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On vanishing of cohomology attached to certain many valued meromorphic functions

By Kazuhiko AOMOTO

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1. Let V be a compact Kähler manifold of the complex dimension n. Let $D=\sum\lambda_k\Gamma_k$ ($\lambda_k\in\boldsymbol{C}$) be a bounding $(2n-2)$-cycle on V consisting of a finite number of irreducible closed analytic divisors Γ_k. Let $F_D(z)=\exp\Phi_D(z)$ be a multiplicative meromorphic function having D as its divisor. It is known that this function is determined uniquely up to a multiplicative constant (see [**7**]). Let ρ_λ be the scalar representation of the fundamental group G of $V-|D|$ into $\boldsymbol{C}^*$ canonically induced by the function $F_D(z)$, where $|D|$ denotes the polyhedron representing D. We denote by $\mathcal{S}_\lambda$ the local system over $\boldsymbol{C}$ defined by ρ_λ. Let $T(|D|)$ be a small tubular neighbourhood of $|D|$ in V associated with Whitney stratification of $|D|$ constructed by R. Thom (see [**13**] Théorème 1.D.1, page 248). We make the three following assumptions:

I. *The cohomology with local coefficients* $\mathcal{S}_\lambda$, $H^p(T(|D|)-|D|,\mathcal{S}_\lambda)$ *on* $T(|D|)-|D|$ *vanish for all* $p\geqq 0$.

II. *The critical points of* $\mathrm{Re}\,\Phi_D(z)$ *are all isolated and non-degenerate on* $V-|D|$.

III. *There exists a complete Kähler metric* $(ds)^2$ *on* $V-|D|$ *so that* $V-|D|$ *becomes a symplectic manifold.*

Then we have the following theorem:

THEOREM 1. $H^p(V-|D|,\mathcal{S}_\lambda)=(0)$ *for* $p\neq n$.

This theorem was stated in [**1**] for special cases. See also [**6**].

2. For the proof of the preceding theorem we make use of Morse theory. Let $\sum g_{i\bar{j}}dz^i\cdot d\bar{z}^j$ be a local expression of the Kähler metric $(ds)^2$ of $V-|D|$ with respect to local coordinates $(z^1, z^2, \cdots, z^n)$ of V. We consider a vector field on $V-|D|$ of the following form:

$$dz^i/dt=\sum_j(\mathrm{Re}\,\Phi_D/\partial\bar{z}^j)\cdot g^{i\bar{j}}. \tag{1}$$

It can be verified that this expression is independent of the choice of local coordinates and so defines a well defined vector field X on $V-|D|$. This vector field is closely related to the symplectic structure on $V-|D|$. Consider the following 2-form ω on $(V-|D|)\times\boldsymbol{R}$:

(2) $$\omega=\sqrt{-1}\sum g_{i\bar{j}}dz^{i}\wedge d\bar{z}^{j}-d(\mathrm{Im}\,\Phi_{D})\wedge dt$$

for $(z, t)\in(V-|D|)\times\boldsymbol{R}$. Then the characteristic Hamiltonian system of ω in E. Cartan's sense is written as follows:

(3) $$\partial(\omega)/\partial(d\bar{z}^{i})=0\,,\qquad \partial(\omega)/\partial(dz^{i})=0\,,\qquad \partial(\omega)/\partial(dt)=0\,,$$

namely,

(4) $$\sqrt{-1}\,g_{i\bar{j}}d\bar{z}^{j}-(\mathrm{Im}\,\Phi_{D}/\partial z^{i})dt=0\,,$$

which defines the vector field X on $V-|D|$. As a result $\mathrm{Im}\,\Phi_{D}(z)$ turns out to be an invariant of X.

Let $\mathcal{S}_{-\lambda}$ be the dual local system of $\mathcal{S}_{\lambda}$. Then the Eilenberg-Maclane homology $H_{p}(V-|D|, \mathcal{S}_{-\lambda})$ can be regarded as dual of $H^{p}(V-|D|, \mathcal{S}_{\lambda})$. We call "twisted chain" or "twisted cycle" a chain or a cycle with coefficients $\mathcal{S}_{-\lambda}$. We denote by V^{c} the subspace of $V-|D|$ defined by the inequality $\mathrm{Re}\,\Phi_{D}<c$. Let $\alpha_{1}, \alpha_{2}, \cdots, \alpha_{\mu}$ be all the critical points of $\mathrm{Re}\,\Phi_{D}$ on $V-|D|$ and $a_{1}>a_{2}>\cdots>a_{\mu}$ be their respective values. Then we have the following:

LEMMA 1. *Let $\mathfrak{z}$ be a twisted cycle in $V-|D|$ of dimension p different from n. Then there exists a suitable twisted cycle $\mathfrak{z}^{*}$ of the same dimension which is homologous to $\mathfrak{z}$ in $V-|D|$ and lies in $T(|D|)-|D|$.*

PROOF. By Assumption II and a property of a holomorphic function the index of $\mathrm{Re}\,\Phi_{D}$ is always n at each critical point α_{j}. Now we want to show by induction with respect to i that $\mathfrak{z}$ is homologous to a twisted cycle $\mathfrak{z}_{i}$ lying in $V^{a_{i}-\varepsilon}$ for a small positive ε. In fact we may assume that $\mathfrak{z}$ lies in V^{c} for some $c>a_{1}$. Along any trajectory of (1) we have

(5) $$\begin{aligned}(ds/dt)^{2}&=\sum g_{i\bar{j}}dz^{i}/dt\cdot d\bar{z}^{j}/dt\\&=\sum \mathrm{Re}\,\Phi_{D}/\partial\bar{z}^{i}\cdot\mathrm{Re}\,\Phi_{D}/\partial z^{j}\cdot g^{\bar{i}j}\\&=\frac{1}{2}d\,\mathrm{Re}\,\Phi_{D}/dt\,.\end{aligned}$$

In particular if we take as time parametre t the function $2\,\mathrm{Re}\,\Phi_{D}$ then

(6) $$ds/d(\mathrm{Re}\,\Phi_{D})=1\,.$$

Therefore the distance $s(P, Q)$ between two any points P and Q on the same trajectory becomes infinite if and only if $\mathrm{Re}\,\Phi_{D}$ becomes $\pm\infty$. Now suppose that $\mathfrak{z}$ is homologous to $\mathfrak{z}_{i-1}$. We want to show $\mathfrak{z}$ is homologous to a twisted cycle $\mathfrak{z}_{i}$. $\mathfrak{z}_{i-1}$ being compact we can retract $\mathfrak{z}_{i-1}$ to a twisted cycle $\tilde{\mathfrak{z}}_{i-1}$ lying in $V^{a_{i}+\varepsilon}$ by the retraction defined by the one parametre diffeomorphism group corresponding to X. This is possible because of (6) and the invariance of $\mathrm{Im}\,\Phi_{D}$ along any trajectory of X. On the other hand Morse theory (see J. Milnor [**9**] Theorem 3.2, page 14) shows that there exists a twisted cycle $\mathfrak{z}_{i}$

lying in $V^{a_i-\varepsilon}$ and which is near and homologous to $\tilde{z}_{i-1}$. In consequence there exists a twisted cycle z_i lying in $V^{a_i-\varepsilon}$ homologous to z in $V-|D|$. For a large L, V^{-L} is obviously contained in $T(|D|)$. So we can retract the cycle z_μ to a twisted cycle z^* lying in $T(|D|)-|D|$ by using the vector field X in view of (6). The lemma is proved.

PROOF OF THEOREM 1. Le z be a twisted cycle in $V-|D|$ of dimension $p\neq n$. Then by the above lemma we can find a twisted cycle z^* in $T(|D|)-|D|$ homologous to z in $V-|D|$. By Assumption I there exists a twisted chain c in $T(|D|)-|D|$ such that $z^*=\partial c$. This implies z is homologous to zero. Q.E.D.

Suppose now that *Γ_k are all normally crossing with each other*. Let $T(S)$ be a tube of a stratum S of $|D|$ in $V-|D|$ which is a fibre bundle on S. Each fibre F is differentiably isomorphic to $\boldsymbol{C}^m-(z_1=0)\cup\cdots\cup(z_m=0)$ where $(z_1,\cdots,z_m)$ denote affine coordinates of $\boldsymbol{C}^m$. Let ι denote the inclusion of F into $V-|D|$. Suppose

I′. $\qquad H^p(F,\iota^*\mathcal{S}_\lambda)=(0)\quad$ *for all* $p\geqq 0$.

Then using Mayer-Vietoris sequence and spectral sequence argument we can prove by induction with respect to dimensions of strata of $|D|$ that Assumption I holds. In this way we have

THEOREM 2. *Under I′, II and III* $H^p(V-|D|,\mathcal{S}_\lambda)=(0)$ *for* $p\neq n$.

3. We assume further the following:

IV. $\operatorname{Arg} F(\alpha_j)$ *are different from each other for* $1\leqq j\leqq\mu$. We want to construct μ twisted cycles corresponding to each critical point α_j which form a basis of $H_p(V-|D|,\mathcal{S}_{-\lambda})$.

LEMMA 2. *Let f be holomorphic and* $\operatorname{Re} f$ *have a non-degenerate critical point at the origin in* $\boldsymbol{C}^n$. *Consider the vector field X as follows:*

$$\frac{dz^i}{dt}=\sum_1^n\frac{\partial(\operatorname{Re} f)}{\partial\bar{z}^j}g^{i\bar{j}} \tag{7}$$

where $(g_{i\bar{j}})$ denotes a positive definite Kähler metric near the origin. Then the union of all trajectories having the origin as their $(\pm\infty)$-limiting points are smooth manifolds $\mathfrak{M}^\pm$ of dimension n respectively. These are transversal to each other. They become Lagrangean manifolds with respect to the symplectic structure $\sqrt{-1}\sum g_{i\bar{j}}dz^i\wedge d\bar{z}^j$.

PROOF. The first part is well known (see [12] page 113). The second part follows from Lemmas 3 and 4.

LEMMA 3. *There exists real local coordinates $(\xi^1,\cdots,\xi^n,\eta^1,\cdots,\eta^n)$ at the origin such that the followings hold:*

$$\sqrt{-1}\sum g_{i\bar{j}}dz^i\wedge d\bar{z}^j=\sum_i d\xi^i\wedge d\eta^i\,, \tag{8}$$

(9) $$\operatorname{Im} f = \sum \lambda_j \xi^j \eta^j + (\textit{higher degree terms})$$

where $\lambda_j > 0$ *for all* j.

PROOF. After a suitable linear coordinate transformation

(10) $$z^i = \sum_1^n a_j^i w^j$$

we may assume the matrix $(g_{i\bar{j}}(0))$ equal to the identity. The $2n$ by $2n$ matrix

(11) $$\begin{pmatrix} \dfrac{\partial^2 \operatorname{Im} f(0)}{\partial x^i \partial x^j} & \dfrac{\partial^2 \operatorname{Im} f(0)}{\partial x^i \partial y^j} \\ \dfrac{\partial^2 \operatorname{Im} f(0)}{\partial y^i \partial x^j} & \dfrac{\partial^2 \operatorname{Im} f(0)}{\partial y^i \partial y^j} \end{pmatrix}$$

turns out to be symmetric and symplectic where $z^i = x^i + \sqrt{-1}\, y^i$. By well known argument on Lie algebra theory we can find a linear unitary transformation of the form (10) such that the function $\operatorname{Im} f$ may be expressed in the form (9) with respect to the variables $w^i = \xi^i + \sqrt{-1}\, \eta^i$ because the origin is a non-degenerate critical point of $\operatorname{Re} f$.

LEMMA 4. *Let* $H(x, y) = \sum \lambda_j x^j y^j + (\textit{higher degree terms})$ *be real analytique at the origin on* $\boldsymbol{R}^{2n}$. *We assume* λ_j *all positive. Then there exists a real analytique function* $\phi(x)$ *at the origin such that*

(12) $$H(x, \operatorname{grad} \phi(x)) = 0\,.$$

PROOF. This can be proved by the majorant method. Actually this is proved in more general situation in [**11**] (see page 302).

PROOF OF LEMMA 2. By the canonical transformation using the above ϕ:

(13) $$\begin{aligned} \eta^i &= -\partial\phi/\partial x^i + y^i \\ \xi^i &= x^i \end{aligned}$$

the Hamiltonian H is transformed as follows:

(14) $$H(x, y) = \sum_1^n \lambda_j \xi^j \eta^j + H'$$

where H' vanishes provided $\eta^1 = \eta^2 = \cdots = \eta^n = 0$. Therefore we have

(15) $$\Big|\sum_1^n \eta^j\, \partial H'/\partial \xi^j\Big| < \varepsilon \sum_1^n (\eta^j)^2$$

for a small $\varepsilon > 0$ satisfying $2\varepsilon < \min \lambda_j$ near the origin. Now we have

(16) $$\begin{aligned} \frac{d}{dt}\Big(\sum_1^n (\eta^j)^2\Big) &= 2\sum_1^n \dot{\eta}^j \eta^j \\ &= -2\Big\{\sum_1^n \lambda_j (\eta^j)^2 + \sum_1^n \eta^j \frac{\partial H'}{\partial \xi^j}\Big\} < -2\varepsilon \sum_1^n (\eta^j)^2\,. \end{aligned}$$

This implies

(17) $$\sum_{i=1}^{n}(\eta^i)^2 \leqq \text{Cte.}\exp(-2\varepsilon t)\,.$$

Let Λ be a trajectory having 0 as its $(-\infty)$-limiting point and through a point $(\xi^1(t_0), \cdots, \eta^n(t_0))$ at a time t_0. Suppose $\eta(t_0)\neq 0$. Then there exists a time t_1 $(t_1 < t_0)$ such that

(18) $$\sum(\eta^i(t_1))^2 < \sum(\eta^i(t_0))^2\,.$$

This is a contradiction against the mean value theorem of real differentiable functions. So Λ is contained in the subspace $\widetilde{\mathfrak{M}}^-$ defined by $\eta^1=\eta^2=\cdots=\eta^n=0$. $\widetilde{\mathfrak{M}}^-$ is clearly a Lagrangean manifold with respect to the symplectic structure $\sum d\xi^i\wedge d\eta^i$. The transformed manifold $\mathfrak{M}^-$ of $\widetilde{\mathfrak{M}}^-$ by (13) is also Lagrangean.

Q. E. D.

One may ask the following interesting question:

PROBLEM 1. What can be said about Lagrangean manifolds in the case of higher degeneracy of the function f? Can we construct real n dimensional cell structure on the hypersurface $f=c$ as intersection of Lagrangean manifolds in the case of isolated singularities?

By Lemma 2 we can construct a Lagrangean manifold $\mathfrak{M}_j^-$ through α_j generated by the vector field X. We can prolong it by X to a twisted chain Δ_j bounded by the boundary $\partial T(|D|)$ of $T(|D|)$ in view of the invariance of $\operatorname{Im}\Phi_D(z)$ and (6). We denote it by $\partial\Delta_j$. $\partial\Delta_j$ lies in $\partial T(|D|)$. By Assumption I we can find a twisted chain E_j in $T(|D|)-|D|$ such that $\partial\Delta_j=\partial E_j$, so that $\mathfrak{z}_j=(\Delta_j-E_j)$ defines a twisted n-cycle in $V-|D|$. $|\mathfrak{z}_j|\cap(V-T(|D|))$ is Lagrangean and contained in the real hypersurface $\operatorname{Im}\Phi_D(z)=\operatorname{Im}\Phi_D(\alpha_j)$. These μ twisted cycles define classes of $H_n(V-|D|, \mathcal{S}_{-\lambda})$. On the other hand Euler number of $\mathcal{S}_\lambda$, $\sum_\nu(-1)^\nu \operatorname{rank} H^\nu(V-|D|, \mathcal{S}_\lambda)$ is independent of λ and equal to that of $V-|D|$, $\chi(V-|D|)$ which is equal to μ according to Morse theory in view of the following lemma (see [4] Hypotheses I and II, page 26-27).

LEMMA 5. *There exists a neighbourhood $U_{\delta_1\delta_2}(|D|)$ of $|D|$ having the following property: For any $\delta\in\boldsymbol{R}$ the set $U_{\delta_1\delta_2}(|D|)=\{z\in V-|D|,\ \delta_1<|\operatorname{Re}\Phi_D|<\delta_2\}$ is homeomorphic to the product $(\delta_1,\delta_2)\times(\{\operatorname{Re}\Phi_D=\delta\}\cap U_{\delta_1\delta_2}(|D|))$ where δ_1 and δ_2 denote two real numbers such that $\delta_1<\delta<\delta_2$.*

PROOF. By resolution of singularities due to Hironaka we may assume the irreducible components are all non-singular and normally crossing with each other. We have only to prove the lemma on each stratum of $|D|$, because then we can construct a neighbourhood $U_{\delta_1\delta_2}(|D|)$ satisfying the above property by patching the neighbourhoods of strata. Let 0 be a point of a stratum S of $|D|$. There exist local coordinates $(z_1, z_2, \cdots, z_n)$ at 0 such that D and $F_D(z)$ are locally defined respectively as follows: $D=\{z_1=0\}\cup\{z_2=0\}\cup\cdots$

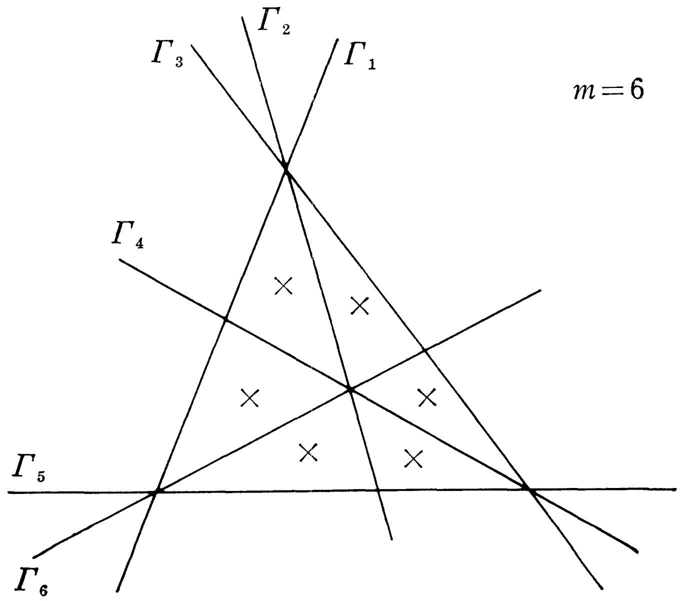
$\cup\{z_r=0\}$ $(r\leqq n)$ and $F_D(z)=z_1^{\lambda_1}\cdots z_s^{\lambda_s}\cdot z_{s+1}^{\lambda_{s+1}}\cdots z_r^{\lambda_r}$ where $\lambda_1>0,\cdots,\lambda_s>0,\lambda_{s+1}<0,\cdots,\lambda_r<0$. Then we easily see that $\left\{\frac{1}{2}<\frac{|z_1z_2\cdots z_s|}{|z_{s+1}\cdots z_n|}<1\right\}\cap\{|z_1|<1,\cdots,|z_{r+1}|<1,\cdots,|z_n|<1\}$ is homeomorphic to $\{\delta_1<\mathrm{Re}\,\Phi_D<\delta_2\}\cap\{|z_2|<1,\cdots,|z_r|<1,|z_{r+1}|<1,\cdots,|z_n|<1\}$. This is obviously homeomorphic to the product $(\delta_1,\delta_2)\times(\{\mathrm{Re}\,\Phi_D=\delta\}\cap\{|z_1|<1,\cdots,|z_n|<1\})$. The lemma is proved. As a result we have

THEOREM 3. *Under Assumptions I, II and III* rank $H_n(V-|D|,\mathcal{S}_{-\lambda})$ *is equal to* μ.

Like Problem 1 one may ask the following question:

PROBLEM 2. Can we realize cycles of a basis of $H_n(V-|D|,\mathcal{S}_{-\lambda})$ as Lagrangean manifolds bounded by $|D|$? One will see it is possible in the following first example.

4. EXAMPLE 1. Let V be a projective space of the complex dimension n and $\Gamma_0,\Gamma_1,\cdots,\Gamma_m$ be hyperplanes. We assume Γ_0 the hyperplane at infinity. Suppose $\Gamma_1,\cdots,\Gamma_m$ all real. Then $F_D(z)$ is equal to $\prod_1^m F_j^{\lambda_j}$, where F_j denotes a linear function on $\boldsymbol{C}^n$ defining Γ_j. Suppose that *any line L intersects with* $|D|$ *in at least two points.* Then it can be verified there exists a λ satisfying Assumptions I and II (see [1] and [6]). On the other hand $V-|D|$ being affine and invariant with respect to the complex conjugation there exists a real complete Kähler metric on $V-|D|$. Suppose λ_j all positive. Then the critical points $\alpha_1,\alpha_2,\cdots,\alpha_\mu$ are all real and lie one by one in each compact chamber C_j divided by real hyperplanes $\mathrm{Re}\,\Gamma_k$ in $\boldsymbol{R}^n$. These μ chambers form just a basis of $H_n(V-|D|,\mathcal{S}_{-\lambda})$ (see the following figure).



The pairing $H_n(V-|D|, \mathcal{S}_{-\lambda})\times H^n(V-|D|, \mathcal{S}_\lambda)\to \boldsymbol{C}$ corresponds to an integral of the following type:

$$\int \prod_1^m F_j(x)^{\lambda_j}\varphi(x)dx_1\wedge\cdots\wedge dx_n \tag{19}$$

where $\varphi(x)$ denotes a rational function with poles on $|D|$. This can be regarded as a generalization of classical hypergeometric functions.

EXAMPLE 2. Let $G=ANK$ be an Iwasawa decomposition of a real semi-simple Lie group with finite centre G, where A, denotes a maximal split torus, N a maximal unipotent subgroup and K a maximal compact subgroup of G. We denote by $\bar{N}$ a hermitien conjugate of N. Let χ_λ be a character of A and $\rho(a)$ $(a\in A)$ denote the jacobian of Ad a on N, then it is known by Harish-Chandra a zonal spherical function $\varphi_\lambda(a)$ can be expressed as follows:

$$\varphi_\lambda(a)=\beta_\lambda(a)\cdot\int_{\bar{N}}\frac{\beta_\lambda(a(a^{-1}\bar{n}a))}{\beta_\lambda(a(\bar{n}))}\rho(a(\bar{n}))^{\frac{1}{2}}d\bar{n} \tag{20}$$

where $\beta_\lambda(a)=\rho(a)^{\frac{1}{2}}$, $\chi_\lambda(a)$ and $\bar{n}=a(\bar{n})\cdot n(\bar{n})\cdot u(\bar{n})\in\bar{N}$ with $a(\bar{n})\in A$, $n(\bar{n})\in N$ and $u(\bar{n})\in K$. Let $V=M^c\cdot A^c\cdot N^c\backslash G^c$ be a generalized flag manifold and we take $F_D(\bar{n})=\beta_\lambda(a(a^{-1}\bar{n}a))/\beta_\lambda(a(\bar{n}))$ where G^c, A^c, N^c and M^c denote the complexification of G, A, N and the centralizer M of A in K respectively. Then *$H^p(V-|D|, \mathcal{S}_\lambda)$ vanish for $p\neq n$ if λ is a general character of A*, and rank *$H^n(V-|D|, \mathcal{S}_\lambda)$ seems to be equal to the order of Weyl group W with respect to A*. The pairing of $H_n(V-|D|, \mathcal{S}_{-\lambda})\times H^n(V-|D|, \mathcal{S}_\lambda)\to\boldsymbol{C}$ is nothing but the above integral.

The further structure of the above integrals will be studied elsewhere.

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Kazuhiko AOMOTO
Department of Mathematics
College of General Education
University of Tokyo
Komaba, Meguro-ku
Tokyo, Japan