J. Math. Soc. Japan
Vol. 27, No. 2, 1975

On mean ergodic theorems for positive operators in Lebesgue space

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(Received March 5, 1973)
(Revised Oct. 22, 1974)

§ 1. Introduction.

Let $(X, \mathcal{M}, m)$ be a σ-finite measure space and $L_p(X)=L_p(X, \mathcal{M}, m)$, $1 \leqq p \leqq \infty$, the usual (complex) Banach spaces. Let T be a bounded linear operator on $L_1(X)$ and τ its linear modulus [2]. In [9] (see also Akcoglu and Sucheston [1]) the author proved that if the adjoint of τ has a strictly positive sub-invariant function in $L_\infty(X)$ then the following two conditions are quivalent: (i) T^n converges weakly; (ii) $\frac{1}{n}\sum_{i=1}^{n} T^{k_i}$ converges strongly for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers. In the present paper we shall prove that if T is positive and satisfies $Tf=f$ whenever $0 \leqq f \in L_1(X)$ and $Tf \geqq f$, then the equivalence of (i) and (ii) still holds. Applying this result, we obtain that if, in addition, $\sup_n \|T^n\|_1 < \infty$ and if $T^n f$ converges weakly for any $f \in L_1(X)$ with $\int f\,dm=0$, then $\frac{1}{n}\sum_{i=1}^{n} T^{k_i} f$ converges strongly for any $f \in L_1(X)$ with $\int f\,dm=0$ and for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers.

§ 2. Mean ergodic theorems.

In this section we shall assume that T is a *positive* linear operator on $L_1(X)$. T^* denotes the adjoint of T. Thus T^* acts on $L_\infty(X)$, and $\int (Tf)u\,dm = \int f(T^*u)\,dm$ for all $f \in L_1(X)$ and all $u \in L_\infty(X)$. If $A \in \mathcal{M}$ then 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a. e. on $X-A$. A set $A \in \mathcal{M}$ is called *closed* under T if $f \in L_1(A)$ implies $Tf \in L_1(A)$.

The following proposition is stated with more generality than what is needed for applications in this paper. In particular, it extends a result of Lin [7, Theorem 1.1] (see also Krengel and Sucheston [5] and Lin [6]).

PROPOSITION. *Let T be a positive linear operator on $L_1(X)$. Assume that $\sup_n \|T^n\|_1 < \infty$ and that T has no nonzero nonnegative invariant function in $L_1(X)$. Let $f \in L_1(X)$ and suppose that there exists a subset J of the nonnegative integers such that* weak-$\lim_{n \notin J} T^n f$ *exists and* $\liminf_n \frac{1}{n} |\{j \in J : j < n\}| = 0$, *where* $|\{j \in J : j < n\}|$ *denotes the cardinality of the set* $\{j \in J : j < n\}$. *Then we have* $\lim_n \|T^n f\|_1 = 0$.

PROOF. Since the L_1 of a σ-finite measure space is isometric to the L_1 of a finite measure space, we may and will assume without loss of generality that $(X, \mathcal{M}, m)$ is a finite measure space. The Vitali-Hahn-Saks theorem (cf. [3, Theorem III.7.2]) implies that given an $\varepsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{M}$ and $m(A) < \delta$ then $\int_A |T^n f| dm < \varepsilon$ for all $n \notin J$. Let $k_1, k_2, \cdots$ be a strictly increasing sequence of positive integers such that

$$\lim_n \frac{1}{k_n} |\{j \in J : j < k_n\}| = 0,$$

and let L be any Banach limit (cf. [11]). Define, for $A \in \mathcal{M}$,

$$\mu(A) = L\Big(\frac{1}{k_n} \sum_{i=0}^{k_n - 1} \int_A |T^i f| dm\Big).$$

It is easily checked that μ is a finite measure on $(X, \mathcal{M})$ and absolutely continuous with respect to m (cf. [1, p. 239]). Let $g = d\mu/dm$. Then $0 \leqq g \in L_1(X)$ and, for any $A \in \mathcal{M}$,

$$\begin{aligned}\int_A Tg\, dm &= \int g(T^* 1_A) dm = L\Big(\frac{1}{k_n} \sum_{i=0}^{k_n - 1} \int |T^i f| (T^* 1_A) dm\Big) \\ &\geqq L\Big(\frac{1}{k_n} \sum_{i=1}^{k_n} \int (|T^i f|) 1_A dm\Big) \\ &= L\Big(\frac{1}{k_n} \sum_{i=0}^{k_n - 1} \int_A |T^i f| dm\Big) = \int_A g\, dm.\end{aligned}$$

Thus $Tg \geqq g$. Let $h = \lim_n T^n g$. Since $\sup_n \|T^n\|_1 < \infty$, we have $\lim_n \|h - T^n g\|_1 = 0$. Hence $Th = h$, and $h = g = 0$ by the nonexistence of nonzero nonnegative invariant functions. This shows that $\liminf_n \|T^n f\|_1 = 0$, and hence $\lim_n \|T^n f\|_1 = 0$, since $\sup_n \|T^n\|_1 < \infty$. The proof is complete.

In what follows we shall assume that T satisfies the following condition:

$$(*) \qquad Tf = f \quad \text{whenever} \quad 0 \leqq f \in L_1(X) \quad \text{and} \quad Tf \geqq f.$$

It may be easily seen that if T^* has a strictly positive subinvariant function in $L_\infty(X)$, then T satisfies the condition (*). To see that there exists a T which satisfies the condition (*) but whose adjoint has no strictly positive subinvariant function in $L_\infty(X)$, let $(X, \mathcal{M}, m)$ be the space of nonnegative integers with counting measure and define, as in Fong [4, p. 82], an operator T on $L_1(X)$ by

$$Tf(j)=\begin{cases} \sum_{i=1}^{\infty} f(i) & \text{if } j=0, \\ f(j+1) & \text{if } j\geqq 1. \end{cases}$$

It follows immediately that $\lim_n \|T^n f\|_1=0$ for any $f\in L_1(X)$. Therefore if $0\leqq f\in L_1(X)$ and $Tf\geqq f$, then $f=0$. Let $0\leqq u\in L_\infty(X)$ satisfy $T^*u\leqq u$. Then, since

$$T^*u(j)=\begin{cases} 0 & \text{if } j=0, \\ u(0) & \text{if } j=1, \\ u(0)+u(j-1) & \text{if } j\geqq 2, \end{cases}$$

we have $u(0)+u(j-1)\leqq u(j)$ for all $j\geqq 2$. Hence $u(0)=0$, since $\sup_j u(j)<\infty$.

THEOREM 1. *Let T be a positive linear operator on $L_1(X)$ which satisfies the condition* (*). *Then the following two conditions are equivalent:*

(i) *If $f\in L_1(X)$ then $T^n f$ converges weakly;*

(ii) *If $f\in L_1(X)$ then $\frac{1}{n}\sum_{i=1}^{n} T^{k_i} f$ converges strongly for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers.*

PROOF. If (i) holds, then the uniform boundedness principle (cf. [3, Corollary II.3.21]) implies that $\sup_n \|T^n\|_1<\infty$. Hence it follows from Sucheston [**10**, Theorems 1 and 2] (see also [8]) that the space X decomposes into two disjoint measurable sets, the remaining part Y and the disappearing part Z, such that

(a) $f\in L_1(Z)$ implies $Tf\in L_1(Z)$ and $\lim_n \|T^n f\|_1=0$;

(b) there exists a nonnegative function s in $L_\infty(Y)$ with $s>0$ a.e. on Y and $T^*s=s$.

Since Z is closed under T, if we define an operator U on $L_1(Y)$ by

$$Uf=(Tf)1_Y \qquad \text{for } f\in L_1(Y),$$

then $U^n f=(T^n f)1_Y$ for all $n\geqq 0$ and all $f\in L_1(Y)$. It follows from the condition (*) that if $0\leqq f\in L_1(Y)$ and $Uf=f$, then $Tf=f$. Moreover it follows from the definition of U that $\sup_n \|U^n\|_1\leqq \sup_n \|T^n\|_1<\infty$ and that $U^*s=T^*s=s$. Hence we can apply Propositions 1 and 2 in Fong [4] (see also [8]) to U to infer that the remaining part Y decomposes into two disjoint measurable sets P and N such that

(c) there exists an $h \in L_1(P)$ with $h > 0$ a.e. on P and $Th = h$;

(d) N is a union of countably many sets $A_j \in \mathcal{M}$ with

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \int_{A_j} T^k f \, dm = 0$$

for any $0 \leqq f \in L_1(Y)$.

Let us write $E = Z \cup N$ and define an operator V on $L_1(E)$ by

$$Vf = (Tf)1_E \quad \text{for} \quad f \in L_1(E).$$

Here we note that $P = X - E$ is closed under T. In fact, we have $T^*1_E = 0$ a.e. on P, since

$$\int h(T^*1_E)dm = \int_E Th \, dm = \int_E h \, dm = 0.$$

Therefore if $0 \leqq f \in L_1(P)$, then $\int_E Tf \, dm = \int f(T^*1_E)dm = 0$, and hence $Tf \in L_1(P)$. It follows that $V^n f = (T^n f)1_E$ for all $n \geqq 0$ and all $f \in L_1(E)$. Hence V has no nonzero nonnegative invariant function in $L_1(E)$ by (d) and (a), and $V^n f$ converges weakly for any $f \in L_1(E)$. Thus Proposition implies that

$$\lim_n \int_E |T^n f| \, dm = \lim_n \int |V^n f| \, dm = 0$$

for any $f \in L_1(E)$.

Next let $k_1, k_2, \cdots$ be any strictly increasing sequence of nonnegative integers. Since $U^*s = s$ and $U^n f$ converges weakly for any $f \in L_1(Y)$, it follows from Sato [**9**, Theorem 1] that

$$\frac{1}{n} \sum_{i=1}^{n} (T^{k_i} f)1_Y = \frac{1}{n} \sum_{i=1}^{n} U^{k_i} f$$

converges strongly for any $f \in L_1(Y)$.

Let $f \in L_1(X)$ and write $f = g + g'$, where $g = f(1_Z + 1_P)$ and $g' = f(1_N)$. Since Z and P are closed under T, the above arguments show that

$$\frac{1}{n} \sum_{i=1}^{n} T^{k_i} g$$

converges strongly. Thus, to prove (ii), it suffices to show the strong convergence of $\frac{1}{n} \sum_{i=1}^{n} T^{k_i} g'$. But this follows easily, since $\lim_n \int_E |T^n g'| \, dm = 0$ and $\frac{1}{n} \sum_{i=1}^{n} (T^{k_i} g')1_Y$ converges strongly.

Conversely if (ii) holds, then it follows that $\sup \|T^n f\|_1 < \infty$ for any $f \in L_1(X)$ (cf. [**1**, p. 237]). Let $0 \leqq f \in L_1(X)$ and $A \in \mathcal{M}$. Write

$$a=\liminf_n \int_A T^n f\,dm \quad \text{and} \quad b=\limsup_n \int_A T^n f\,dm\,.$$

If $a<b$, then we can choose a strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers such that

$$a=\liminf_n \frac{1}{n}\sum_{i=1}^{n}\int_A T^{k_i} f\,dm<\limsup_n \frac{1}{n}\sum_{i=1}^{n}\int_A T^{k_i} f\,dm=b$$

(cf. [1, p. 236]). But this contradicts (ii). Hence it must follow that $a=b$, which shows that $T^n f$ converges weakly for any $0\leqq f\in L_1(X)$, and hence for any $f\in L_1(X)$. This completes the proof.

THEOREM 2. *Let T be a positive linear operator on $L_1(X)$ which satisfies the condition* (*). *Suppose that* $\sup_n \|T^n\|_1<\infty$. *Then the following two conditions are equivalent:*

(i) *If $f\in L_1(X)$ and $\int f\,dm=0$, then $T^n f$ converges weakly;*

(ii) *If $f\in L_1(X)$ and $\int f\,dm=0$, then $\frac{1}{n}\sum_{i=1}^{n} T^{k_i} f$ converges strongly for any strictly increasing sequence $k_1, k_2, \cdots$ of nonnegative integers.*

PROOF. Suppose (i) holds. If T has no nonzero nonnegative invariant function in $L_1(X)$, then (ii) follows from Proposition. If there exists a nonnegative function $h\in L_1(X)$ with $Th=h$ and $\|h\|_1>0$, it follows from Akcoglu and Sucheston [1, p. 243] that for any $f\in L_1(X)$, $T^n f$ converges weakly. Hence, in this case, (ii) follows from Theorem 1.

The proof of (ii)$\Rightarrow$(i) is similar to that of (ii)$\Rightarrow$(i) in Theorem 1.

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