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Remarks on linear m-accretive operators in a Hilbert space

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Introduction.

A linear operator A with domain $D(A)$ and range $R(A)$ in a Hilbert space H is said to be *accretive* if

$$\mathrm{Re}\,(Au, u) \geqq 0 \qquad \text{for every } u \in D(A)\,,$$

or equivalently if

$$\|(A+\xi)u\| \geqq \xi\|u\| \qquad \text{for all } u \in D(A) \text{ and } \xi > 0\,.$$

If in particular $R(A+\xi)=H$ for some (and hence for every) $\xi>0$, we say that A is *m-accretive.* A linear m-accretive operator in H is closed and densely defined; its adjoint is also m-accretive (see Kato [**4**], V-§3.10).

The purpose of this note is to give some remarks on linear m-accretive operators in H. §1 contains a criterion for a closed linear accretive operator in H to be m-accretive. In §2, we prove some perturbation theorems. §3 is concerned with the real part of a linear m-accretive operators in H. We shall mention that a theorem of Kato [**3**] can be proved also by making use of the result obtained in §2.

§1. A criterion for m-accretiveness.

Let B be a closed linear operator in a Hilbert space H and suppose that

(S) for $n=1, 2, \cdots$, $R(1+n^{-1}B)=H$ and $(1+n^{-1}B)^{-1}$ exists, and moreover $\|(1+n^{-1}B)^{-1}\|$ is bounded as n tends to infinity.

Then for every $v \in H$,

$$\|(1+n^{-1}B)^{-1}v-v\| \longrightarrow 0 \qquad \text{as } n\to\infty\,; \tag{1.1}$$

note that B is densely defined (see Yosida [**9**], VIII-§4).

Let us start with

PROPOSITION 1.1. *Let A be a linear m-accretive operator in H. Then there exists a closed linear operator B with $D(B)\subset D(A)$, satisfying condition* (S), *such that*

$$\mathrm{Re}\,(Au, Bu) \geqq 0 \qquad \textit{for every } u \in D(B)\,. \tag{1.2}$$

PROOF. Let A^* be the adjoint of A. Then, since A is closed and densely defined, A^*A is a nonnegative selfadjoint operator in H (see [4], Theorem V-3.24). Setting $B=A^*A$, B has the required properties; note that A^* is also accretive. Q. E. D.

Conversely, we have

PROPOSITION 1.2. *Let A be a linear accretive operator in H. Then A has the m-accretive closure if there exists a linear operator B with the properties stated in Proposition* 1.1.

To prove Proposition 1.2, we use the following

LEMMA 1.3 (cf. Krein [5], Theorem I-4.4). *Let A be a densely defined linear accretive operator in H. Then A has the m-accretive closure if and only if A^* is accretive.*

PROOF OF PROPOSITION 1.2. Since $D(A)\supset D(B)$ and $D(B)$ is dense in H, A is densely defined and so A is closable (see [4], Theorem V-3.4). Consequently, it suffices by Lemma 1.3 to show that A^* is accretive. Since A is accretive, it follows from (1.2) that

$$\mathrm{Re}\,(Au, (1+n^{-1}B)u)\geqq 0 \qquad \text{for every } u\in D(B)\,. \tag{1.3}$$

Now let $v\in D(A^*)$. Then $(1+n^{-1}B)^{-1}v\in D(B)$. Setting $u=(1+n^{-1}B)^{-1}v$ in (1.3), we have that for every $v\in D(A^*)$,

$$\mathrm{Re}\,((1+n^{-1}B)^{-1}v, A^*v)\geqq 0\,.$$

Going to the limit $n\to\infty$, we see by (1.1) that A^* is accretive. Q. E. D.

In view of these propositions, we obtain

THEOREM 1.4. *Let A be a closed linear accretive operator in H. Then A is m-accretive if and only if there exists a closed linear operator B with $D(B)\subset D(A)$, satisfying condition* (S), *such that* $\mathrm{Re}\,(Au, Bu)\geqq 0$ *for every* $u\in D(B)$.

REMARK 1.5. Theorem 1.4 is a slight refinement of a result of de Graaf (see [2], Theorems 1 and 7).

REMARK 1.6. Let A be a bounded linear accretive operator with $D(A)=H$. Then A is m-accretive. In fact, we can take the identity operator as B in Theorem 1.4.

§ 2. Perturbations.

Let A be a linear m-accretive operator in a Hilbert space H. Set

$$A_n=A(1+n^{-1}A)^{-1}=n(1-(1+n^{-1}A)^{-1})\,, \qquad n=1, 2, \cdots.$$

Then A_n is also m-accretive and is called the *Yosida approximation* of A in the sense that for every $u\in D(A)$,

$$\|A_nu-Au\| \longrightarrow 0 \qquad \text{as } n\to\infty\,. \tag{2.1}$$

The first result is given by

THEOREM 2.1. *Let A and B be linear m-accretive operators in H. Assume that there exist nonnegative constants a and $b \leqq 1$ such that for all $u \in D(B)$,*

$$0 \leqq \operatorname{Re}(A_n u, Bu) + a\|u\|^2 + b\|A_n u\|^2 . \tag{2.2}$$

If $b<1$ then $A+B$ is also m-accretive. If $b=1$ then the closure of $A+B$ is m-accretive.

To prove Theorem 2.1, the following lemma is useful.

LEMMA 2.2 (see [**1**]; cf. also [**8**]). *Let A and B be as in Theorem 2.1. Then $A+B$ is m-accretive if and only if $\|A_n(A_n+B+1)^{-1}\|$ is bounded as n tends to infinity.*

PROOF OF THEOREM 2.1. Since A_n is also accretive, it follows from (2.2) that for all $u \in D(B)$,

$$0 \leqq \operatorname{Re}(A_n u + au, Bu + u) + b\|(A_n + a)u\|^2 . \tag{2.3}$$

Let $v \in H$. Since $A_n + B$ is m-accretive, $u_n = (A_n + B + 1)^{-1} v$ is defined and we have

$$(A_n u_n + a u_n) + (B u_n + u_n) = v + a u_n .$$

First let us consider the case of $b<1$. In view of (2.3), it follows that

$$\operatorname{Re}(A_n u_n + a u_n, v + a u_n) \geqq (1-b)\|A_n u_n + a u_n\|^2 .$$

Since $\|u_n\| \leqq \|v\|$, we obtain

$$\begin{aligned} \|A_n u_n\| &\leqq \|A_n u_n + a u_n\| + a\|u_n\| \\ &\leqq [(1-b)^{-1}(1+a) + a]\|v\| . \end{aligned}$$

Thus, by Lemma 2.2, $A+B$ is m-accretive.

Next, suppose that $b=1$. Then we have by (2.2) that for all $u \in D(B)$,

$$0 \leqq \operatorname{Re}(A_n u, (B/2)u) + (a/2)\|u\|^2 + \frac{1}{2}\|A_n u\|^2 . \tag{2.4}$$

Therefore, $A+B/2$ is m-accretive as shown above. Thus, we see that $D(A+B)$ is dense in H. Going to the limit in (2.4) with $u \in D(A+B)$, we obtain by (2.1)

$$0 \leqq \operatorname{Re}(Au, Bu) + a\|u\|^2 + \|Au\|^2 , \qquad u \in D(A+B) . \tag{2.5}$$

Hence it follows that

$$\begin{aligned} \|(B/2)u\|^2 &\leqq \|(B/2)u\|^2 + \operatorname{Re}(Au, Bu) + a\|u\|^2 + \|Au\|^2 \\ &= a\|u\|^2 + \|(A+B/2)u\|^2 , \qquad u \in D(A+B) . \end{aligned}$$

This implies that the closure of $(A+B/2)+B/2 = A+B$ is m-accretive (see e.g. [**6**] or [**7**]). Q. E. D.

COROLLARY 2.3. *If $b<1$ in Theorem 2.1, then $D(A+B)$ is a core of A. If in particular $b=0$, then $D(A+B)$ is a core of both A and B.*

PROOF. To see that $D(A+B)$ is a core of A, it suffices to show that $(A+1)D(A+B)$ is dense in H (see [4], III-§5.3). To this end, we shall show that an element v of H orthogonal to $(A+1)D(A+B)$ should be zero. Let $c>a$. Then, since $D(A+B)=(A+B+c)^{-1}H$, it follows that for all $w\in H$,

$$((A+1)(A+B+c)^{-1}w, v)=0. \tag{2.6}$$

Setting $w=v$ and $(A+B+c)^{-1}v=u$, we have $(Au+u, (A+B+c)u)=0$. So, we see from (2.5) that

$$0\geqq \operatorname{Re}(Au, (A+B)u)+c\|u\|^2\geqq (c-a)\|u\|^2.$$

Consequently, $u=0$ and hence $v=0$.

Now let $b=0$ and suppose that v in H is orthogonal to $(B+1)D(A+B)$. Then we have instead of (2.6)

$$((B+1)(A+B+c)^{-1}w, v)=0, \qquad w\in H.$$

In the same way as above we can show that $v=0$. Q.E.D.

REMARK 2.4. Let A and B be as in Theorem 2.1. Suppose that there exist nonnegative constants a and $b<1$ such that for all $u\in D(B)$,

$$0\leqq \operatorname{Re}(A_nu, Bu)+a\|u\|^2+b\|Bu\|^2.$$

Then $A+B$ is also m-accretive.

In this connection note further that $A+B$ is m-accretive if and only if there are nonnegative constants a and $b<1$ such that for all $u\in D(B)$,

$$0\leqq 2\operatorname{Re}(A_nu, Bu)+a\|u\|^2+b(\|A_nu\|^2+\|Bu\|^2);$$

see [8].

Our second result is the following

THEOREM 2.5. *Let A and B be linear m-accretive operators in H. Let D be a linear manifold invariant under $(1+n^{-1}A)^{-1}$ for $n=1, 2, \cdots$. Assume that D is a core of B and there exist nonnegative constants a and $b\leqq 1$ such that for all $u\in D_0=(1+A)^{-1}D$,*

$$0\leqq \operatorname{Re}(Au, Bu)+a\|u\|^2+b\|Au\|^2. \tag{2.7}$$

If $b<1$ then $A+B$ is also m-accretive. If $b=1$ then the closure of $A+B$ is m-accretive.

PROOF. Let $u\in D_0$. Then $Au\in D$. Since $\operatorname{Re}(Au, BAu)\geqq 0$, we see from (2.7) that for all $u\in D_0$,

$$0\leqq \operatorname{Re}(Au, B(1+n^{-1}A)u)+a\|u\|^2+b\|Au\|^2, \qquad n\geqq 1. \tag{2.8}$$

Now let $v\in D$. Then $(1+n^{-1}A)^{-1}v\in D_0$ (note that $D_0=(1+n^{-1}A)^{-1}D$). Setting

$u=(1+n^{-1}A)^{-1}v$ in (2.8), we have that for all $v\in D$,

$$0\leqq \mathrm{Re}\,(A_n v,\ Bv)+a\|v\|^2+b\|A_n v\|^2 .$$

Since D is a core of B, we obtain (2.2). Q. E. D.

COROLLARY 2.6. *Let A and B be selfadjoint operators in H satisfying the inequality* (2.7) *with $u\in D(A^\infty)$. Assume that $D(A^\infty)$ is a core of B. If $b<1$ then $A+B$ is also selfadjoint. If $b=1$ then $A+B$ is essentially selfadjoint, i.e., the closure of $A+B$ is selfadjoint.*

PROOF. Since $A+B$ is symmetric, it suffices to show that $\pm i(A+B)$ are m-accretive. To this end, we can apply Theorem 2.5 to $\pm iA$ and $\pm iB$ (cf. [8], Corollary 3.5). Q. E. D.

REMARK 2.7. Theorem 2.5 improves Theorem 3.4 of [8] in which b is assumed to be smaller than 1/2. The improvement was suggested by Professor T. Kato (private communication)[1].

REMARK 2.8. In Theorem 2.5 assume further that $D(A)\subset D(B)$. Then D can be replaced by $D(B)$. In fact, we have that for all $v\in D$,

$$0\leqq \mathrm{Re}\,(A(1+A)^{-1}v,\ B(1+A)^{-1}v)+a\|(1+A)^{-1}v\|^2+b\|A(1+A)^{-1}v\|^2 .$$

But, this inequality holds for all $v\in H$ since D is dense in H and $B(1+A)^{-1}$ is bounded by assumption. Thus, (2.7) holds for all $u\in D(A)$.

§ 3. Real parts.

Let A be a linear m-accretive operator in a Hilbert space H, and A^* be its adjoint. Then $\frac{1}{2}(A+A^*)$ may be regarded as the *real part* of A (and also of A^*) if the intersection of $D(A)$ and $D(A^*)$ is wide enough. Let A_n be the Yosida approximation of A. Then $(A_n)^*=(A^*)_n$.

THEOREM 3.1. *Let A be m-accretive in H. Assume that A_n-A_m is accretive for each pair of integers m and n satisfying $n\geqq m$. Then $\frac{1}{2}(A+A^*)$ is selfadjoint and $D(A+A^*)$ is a core of both A and A^*.*

PROOF. Since $A_n-A_m=(m^{-1}-n^{-1})A_nA_m$, we obtain

$$\mathrm{Re}\,(A_m v,\ A_n^* v)\geqq 0\,, \qquad n\geqq m\,.$$

Going to the limit $n\to\infty$, it follows that for all $u\in D(A^*)$,

$$\mathrm{Re}\,(A_m u,\ A^* u)\geqq 0\,, \qquad m\geqq 1\,.$$

Therefore, by Theorem 2.1 and Corollary 2.3, $A+A^*$ is m-accretive and $D(A+A^*)$ is a core of both A and A^*. Consequently, we see that $D(A+A^*)$ is dense in H and $A+A^*$ is symmetric. Thus, $A+A^*$ is selfadjoint. Q.E.D.

1) The writer would like to thank Professor T. Kato for his kind suggestions.

Now let $A^{1/2}$ be the square root of A. Then $A^{1/2}$ is also m-accretive and $A^{*1/2}=A^{1/2*}$. The following corollary is Theorem 5.1 in [3].

COROLLARY 3.2. *Let A be m-accretive in H. Then $\frac{1}{2}(A^{1/2}+A^{*1/2})$ is self-adjoint and $D(A^{1/2}+A^{*1/2})$ is a core of both $A^{1/2}$ and $A^{*1/2}$.*

PROOF. Let B_n be the Yosida approximation of $A^{1/2}$. Then it suffices to show that for $n\geqq m$, B_n-B_m is accretive. But, this is shown in the first step of the proof of Theorem 5.1 in [3] as follows. We first note that

$$B_n-B_m=(m^{-1}-n^{-1})A(1+n^{-1}A^{1/2})^{-1}(1+m^{-1}A^{1/2})^{-1}.$$

Setting $u=(1+n^{-1}A^{1/2})^{-1}(1+m^{-1}A^{1/2})^{-1}v$, $v\in H$, we have

$$\begin{aligned}&((B_n-B_m)v, v)\\&=(m^{-1}-n^{-1})(Au, (1+m^{-1}A^{1/2})(1+n^{-1}A^{1/2})u)\\&=(m^{-1}-n^{-1})[(Au, u)+(m^{-1}+n^{-1})(Au, A^{1/2}u)+(mn)^{-1}\|Au\|^2].\end{aligned}$$

Consequently, we obtain

$$\mathrm{Re}\,((B_n-B_m)v, v)\geqq(mn)^{-1}(m^{-1}-n^{-1})\|Au\|^2\geqq 0.$$

Q. E. D.

References

[1] H. Brezis, M. G. Crandall and A. Pazy, Perturbations of nonlinear maximal monotone sets in Banach space, Comm. Pure Appl. Math., **23** (1970), 123–144.

[2] J. de Graaf, A constructive approach to one-parameter semi-groups of operators in Hilbert space, Arch. Rational Mech. Anal., **43** (1971), 125–153.

[3] T. Kato, Fractional powers of dissipative operators, J. Math. Soc. Japan, **13** (1961), 246–274.

[4] T. Kato, Perturbation Theory for Linear Operators, Die Grundlehren der math. Wissenschaften, Band 132, Springer-Verlag, Berlin and New York, 1966.

[5] S. G. Krein, Linear Differential Equations in Banach Space, Nauka, Moscow, 1967 (Russian); English Translation, Transl. Math. Monographs, vol. 29, Amer. Math. Soc., Providence, R. I., 1972.

[6] N. Okazawa, Two perturbation theorems for contraction semigroups in a Hilbert space, Proc. Japan Acad., **45** (1969), 850–853.

[7] N. Okazawa, A perturbation theorem for linear contraction semigroups on reflexive Banach spaces, Proc. Japan Acad., **47** (1971), 947–949.

[8] N. Okazawa, Perturbations of linear m-accretive operators, Proc. Amer. Math. Soc., **37** (1973), 169–174.

[9] K. Yosida, Functional Analysis, 3rd ed., Die Grundlehren der math. Wissenschaften, Band 123, Springer-Verlag, Berlin and New York, 1971.

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