\xi < \alpha)$ satisfy the desired condition because of Lemmas 5 and 6.

(i) $\Rightarrow$ (ii): Let $\eta$ be an arbitrary countable ordinal and $f$ be an injective map from $\eta$ into $\omega$. Define $U \subseteqq \lambda$ by $\langle \lambda, <, U \rangle = \sum_{\zeta < \lambda} (\sum_{\xi < \eta} \langle \omega, <, f(\xi) \rangle)$, the concatenation of $\lambda$ copies of $\sum_{\xi < \eta} \langle \omega, <, f(\xi) \rangle$. By the assumption, there is $V \subseteqq \mu$ such that $\langle \lambda, <, U \rangle \prec \langle \lambda, <, U \rangle + \langle \mu, <, V \rangle$. Now, the set $W = \{\omega \cdot \eta \cdot \zeta : \zeta < \lambda\}$ is definable in $\langle \lambda, <, U \rangle$. If $\gamma \in W$, then every $\xi$ such that $\gamma \leqq \xi < \gamma + \omega \cdot \eta$ is definable from $\gamma$ in $\langle\lambda, <, U\rangle$. $W$ is unbounded in $\lambda$. So, for every $\delta<\mu$ there exists $\gamma>\delta$ such that $[\gamma, \gamma+\omega\cdot\eta)\subseteqq\mu$. This implies (ii).

THEOREM 2. *Let* $A=\langle|A|, <, U_\xi\rangle_{\xi<\alpha}$ *where* $U_\xi\subseteqq|A|$ *for each* $\xi<\alpha$. *If a function* $f$ *from* $|A|^2$ *into* $|A|$ *is definable in* $A$, *then for any cardinal* $\kappa<\overline{\overline{|A|}}$, *there exists* $a\in|A|$ *such that* $\overline{\overline{f^{-1}(a)}}\geqq\kappa$.

PROOF. It suffices to prove this assuming that $\alpha$ is finite. Let $\Phi(v_0, v_1, v_2)$ be a formula such that for any $a, b, c\in|A|$, $A\models\Phi[a, b, c]\Leftrightarrow f(a, b)=c$. Let $j$ be a number of quantifiers contained in $\Phi$. By Lemma 2, the number of equivalence classes of $\equiv_{j+3}^{\alpha+2}$ is finite. Let it be $n$.

CASE 1. $\overline{\overline{|A|}}<\aleph_0$: obvious.

CASE 2. $\overline{\overline{|A|}}\geqq\aleph_0$: If $\kappa<\overline{\overline{|A|}}$, there exist substructures $B$ and $C$ of $A$ such that $A=B+C$ and $\overline{\overline{n\cdot\kappa}}\leqq\overline{\overline{|B|}}, \overline{\overline{|C|}}$. Then there exists a subset $X$ of $|B|$ such that $\overline{\overline{X}}=\kappa$ and for any $b, b'$ in $X$, $\langle B, \{b\}, \emptyset\rangle\equiv_{j+3}^{\alpha+2}\langle B, \{b'\}, \emptyset\rangle$. Similarly, there exists a subset $Y$ of $|C|$ such that $\overline{\overline{Y}}=\kappa$ and for any $c, c'$ in $Y$, $\langle C, \{c\}, \emptyset\rangle\equiv_{j+3}^{\alpha+2}\langle C, \{c'\}, \emptyset\rangle$. Fix $b_0\in X$ and $c_0\in Y$ arbitrarily and put $d_0=f(b_0, c_0)$. Without loss of generality, assume $d_0\in|C|$. Then, by Lemma 3, for any $b\in X$, $\langle A, \{b, c_0\}, \{d_0\}\rangle=\langle B, \{b\}, \emptyset\rangle+\langle C, \{c_0\}, \{d_0\}\rangle\equiv_{j+3}^{\alpha+2}\langle B, \{b_0\}, \emptyset\rangle+\langle C, \{c_0\}, \{d_0\}\rangle=\langle A, \{b_0, c_0\}, \{d_0\}\rangle$. So, by Lemma 1 (ii), for every $b\in X$, $\langle A, \{b, c_0\}, \{d_0\}\rangle\models\exists x\exists y\exists z[x<y\wedge P_\alpha(x)\wedge P_\alpha(y)\wedge P_{\alpha+1}(z)\wedge\Phi(x, y, z)]$. Hence $X\times\{c_0\}\subseteqq f^{-1}(d_0)$.

**Appendix.**

In Chapter 1 of Silver [**7**], he states, "Whenever $\langle\kappa, <, S\rangle_{S\subseteqq\kappa}$ has an elementary extension of the form $\langle\alpha, <, S'\rangle_{S\subseteqq\kappa}$ where $\kappa<\alpha$, $\kappa$ is measurable." But it would be a misquotation of Keisler [**5**]. As an example of refutations of the statement, our results give the following:

(*) If $\kappa$ is measurable, then even $\langle\kappa+\kappa, <, U\rangle_{U\subseteqq\kappa+\kappa}$ has a well-ordered elementary end-extension.

PROOF. Let $F$ be a $\kappa$-complete non-principal ultrafilter over $\kappa$ and $\mathfrak{A}$ be the structure $\langle\kappa+\kappa, <, U\rangle_{U\subseteqq\kappa+\kappa}$. Put $\mathfrak{B}=\mathfrak{A}^\kappa/F$, the ultrapower of $\mathfrak{A}$ modulo $F$. Let $\mathfrak{C}$ be the initial segment of $\mathfrak{B}$ which is cofinal with $\mathfrak{A}$ in $\mathfrak{B}$. Then $\mathfrak{C}$ is a proper initial segment of $\mathfrak{B}$, and by Lemma 7, $\mathfrak{A}\prec\mathfrak{B}$ implies $\mathfrak{C}\prec\mathfrak{B}$. Define $\mathfrak{D}$ by $\mathfrak{C}+\mathfrak{D}=\mathfrak{B}$. Then by Lemma 5 $\mathfrak{A}\prec\mathfrak{A}+\mathfrak{D}$ since $\mathfrak{C}\prec\mathfrak{C}+\mathfrak{D}$ and $\mathfrak{A}\equiv\mathfrak{C}$.

OPEN PROBLEM. For arbitrarily given subsets $U_\xi(\xi<\omega_1)$ of $\omega_1$, does the structure $\langle\omega_1, <, U_\xi\rangle_{\xi<\omega_1}$ have a well-ordered elementary end-extension?

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