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Pseudomonotone operators and nonlinear elliptic boundary value problems

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Introduction.

In this paper, from a view-point of the nonlinear operator theory we study nonlinear elliptic partial differential equations of the form

$$\text{(P)} \qquad -\sum_{k=1}^{N} \frac{\partial}{\partial x_k} A_k(x, u, \nabla u) + A_0(x, u, \nabla u) = f \quad \text{in } \Omega$$

with some boundary conditions, and nonlinear elliptic variational inequalities of the form

$$\text{(V)} \qquad \begin{cases} u \in K, \\ a(u, u-v) - \int_\Omega f(u-v)dx \leqq \Phi(v) - \Phi(u) \quad \text{for all } v \in K, \end{cases}$$

where Ω is a bounded domain in R^N with smooth boundary Γ, $f \in L^{p'}(\Omega)$ $(1/p+1/p'=1,\ 1<p<\infty)$, K is a convex closed subset of the Sobolev space $W^{1,p}(\Omega)$, Φ is a lower semicontinuous convex function on K and $a(\cdot, \cdot)$ is the functional on $W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ given by

$$a(v, w) = \sum_{k=1}^{N} \int_\Omega A_k(x, v, \nabla v) \frac{\partial w}{\partial x_k} dx + \int_\Omega A_0(x, v, \nabla v) w dx .$$

In order to find a solution of (P) with a boundary condition of the Dirichlet type:

$$u|_\Gamma = \psi \quad \text{on } \Gamma,$$

one considers a variational inequality of type (V) with Φ and K associated with this boundary condition. Existence theorems for variational inequalities of type (V) were established by many authors (e.g., [1], [3], [4], [5], [6], [12], [14], [17], [21], [22]).

We treat partial differential equations of the form (P) with boundary conditions of mixed type. In particular, in case (P) has a smooth solution u, our boundary condition of mixed type is given by

$$\begin{cases} u|_\Gamma=\psi & \text{on } \Gamma_0, \\ \sum_{k=1}^{N}(A_k(x, u, \nabla u)|_\Gamma)\nu_k=\psi^*+h(u|_\Gamma) & \text{on } \Gamma\setminus\Gamma_0, \end{cases}$$

where Γ_0 is a closed subset of Γ, ψ, ψ^* and h are given functions on Γ_0, $\Gamma\setminus\Gamma_0$ and R^1, respectively, and $\nu=(\nu_1, \nu_2, \cdots, \nu_N)$ is the unit exterior normal to Γ. However, generally, solutions of (P) need not be smooth, so we have to construct boundary conditions in a generalized sense. For this purpose, we introduce a continuous linear operator $\boldsymbol{B}$ from the Banach space $\boldsymbol{E}^{p'}(\Omega) = \{\boldsymbol{v}=(v_1, v_2, \cdots, v_N)\,;\, v_k\in L^{p'}(\Omega),\, k=1, 2, \cdots, N, \operatorname{div} \boldsymbol{v}\in L^{p'}(\Omega)\}$ into $W^{-1/p',p'}(\Gamma)$ (=the dual space of $W^{1/p',p}(\Gamma)$) such that

$$\boldsymbol{Bv}=\sum_{k=1}^{N}(v_k|_\Gamma)\nu_k$$

if $\boldsymbol{v}=(v_1, v_2, \cdots, v_N)$ with $v_k\in\mathcal{D}(\bar{\Omega})$ for all k. Then our boundary condition is given by means of the operator $\boldsymbol{B}$ as follows:

$$\begin{cases} u|_\Gamma=\psi & \text{a.e. on } \Gamma_0, \\ \boldsymbol{Ba}(u)=\psi^*+h(u|_\Gamma) & \text{on } \Gamma\setminus\Gamma_0 \text{ (in the distribution sense)}, \end{cases}$$

where $\boldsymbol{a}(u)=(A_1(x, u, \nabla u), \cdots, A_N(x, u, \nabla u))$.

One aim of this paper is to show that equation (P) with generalized boundary conditions of the above type is equivalent to the variational inequality of type (V) associated with it. Another aim is to investigate the continuous dependence of solutions of boundary value problems as formulated above on boundary conditions by using results in [**19**] and [**8**].

§ 1. Preliminaries.

Throughout this paper, let Ω be a bounded domain in R^N, $N\geqq 2$, and assume that the boundary Γ of Ω is very regular, that is, it consists of a finite number of C^∞ compact $(N-1)$-dimensional connected manifolds with Ω lying on one side of Γ. In this section, let $1<p<\infty$ and $1/p+1/p'=1$.

1.1. The Sobolev space $W^{1,p}(\Omega)$ and its trace space $W^{1/p',p}(\Gamma)$.

Let us consider the Sobolev space

$$W^{1,p}(\Omega)=\left\{v\in L^p(\Omega)\,;\frac{\partial v}{\partial x_k}\in L^p(\Omega),\ k=1, 2, \cdots, N\right\}$$

and the trace space of $W^{1,p}(\Omega)$

$$W^{1/p',p}(\Gamma)=\{\hat{v}\in L^p(\Gamma)\,;\,(\hat{v})_p<\infty\},$$

where

$$(\hat{v})_p=\int_\Gamma\int_\Gamma\frac{|\hat{v}(x')-\hat{v}(y')|^p}{|x'-y'|^{p+N-2}}d\Gamma_{x'}d\Gamma_{y'},$$

where $d\Gamma_{x'}$ and $d\Gamma_{y'}$ mean the surface measure. Norms in these Banach spaces are defined by

$$\|v\|_{1,p}=\|v\|_{L^p(\Omega)}+\sum_{k=1}^{N}\left\|\frac{\partial v}{\partial x_k}\right\|_{L^p(\Omega)}$$

and

$$[\hat{v}]_{1/p',p}=\|\hat{v}\|_{L^p(\Gamma)}+(\hat{v})_p^{1/p},$$

respectively. The space of all C^∞-functions on R^N with compact support in Ω is denoted by $\mathcal{D}(\Omega)$ and the space of the restrictions of all C^∞-functions on R^N to Ω is denoted by $\mathcal{D}(\bar{\Omega})$. It is well-known that the operator $\gamma: u\in\mathcal{D}(\bar{\Omega})\to$(the boundary values of u) is a linear and continuous operator from $\mathcal{D}(\bar{\Omega})$ equipped with the topology of $W^{1,p}(\Omega)$ into $W^{1/p',p}(\Gamma)$. Since $\mathcal{D}(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$, there is a unique continuous extension of γ to all of $W^{1,p}(\Omega)$. This extension is also denoted by γ. Then we know that the range of γ is all of $W^{1/p',p}(\Gamma)$ and there are positive constants λ_1 and λ_2 such that for all $\hat{v}\in W^{1/p',p}(\Gamma)$

$$\begin{aligned}&\inf\{\|v\|_{1,p};\ v\in W^{1,p}(\Omega),\ \gamma v=\hat{v}\}\\ &\leqq\lambda_1[\hat{v}]_{1/p',p}\\ &\leqq\lambda_2\inf\{\|v\|_{1,p};\ v\in W^{1,p}(\Omega),\ \gamma v=\hat{v}\}\,.\end{aligned}\tag{1.1}$$

For a detailed discussion on the operator γ, see Gagliardo [7] and Lions-Magenes [16].

1.2. Equalities and inequalities for functions in $W^{1,p}(\Omega)$ and in $W^{1/p',p}(\Gamma)$.

We now recall notions of equalities and inequalities for functions in $W^{1,p}(\Omega)$ and in $W^{1/p',p}(\Gamma)$ (cf. Littman-Stampacchia-Weinberger [18]).

Let Γ_0 be a compact subset of Γ. Then we say that $\phi\in W^{1/p',p}(\Gamma)$ is non-negative on Γ_0 in the sense of $W^{1/p',p}(\Gamma)$, if there is a sequence $\{\phi_k\}\subset\mathcal{D}(\bar{\Omega})$ such that $\gamma\phi_k\geqq 0$ on Γ_0 for all k and $\gamma\phi_k\overset{s}{\to}\phi$ in $W^{1/p',p}(\Gamma)$ as $k\to\infty$, where we mean by "$\overset{s}{\to}$" the convergence in the strong topology. For two functions ϕ and η in $W^{1/p',p}(\Gamma)$, we define "$\phi\geqq\eta$ on Γ_0 in the sense of $W^{1/p',p}(\Gamma)$" by "$\phi-\eta\geqq 0$ on Γ_0 in the sense of $W^{1/p',p}(\Gamma)$".

Next, let F be a compact subset of Ω and $v\in W^{1,p}(\Omega)$. We then say that $v=0$ on F in the sense of $W^{1,p}(\Omega)$, if there is a sequence $\{\phi_k\}\subset\mathcal{D}(\bar{\Omega})$ such that $\phi_k=0$ on a neighborhood of F for all k and $\phi_k\overset{s}{\to}v$ in $W^{1,p}(\Omega)$ as $k\to\infty$. For two functions v and w in $W^{1,p}(\Omega)$, we define "$v=w$ on F in the sense of $W^{1,p}(\Omega)$" in a way similar to the above. Note that if $v=w$ on F in the sense of $W^{1,p}(\Omega)$, then $v=w$ a.e. on F and $\frac{\partial v}{\partial x_k}=\frac{\partial w}{\partial x_k}$ a.e. on F, $k=1,2,\cdots,N$.

1.3. The space $\boldsymbol{E}^{p'}(\Omega)$.

Let us consider the Banach space

$$\boldsymbol{E}^{p'}(\Omega)=\{\boldsymbol{v}=(v_1, v_2, \cdots, v_N); v_k \in L^{p'}(\Omega),\ k=1, 2, \cdots, N, \operatorname{div} \boldsymbol{v} \in L^{p'}(\Omega)\}$$

with the norm

$$|||\boldsymbol{v}|||_{p'}=\| \operatorname{div} \boldsymbol{v}\|_{L^{p'}(\Omega)}+\sum_{k=1}^{N}\|v_k\|_{L^{p'}(\Omega)}.$$

We denote by $\boldsymbol{D}(\bar{\Omega})$ the space $\{\boldsymbol{v}=(\phi_1, \phi_2, \cdots, \phi_N); \phi_k \in \mathcal{D}(\bar{\Omega}),\ k=1, 2, \cdots, N\}$. We then prove the following:

PROPOSITION 1.1. *$\boldsymbol{D}(\bar{\Omega})$ is a dense subspace of $\boldsymbol{E}^{p'}(\Omega)$.*

Before proving this proposition we give two lemmas.

LEMMA 1.2. *Let $\boldsymbol{u}=(u_1, u_2, \cdots, u_N)$ be a function such that $u_k \in L^{p'}(R^N)$, $k=1, 2, \cdots, N$, and $\operatorname{div} \boldsymbol{u} \in L^{p'}(R^N)$. Then the restriction $\boldsymbol{u}|_\Omega$ of $\boldsymbol{u}$ to Ω belongs to $\boldsymbol{E}^{p'}(\Omega)$ and can be approximated by functions in $\boldsymbol{D}(\bar{\Omega})$ in the topology of $\boldsymbol{E}^{p'}(\Omega)$.*

PROOF. For each positive integer k, let ρ_k be a non-negative C^∞-function on R^N such that $\int_{R^N}\rho_k(x)dx=1$ and $\rho_k(x)=0$ for $x \in R^N$ with $|x| \geqq 1/k$. Set

$$\boldsymbol{v}_k=(u_1*\rho_k, u_2*\rho_k, \cdots, u_N*\rho_k),$$

where $u_j*\rho_k$ denotes the convolution of u_j and ρ_k. Then, clearly,

$$\boldsymbol{v}_k|_\Omega \xrightarrow{s} \boldsymbol{u}|_\Omega \quad \text{in} \quad \boldsymbol{E}^{p'}(\Omega)$$

as $k \to \infty$.

LEMMA 1.3. *Let ν_0 be a unit vector and Γ_0 be an open subset of Γ such that for a positive number τ*

$$\{x'+\lambda\nu_0;\ x' \in \Gamma_0,\ -\tau<\lambda<0\} \subset \Omega$$

and

$$\{x'+\lambda\nu_0;\ x' \in \Gamma_0,\ 0<\lambda<\tau\} \subset R^N \setminus \bar{\Omega}.^{1)}$$

Let $\boldsymbol{u}=(u_1, u_2, \cdots, u_N)$ be a function in $\boldsymbol{E}^{p'}(\Omega)$ such that

$$S(\boldsymbol{u}) \subset \{x'+\lambda\nu_0;\ x' \in \Gamma_0,\ -\tau<\lambda \leqq 0\},$$

where

$$S(\boldsymbol{u})=\bigcup_{k=1}^{N}\{\text{the support of } u_k\}.$$

Then $\boldsymbol{u}$ can be approximated by functions in $\boldsymbol{D}(\bar{\Omega})$ in the topology of $\boldsymbol{E}^{p'}(\Omega)$.

PROOF. First, let us choose a positive number τ' with $0<\tau'<\tau$ and an open subset Γ_0' of Γ with $\bar{\Gamma}_0' \subset \Gamma_0$ such that

$$S(\boldsymbol{u}) \subset \{x'+\lambda\nu_0;\ x' \in \Gamma_0',\ -\tau'<\lambda \leqq 0\}. \tag{1.2}$$

For small $\varepsilon>0$ we define $\boldsymbol{u}_\varepsilon=(u_{\varepsilon,1}, u_{\varepsilon,2}, \cdots, u_{\varepsilon,N})$ on

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1) $A \setminus B=\{x;\ x \in A,\ x \notin B\}$.

$$\Omega_\varepsilon = \{x' + \lambda\nu_0\,;\ x' \in \Gamma_0,\ -\tau < \lambda < \varepsilon\} \cup \Omega$$

by

$$u_{\varepsilon,k}(x) = \begin{cases} u_k(x - \varepsilon\nu_0) & \text{if } x \in \{x' + \lambda\nu_0\,;\ x' \in \Gamma_0,\ -\tau' < \lambda < \varepsilon\}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For small $\varepsilon > 0$, $\boldsymbol{u}_\varepsilon$ is well defined on Ω_ε by (1.2). Clearly,

$$(\operatorname{div} \boldsymbol{u}_\varepsilon)(x) = \begin{cases} (\operatorname{div} \boldsymbol{u})(x - \varepsilon\nu_0) & \text{if } x \in \{x' + \lambda\nu_0\,;\ x' \in \Gamma_0,\ -\tau' < \lambda < \varepsilon\}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and hence, $\boldsymbol{u}_\varepsilon|_\Omega \in \boldsymbol{E}^{p'}(\Omega)$. Furthermore, by a theorem of Lebesgue,

$$\text{(1.3)} \qquad \boldsymbol{u}_\varepsilon|_\Omega \xrightarrow{s} \boldsymbol{u} \quad \text{in} \quad \boldsymbol{E}^{p'}(\Omega) \quad \text{as } \varepsilon \downarrow 0\,.$$

We next observe from (1.2) that for small $\varepsilon > 0$

$$(S(\boldsymbol{u}_\varepsilon|_\Omega) \cup S(\boldsymbol{u})) \subset \{x' + \lambda\nu_0\,;\ x' \in \Gamma_0',\ -\tau' < \tau \leqq 0\}\,.$$

Let $\alpha_\varepsilon(x)$ be a C^∞-function on R^N such that

$$\alpha_\varepsilon(x) = 1 \qquad \text{if} \quad x \in \{x' + \lambda\nu_0\,;\ x' \in \Gamma_0',\ -\tau' < \lambda < \varepsilon/2\}$$

and

$$\{\text{the support of } \alpha_\varepsilon\} \subset \{x' + \lambda\nu_0\,;\ x' \in \Gamma_0,\ -\tau < \lambda < \varepsilon\}\,,$$

and define $\boldsymbol{v}_\varepsilon = (v_{\varepsilon,1}, v_{\varepsilon,2}, \cdots, v_{\varepsilon,N})$ on R^N by

$$v_{\varepsilon,k}(x) = \begin{cases} \alpha_\varepsilon(x) u_k(x - \varepsilon\nu_0) & \text{if } x \in \{x' + \lambda\nu_0\,;\ x' \in \Gamma_0,\ -\tau' < \lambda < \varepsilon\}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then, $\boldsymbol{v}_\varepsilon \in [L^{p'}(R^N)]^N$, $\operatorname{div} \boldsymbol{v}_\varepsilon \in L^{p'}(R^N)$ and $\boldsymbol{v}_\varepsilon|_\Omega = \boldsymbol{u}_\varepsilon|_\Omega$ on Ω. Hence, by (1.3),

$$\boldsymbol{v}_\varepsilon|_\Omega \xrightarrow{s} \boldsymbol{u} \quad \text{in} \quad \boldsymbol{E}^{p'}(\Omega) \quad \text{as } \varepsilon \downarrow 0\,.$$

Since each $\boldsymbol{v}_\varepsilon|_\Omega$ is approximated by functions in $\boldsymbol{D}(\bar{\Omega})$ in the topology of $\boldsymbol{E}^{p'}(\Omega)$ by Lemma 1.2, so is $\boldsymbol{u}$. q. e. d.

PROOF OF PROPOSITION 1.1. Since Γ is very regular, we can find an open covering $\{U_i\}_{i=1}^n$ of Γ and a system $\{\theta_i\}_{i=1}^n$ of functions in $\mathcal{D}(R^N)$ with the following properties:

(i) For each i there are a unit vector $\nu^{(i)}$, an open subset $\Gamma^{(i)}$ of Γ and a positive number $\tau^{(i)}$ such that

$$U_i = \{x' + \lambda\nu^{(i)}\,;\ x' \in \Gamma^{(i)},\ |\lambda| < \tau^{(i)}\}\,,$$

$$\{x' + \lambda\nu^{(i)}\,;\ x' \in \Gamma^{(i)},\ 0 < \lambda < \tau^{(i)}\} \subset R^N \setminus \bar{\Omega}$$

and

$$\{x' + \lambda\nu^{(i)}\,;\ x' \in \Gamma^{(i)},\ -\tau^{(i)} < \lambda < 0\} \subset \Omega\,.$$

(ii) The support of θ_i is contained in U_i, $i=1, 2, \cdots, n$, and

$$\sum_{k=1}^{n}\theta_i(x)=1 \qquad \text{in a neighborhood of } \Gamma .$$

Let $\boldsymbol{u}=(u_1, u_2, \cdots, u_N)$ be an arbitrary function in $\boldsymbol{E}^{p'}(\Omega)$, and set

$$u_k^{(j)}(x)=\theta_j(x)u_k(x) , \qquad k=1, 2, \cdots, N,$$

and

$$u_k^{(0)}(x)=u_k(x)-\sum_{j=1}^{n}u_k^{(j)}(x) .$$

Then

$$\boldsymbol{u}=\sum_{j=0}^{n}\boldsymbol{u}^{(j)} ,$$

where $\boldsymbol{u}^{(j)}=(u_1^{(j)}, u_2^{(j)}, \cdots, u_N^{(j)})$. Since $\boldsymbol{u}^{(j)}$, $j=0, 1, 2, \cdots, n$, can be approximated by functions in $\boldsymbol{D}(\bar{\Omega})$ in the topology of $\boldsymbol{E}^{p'}(\Omega)$ by Lemmas 1.2 and 1.3, so can $\boldsymbol{u}$. q. e. d.

1.4. The operator $\boldsymbol{B}$.

We now define a linear continuous operator from $\boldsymbol{E}^{p'}(\Omega)$ into $W^{-1/p',p'}(\Gamma)$ (=the dual space of $W^{1/p',p}(\Gamma)$).

Let $\boldsymbol{w}=(\phi_1, \phi_2, \cdots, \phi_n)\in\boldsymbol{D}(\bar{\Omega})$ and $\phi\in\mathcal{D}(\bar{\Omega})$. Then, by the divergence theorem,

$$\int_{\Omega}(\operatorname{div}\boldsymbol{w})\phi dx+\int_{\Omega}(\boldsymbol{w}, \nabla\phi)dx=\int_{\Gamma}\sum_{k=1}^{N}(\gamma\phi_k)\nu_k(\gamma\phi)d\Gamma , \tag{1.4}$$

where $(\cdot, \cdot)$ denotes the inner product in R^N, $\nabla\phi=\operatorname{grad}\phi$ and $\nu(x')=(\nu_1(x'), \nu_2(x'), \cdots, \nu_N(x'))$ is the unit vector which is normal to Γ at $x'\in\Gamma$ and oriented toward the exterior of Ω. Since $\mathcal{D}(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$, for each $v\in W^{1,p}(\Omega)$ there is a sequence $\{\eta_j\}\subset\mathcal{D}(\bar{\Omega})$ such that $\eta_j\xrightarrow{s}v$ in $W^{1,p}(\Omega)$. Substituting η_j for ϕ in (1.4) and letting $j\to\infty$, we have

$$\int_{\Omega}(\operatorname{div}\boldsymbol{w})vdx+\int_{\Omega}(\boldsymbol{w}, \nabla v)dx=\int_{\Gamma}\sum_{k=1}^{N}(\gamma\phi_k)\nu_k(\gamma v)d\Gamma . \tag{1.5}$$

The functional

$$\hat{v}\in W^{1/p',p}(\Gamma)\longrightarrow\int_{\Gamma}\sum_{k=1}^{N}(\gamma\phi_k)\nu_k\hat{v}d\Gamma$$

is linear and continuous in $W^{1/p',p}(\Gamma)$. Therefore there exists a unique element $\delta(\boldsymbol{w})\in W^{-1/p',p'}(\Gamma)$ such that

$$\int_{\Gamma}\sum_{k=1}^{N}(\gamma\phi_k)\nu_k\hat{v}d\Gamma=\langle\delta(\boldsymbol{w}), \hat{v}\rangle_{\Gamma} \qquad \text{for all } \hat{v}\in W^{1/p',p}(\Gamma) ,$$

where $\langle\,,\rangle_{\Gamma}$ denotes the natural pairing between $W^{-1/p',p'}(\Gamma)$ and $W^{1/p',p}(\Gamma)$. It follows from (1.5) that for each $\hat{v}\in W^{1/p',p}(\Gamma)$,

$$|\langle\delta(\boldsymbol{w}), \hat{v}\rangle_\Gamma| \leqq \|\operatorname{div} \boldsymbol{w}\|_{L^{p'}(\Omega)} \cdot \|v\|_{L^p(\Omega)} + \sum_{k=1}^N \|\phi_k\|_{L^{p'}(\Omega)} \cdot \left\| \frac{\partial v}{\partial x_k} \right\|_{L^p(\Omega)}$$

$$\leqq \text{const.} |||\boldsymbol{w}|||_{p'} \cdot \|v\|_{1,p} \qquad \text{for all} \quad v \in W^{1,p}(\Omega) \text{ with } \gamma v = \hat{v}\,.$$

Hence, by (1.1),

$$|\langle\delta(\boldsymbol{w}), \hat{v}\rangle_\Gamma| \leqq \text{const.} |||\boldsymbol{w}|||_{p'} \cdot [\hat{v}]_{1/p',p} \qquad \text{for all} \quad \hat{v} \in W^{1/p',p}(\Gamma)\,.$$

Thus δ is a linear and continuous operator from $\boldsymbol{D}(\bar{\Omega})$ into $W^{-1/p',p'}(\Gamma)$ in the topology of $\boldsymbol{E}^{p'}(\Omega)$. Therefore, by Proposition 1.1, there is a unique continuous linear operator $\boldsymbol{B}$ from $\boldsymbol{E}^{p'}(\Omega)$ into $W^{-1/p',p'}(\Gamma)$ such that $\delta(\boldsymbol{w}) = \boldsymbol{B}\boldsymbol{w}$ if $\boldsymbol{w} \in \boldsymbol{D}(\bar{\Omega})$. Moreover we have the following:

PROPOSITION 1.4. (a) *For any* $\boldsymbol{v} \in \boldsymbol{E}^{p'}(\Omega)$ *and* $v \in W^{1,p}(\Omega)$

$$\int_\Omega (\operatorname{div} \boldsymbol{v}) v\, dx + \int_\Omega (\boldsymbol{v}, \nabla v)\, dx = \langle \boldsymbol{B}\boldsymbol{v}, \gamma v\rangle_\Gamma\,. \tag{1.6}$$

(b) *If* $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{E}^{p'}(\Omega)$ *and if* $\boldsymbol{v}_1 = \boldsymbol{v}_2$ *a.e. near* Γ, *then*

$$\boldsymbol{B}\boldsymbol{v}_1 = \boldsymbol{B}\boldsymbol{v}_2\,.$$

PROOF. The formula (1.6) immediately follows from (1.5). We now prove (b). Let U be a neighborhood of Γ and assume that $\boldsymbol{v}_1 = \boldsymbol{v}_2$ a.e. on $U \cap \Omega$. Let $\hat{v}$ be an arbitrary element of $W^{1/p',p}(\Gamma)$. We can choose $v \in W^{1,p}(\Omega)$ such that $\gamma \hat{v} = \hat{v}$ and $v = 0$ a.e. on $\Omega \setminus \overline{U \cap \Omega}$. Then by (1.6)

$$\langle \boldsymbol{B}\boldsymbol{v}_1, \hat{v}\rangle_\Gamma = \int_\Omega (\operatorname{div} \boldsymbol{v}_1) v\, dx + \int_\Omega (\boldsymbol{v}_1, \nabla v)\, dx$$

$$= \int_\Omega (\operatorname{div} \boldsymbol{v}_2) v\, dx + \int_\Omega (\boldsymbol{v}_2, \nabla v)\, dx = \langle \boldsymbol{B}\boldsymbol{v}_2, \hat{v}\rangle_\Gamma\,.$$

Thus (b) holds. q.e.d.

If $v \in C^1(\bar{\Omega})$ and if $\boldsymbol{v} = \nabla v$, then $\boldsymbol{B}\boldsymbol{v} = \frac{\partial v}{\partial n}$, the outward normal derivative of v on Γ.

REMARK. Let A be an operator from $W^{1,p}(\Omega)$ into the dual space $(W^{1,p}(\Omega))^*$ defined by

$$\langle Au, v\rangle = \sum_{k=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_k} \right|^{p-2} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k}\, dx\,, \qquad \text{for all} \quad v \in W^{1,p}(\Omega)\,,$$

and let

$$S = \{v \in W^{1,p}(\Omega)\,;\ Av = 0\}\,.$$

Lions [**14**; Chapter 2] introduced a nonlinear operator $\mathfrak{T}$ from S into $W^{-1/p',p'}(\Gamma)$ such that

$$\mathfrak{T}(v) = \sum_{k=1}^N \left| \frac{\partial v}{\partial x_k} \right|^{p-2} \frac{\partial v}{\partial x_k} \nu_k \qquad \text{on} \quad \Gamma\,,$$

if $v \in S$ and $v \in C^2(\bar{\Omega})$. It is easily checked that

$$\mathcal{T}(v)=\boldsymbol{B}\boldsymbol{a}(v) \quad \text{for all} \quad v\in S,$$

where $\boldsymbol{a}(v)=(a_1(v), a_2(v), \cdots, a_N(v))$ is given by

$$a_k(v)=\left|\frac{\partial v}{\partial x_k}\right|^{p-2}\frac{\partial v}{\partial x_k}, \qquad k=1, 2, \cdots, N.$$

Let U be a neighborhood of Γ in R^N. For a function $\boldsymbol{v}=(v_1, v_2, \cdots, v_N)$ with $v_k\in L^{p'}(\Omega\cap U)$, $k=1, 2, \cdots, N$, such that $\operatorname{div}\boldsymbol{v}\in L^{p'}(\Omega\cap U)$, there exists $\tilde{\boldsymbol{v}}\in \boldsymbol{E}^{p'}(\Omega)$ such that $\boldsymbol{v}=\tilde{\boldsymbol{v}}$ a. e. near Γ. The assertion (b) of Proposition 1.4 implies that $\boldsymbol{B}\tilde{\boldsymbol{v}}$ depends only on $\boldsymbol{v}$, that is, it does not depend on the choice of such $\tilde{\boldsymbol{v}}$. Therefore we may define $\boldsymbol{B}\boldsymbol{v}$ by $\boldsymbol{B}\tilde{\boldsymbol{v}}$.

COROLLARY 1.5. *Let F be a compact subset of Ω and set*

$$K(F, 0)=\{v\in W^{1,p}(\Omega)\,;\ v=0 \text{ on } F \text{ in the sense of } W^{1,p}(\Omega)\}.$$

Then for any $v\in K(F, 0)$ and any $\boldsymbol{v}=(v_1, v_2, \cdots, v_N)\in[L^{p'}(\Omega\setminus F)]^N$ with $\operatorname{div}\boldsymbol{v}\in L^{p'}(\Omega\setminus F)$,

$$\int_{\Omega\setminus F}(\operatorname{div}\boldsymbol{v})v\,dx+\int_{\Omega\setminus F}(\boldsymbol{v}, \nabla v)dx=\langle \boldsymbol{B}\boldsymbol{v}, \gamma v\rangle_\Gamma. \tag{1.7}$$

In fact, we see that (1.7) holds for $v\in\mathcal{D}(\bar{\Omega})$ such that $v=0$ on a neighborhood V of F by considering $\tilde{\boldsymbol{v}}\in\boldsymbol{E}^{p'}(\Omega)$, which is equal to $\boldsymbol{v}$ a. e. on $\Omega\setminus V$, and applying (1.6). Since any function of $K(F, 0)$ is the strong limit of such functions v, (1.7) holds for any $v\in K(F, 0)$.

§ 2. Pseudomonotone operators in $W^{1,p}(\Omega)$.

2.1. Definition of pseudomonotone operators.

Let $1<p<\infty$ and $1/p+1/p'=1$. Let $A(x, \zeta, \xi)$ be a real-valued functions on $\Omega\times R\times R^N$. We then say that it satisfies the *Carathéodory conditions*, if for a. e. $x\in\Omega$, $A(x, \zeta, \xi)$ is continuous in (ζ, ξ) and if for all $(\zeta, \xi)\in R\times R^N$, $A(x, \zeta, \xi)$ is measurable in $x\in\Omega$. Furthermore it is said to satisfy Assumption (I), if it satisfies the Carathéodory conditions and there exist a positive constant C and a function $h(x)\in L^{p'}(\Omega)$ such that

$$|A(x, \zeta, \xi_1, \xi_2, \cdots, \xi_N)|\leqq h(x)+C(|\zeta|^{p-1}+\sum_{k=1}^N|\xi_k|^{p-1}).$$

REMARK. Assumption (I) is equivalent to the following condition: for any $v_k\in L^p(\Omega)$, $k=0, 1, \cdots, N$,

$$A(x, v_0(x), v_1(x), \cdots, v_N(x))\in L^{p'}(\Omega).$$

Moreover, Assumption (I) implies that the integral operator:

$$(v_0, v_1, \cdots, v_N)\longrightarrow A(x, v_0, v_1, \cdots, v_N)$$

is a bounded continuous operator from $[L^p(\Omega)]^{N+1}$ into $L^{p'}(\Omega)$ (cf. Krasnosel'skiĭ [**11**; Chapter 1]).

We now consider a functional $a(\cdot, \cdot)$ on $W^{1,p}(\Omega)\times W^{1,p}(\Omega)$ defined by

$$a(v, w)=\sum_{k=1}^{N}\int_{\Omega}A_k(x, v, \nabla v)\frac{\partial w}{\partial x_k}dx+\int_{\Omega}A_0(x, v, \nabla v)wdx. \tag{2.1}$$

We often say that $a(\cdot, \cdot)$ given by (2.1) satisfies Assumption (I), if every $A_j(x, \zeta, \xi)$, $j=0, 1, \cdots, N$, satisfies Assumption (I). Assume Assumption (I). Then for any fixed $v\in W^{1,p}(\Omega)$, the functional $w\to a(v, w)$ is linear and continuous on $W^{1,p}(\Omega)$. Hence there is a unique element $Tv\in(W^{1,p}(\Omega))^*$ (=the dual of $W^{1,p}(\Omega)$) such that

$$\langle Tv, w\rangle=a(v, w) \qquad \text{for all} \quad w\in W^{1,p}(\Omega). \tag{2.2}$$

Here $\langle\cdot, \cdot\rangle$ denotes the natural pairing between $(W^{1,p}(\Omega))^*$ and $W^{1,p}(\Omega)$. Thus we can define a nonlinear operator T from $W^{1,p}(\Omega)$ into $(W^{1,p}(\Omega))^*$ by (2.1) and (2.2). Furthermore we consider the following

ASSUMPTION (II). If $\{v_n\}$ is a sequence weakly convergent to v in $W^{1,p}(\Omega)$ and if

$$\limsup_{n\to\infty} a(v_n, v_n-v)\leqq 0,$$

then

$$a(v, v-w)\leqq\liminf_{n\to\infty} a(v_n, v_n-w) \qquad \text{for all} \quad w\in W^{1,p}(\Omega).$$

Here, we recall the notion of pseudomonotone operators that was originally introduced by Brezis [**1**]. Let X be a real reflexive Banach space and X^* be its dual space. Then an operator T from X into X^* is called *pseudomonotone* if the following two properties are satisfied:

(PM_1) If $\{v_n\}$ is a sequence weakly convergent to v in X and if

$$\limsup_{n\to\infty}\langle Tv_n, v_n-v\rangle\leqq 0,$$

then

$$\langle Tv, v-w\rangle\leqq\liminf_{n\to\infty}\langle Tv_n, v_n-w\rangle \qquad \text{for all} \quad w\in X.$$

(PM_2) For each $v\in X$, the functional $w\to\langle Tw, w-v\rangle$ is bounded below on each bounded subset of X.

Here, $\langle\cdot, \cdot\rangle$ denotes the natural pairing between X^* and X.

It is easy to see that Assumptions (I) and (II) imply that T defined by (2.1) and (2.2) is a bounded and pseudomonotone operator from $W^{1,p}(\Omega)$ into $(W^{1,p}(\Omega))^*$.

Finally, we mention an existence theorem for variational inequalities involving pseudomonotone operators in X.

THEOREM A (Brezis [**1**; Corollary 30]). *Let T be a bounded pseudomonotone* *operator from X into X^*, K be a closed convex subset of X and Φ be a lower semicontinuous convex function on K such that $\Phi(v)\in(-\infty,\infty]$ for all $v\in K$ and $\Phi\not\equiv\infty$ on K. Suppose that for some $v_0\in K$ with $\Phi(v_0)<\infty$*

$$\frac{\langle Tv, v-v_0\rangle+\Phi(v)}{\|v\|}\longrightarrow\infty \qquad as\quad \|v\|\to\infty,\ v\in K,$$

where $\|\cdot\|$ denotes the norm in X. Then there is $u\in K$ such that

$$\langle Tu, u-v\rangle\leqq\Phi(v)-\Phi(u) \qquad for\ all\quad v\in K.$$

If, in particular, T is strictly monotone, i.e.,

$$\langle Tv-Tw, v-w\rangle>0 \qquad for\ any\quad v, w\in X\ with\ v\neq w,$$

then the above variational problem has a unique solution in K.

2.2. Examples.

We give some simple examples of functionals satisfying Assumptions (I) and (II).

EXAMPLE 2.1. Let $2\leqq p<\infty$, $\alpha_k(x)\in L^\infty(\Omega)$, $k=1,2,\cdots,N$, $\beta(x)\in L^\infty(\Omega)$ and assume that for a positive constant C

$$\beta(x)\geqq C,\quad \alpha_k(x)\geqq C,\qquad \text{a.e. on}\quad \Omega,\ k=1,2,\cdots,N.$$

We then define a functional $a_1(\cdot,\cdot)$ on $W^{1,p}(\Omega)\times W^{1,p}(\Omega)$ by

$$a_1(v,w)=\sum_{k=1}^N\int_\Omega\alpha_k\left|\frac{\partial v}{\partial x_k}\right|^{p-2}\frac{\partial v}{\partial x_k}\frac{\partial w}{\partial x_k}dx+\int_\Omega\beta|v|^{p-2}vwdx.$$

It is easy to check Assumptions (I) and (II). Thus the operator T_1 given by

$$\langle T_1v, w\rangle=a_1(v,w)$$

is a continuous, bounded and pseudomonotone operator from $W^{1,p}(\Omega)$ into $(W^{1,p}(\Omega))^*$. Furthermore, it is strictly monotone and for each $v_0\in W^{1,p}(\Omega)$,

$$\frac{a_1(v, v-v_0)}{\|v\|_{1,p}}\longrightarrow\infty \qquad \text{as}\quad \|v\|_{1,p}\to\infty.$$

EXAMPLE 2.2. Let $2\leqq p<\infty$ and $\alpha(x)$, $\beta(x)\in L^\infty(\Omega)$ such that for a positive constant C, $\alpha(x)\geqq C$ and $\beta(x)\geqq C$ a.e. on Ω. Set

$$a_2(v,w)=\sum_{k=1}^N\int_\Omega\alpha|\nabla v|^{p-2}\frac{\partial v}{\partial x_k}\frac{\partial w}{\partial x_k}dx+\int_\Omega\beta|v|^{p-2}vwdx \qquad \text{for}\quad v, w\in W^{1,p}(\Omega),$$

where $|\nabla v|=\left(\sum_{k=1}^N\left|\frac{\partial v}{\partial x_k}\right|^2\right)^{1/2}$. This functional also has the same properties as $a_1(\cdot,\cdot)$.

EXAMPLE 2.3. Let $3\leqq\sigma<p<\infty$ and $\beta_k(x)\in L^\infty(\Omega)$, $k=1,2,\cdots,N$. We define

$$b(v, w)=\sum_{k=1}^{N}\int_{\Omega}\beta_k|v|^{\sigma-3}v\frac{\partial v}{\partial x_k}wdx \qquad \text{for} \quad w\in W^{1,p}(\Omega),$$

and $a_3(v, w)=a_1(v, w)+b(v, w)$ or $a_2(v, w)+b(v, w)$. Functions

$$A_0(x, \zeta, \xi)=\beta|\zeta|^{p-2}\zeta$$

and

$$A_j(x, \zeta, \xi)=\alpha_j|\xi_j|^{p-2}\xi_j+\beta_j|\zeta|^{\sigma-3}\zeta\xi_j$$

or

$$\alpha(\sum_{k=1}^{N}|\xi_k|^2)^{\frac{p-2}{2}}\xi_j+\beta_j|\zeta|^{\sigma-3}\zeta\xi_j, \qquad j=1, 2, \cdots, N,$$

satisfy Assumption (I). Let $\{v_n\}$ be a sequence in $W^{1,p}(\Omega)$ such that $v_n\overset{w}{\to}v$ in $W^{1,p}(\Omega)$ ("$\overset{w}{\to}$" means the weak convergence). Then, noting that the natural injection of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact (see [**13**; Theorem 7.1 in Chapter 2]), we see that $v_n\overset{s}{\to}v$ in $L^p(\Omega)$. Since $\frac{\partial v_n}{\partial x_k}\overset{w}{\to}\frac{\partial v}{\partial x_k}$ in $L^p(\Omega)$, $k=1, 2, \cdots, N$, it follows that $b(v_n, v_n-w)\to b(v, v-w)$ for all $w\in W^{1,p}(\Omega)$, and hence, if

$$\limsup_{n\to\infty} a_3(v_n, v_n-v)\leqq 0,$$

then

$$\limsup_{n\to\infty} a_1(v_n, v_n-v)\leqq 0.$$

Since $a_1(\cdot, \cdot)$ satisfies Assumption (II), so does $a_3(\cdot, \cdot)$. Moreover, we see that for each $v_0\in W^{1,p}(\Omega)$ there are positive constants δ_i, $i=1, 2, 3, 4$, such that for any $v\in W^{1,p}(\Omega)$

$$a_3(v, v-v_0)\geqq\delta_1\|v\|_{1,p}^{p}-\delta_2\|v\|_{1,p}^{\sigma}-\delta_3\|v\|_{1,p}^{p-2}-\delta_4\|v\|_{1,p}^{\sigma-1}.$$

This implies that

$$\frac{a_3(v, v-v_0)}{\|v\|_{1,p}}\longrightarrow\infty \qquad \text{as} \quad \|v\|_{1,p}\to\infty.$$

§ 3. Boundary value problems of mixed type.

3.1. Existence theorem.

Let $1<p<\infty$, $1/p+1/p'=1$, Γ_0 be a compact subset of Γ and F be a compact subset of Ω, and let Φ be a lower semicontinuous convex function on $W^{1/p',p}(\Gamma)$ with values in $(-\infty, \infty]$. The subdifferential $\partial\Phi$[1] of Φ is a (possibly multivalued) operator from $W^{1/p',p}(\Gamma)$ into $W^{-1/p',p'}(\Gamma)$. For functions $w\in W^{1,p}(\Omega)$ and $\phi\in W^{1/p',p}(\Gamma)$, we put

1) Let Φ be a lower semicontinuous convex function on a Banach space X. Then the multivalued operator $\partial\Phi$ from X into X^* defined by $\partial\Phi(u)=\{u^*\in X^*; \langle u^*, w-u\rangle\leqq\Phi(w)-\Phi(u)$ for all $w\in X\}$ for $u\in X$ with $\Phi(u)<\infty$ and by $\partial\Phi(u)=\emptyset$ for $u\in X$ with $\Phi(u)=\infty$ is called the *subdifferential* of Φ.

$$K(\Gamma_0, \psi; F, w)=\{v\in W^{1,p}(\Omega);\ \gamma v=\psi \text{ a.e. on } \Gamma_0 \text{ and } v=w \text{ on } F \text{ in the sense of } W^{1,p}(\Omega)\}$$

and

$$\hat{K}(\Gamma_0, \psi; F, w)=\{\hat{v}=\gamma v;\ v\in K(\Gamma_0, \psi; F, w)\}.$$

Clearly, the sets $K(\Gamma_0, \psi; F, w)$ and $\hat{K}(\Gamma_0, \psi; F, w)$ are closed and convex in $W^{1,p}(\Omega)$ and in $W^{1/p',p}(\Gamma)$, respectively.

Let $A_j(x, \zeta, \xi)$, $j=0, 1, \cdots, N$, be integral operators satisfying Assumption (I) and let $a(\cdot, \cdot)$ be the functional given by (2.1). Let $w\in W^{1,p}(\Omega)$, $\psi\in W^{1/p',p}(\Gamma)$, $\psi^*\in W^{-1/p',p'}(\Gamma)$ and $f\in L^{p'}(\Omega)$ be given. We suppose that

$$\Phi\not\equiv\infty \quad \text{on} \quad \hat{K}(\Gamma_0, \psi; F, w)$$

and

$$\Phi\equiv\infty \quad \text{on} \quad W^{1/p',p}(\Gamma)\setminus\hat{K}(\Gamma_0, \psi; F, w).$$

Then we propose the following problem $P_1[\Phi, w, \psi, \psi^*, f]$: Find $u\in W^{1,p}(\Omega)$ such that

$$-\sum_{k=1}^{N}\frac{\partial}{\partial x_k}A_k(x, u, \nabla u)+A_0(x, u, \nabla u)=f \quad \text{in} \quad \Omega\setminus F \quad \text{(in the distribution sense)}, \tag{3.1}$$

$$\begin{cases} \gamma u=\psi & \text{a.e. on } \Gamma_0, \\ u=w & \text{on } F \text{ in the sense of } W^{1,p}(\Omega), \end{cases} \tag{3.2}$$

$$-\boldsymbol{B}\boldsymbol{a}(u)+\psi^*\in\partial\Phi(\gamma u), \tag{3.3}$$

where $\boldsymbol{a}(u)=(a_1(u), a_2(u), \cdots, a_N(u))$, $a_k(u)=A_k(x, u, \nabla u)$, $k=1, 2, \cdots, N$. We note that $\boldsymbol{B}\boldsymbol{a}(u)$ makes sense, since $\boldsymbol{a}(u)\in[L^{p'}(\Omega)]^N$ and $\operatorname{div}\boldsymbol{a}(u)\in L^{p'}(\Omega\setminus F)$ by Assumption (I) and (3.1).

We first show that the above problem $P_1[\Phi, w, \psi, \psi^*, f]$ is equivalent to the following variational inequality $V_1[\Phi, w, \psi, \psi^*, f]$: Find $u\in W^{1,p}(\Omega)$ such that

$$u\in K(\Gamma_0, \psi; F, w), \tag{3.4}$$

$$a(u, u-v)-\int_{\Omega}f(u-v)dx\leqq\langle\psi^*, \gamma u-\gamma v\rangle_{\Gamma}+\Phi(\gamma v)-\Phi(\gamma u) \quad \text{for all} \quad v\in K(\Gamma_0, \psi; F, w). \tag{3.5}$$

PROPOSITION 3.1. *Problems* $P_1[\Phi, w, \psi, \psi^*, f]$ *and* $V_1[\Phi, w, \psi, \psi^*, f]$ *are equivalent to each other.*

PROOF. First we assume that $u\in W^{1,p}(\Omega)$ is a solution of $V_1[\Phi, w, \psi, \psi^*, f]$. Then u satisfies (3.2) because of (3.4). Substituting $u\pm\phi$ with $\phi\in\mathcal{D}(\Omega\setminus F)$ for v in (3.5), we have

$$a(u, \phi)-\int_{\Omega\setminus F}f\phi dx=0.$$

This means that (3.1) holds. Next we show (3.3). We note that

$$\operatorname{div} \boldsymbol{a}(u) \in L^{p'}(\Omega \setminus F),$$

which follows from (3.1). Therefore we can apply the formula (1.7) for $\boldsymbol{a}(u)$ and any $v \in K(F, 0)$ and have

$$\int_{\Omega \setminus F} (\operatorname{div} \boldsymbol{a}(u)) v dx + \int_{\Omega \setminus F} (\boldsymbol{a}(u), \nabla v) dx = \langle \boldsymbol{B}\boldsymbol{a}(u), \gamma v \rangle_{\Gamma}. \tag{3.6}$$

Using (3.1), (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}\Phi(\gamma v) - \Phi(\gamma u) \geqq & \int_{\Omega \setminus F} (\boldsymbol{a}(u), \nabla(u-v)) dx + \int_{\Omega \setminus F} A_0(x, u, \nabla u)(u-v) dx \\ & - \int_{\Omega \setminus F} f(u-v) dx - \langle \psi^*, \gamma u - \gamma v \rangle_{\Gamma} \\ = & - \int_{\Omega \setminus F} (\operatorname{div} \boldsymbol{a}(u))(u-v) dx + \int_{\Omega \setminus F} A_0(x, u, \nabla u)(u-v) dx \\ & - \int_{\Omega \setminus F} f(u-v) dx + \langle \boldsymbol{B}\boldsymbol{a}(u) - \psi^*, \gamma u - \gamma v \rangle_{\Gamma} \\ = & \langle \boldsymbol{B}\boldsymbol{a}(u) - \psi^*, \gamma u - \gamma v \rangle_{\Gamma} \qquad \text{for all} \quad v \in K(\Gamma_0, \psi; F, w).\end{aligned}$$

Since $\Phi \equiv \infty$ on $W^{1/p',p}(\Gamma) \setminus \hat{K}(\Gamma_0, \psi; F, w)$, we have (3.3) by the definition of $\partial\Phi$.

Conversely, let u be a solution of $P_1[\Phi, w, \psi, \psi^*, f]$. (3.2) shows that (3.4) holds. From (3.1) it follows that $\operatorname{div} \boldsymbol{a}(u) \in L^{p'}(\Omega \setminus F)$, so that as above we have (3.6). We derive from (3.3) together with (3.1) and (3.6) that

$$\begin{aligned}\Phi(\gamma v) - \Phi(\gamma u) \geqq & \langle \boldsymbol{B}\boldsymbol{a}(u), \gamma u - \gamma v \rangle_{\Gamma} - \langle \psi^*, \gamma u - \gamma v \rangle_{\Gamma} \\ = & \int_{\Omega \setminus F} (\operatorname{div} \boldsymbol{a}(u))(u-v) dx + \int_{\Omega \setminus F} (\boldsymbol{a}(u), \nabla(u-v)) dx \\ & - \langle \psi^*, \gamma u - \gamma v \rangle_{\Gamma} \\ = & - \int_{\Omega \setminus F} f(u-v) dx + \int_{\Omega \setminus F} A_0(x, u, \nabla u)(u-v) dx \\ & + \int_{\Omega \setminus F} (\boldsymbol{a}(u), \nabla(u-v)) dx - \langle \psi^*, \gamma u - \gamma v \rangle_{\Gamma} \\ = & a(u, u-v) - \int_{\Omega \setminus F} f(u-v) dx - \langle \psi^*, \gamma u - \gamma v \rangle_{\Gamma} \\ & \text{for all} \quad v \in K(\Gamma_0, \psi; F, w).\end{aligned}$$

Thus we have (3.5). q. e. d.

The following existence theorem for $P_1[\Phi, w, \psi, \psi^*, f]$ is a consequence of Theorem A in paragraph 2.1 and the above proposition.

THEOREM 3.2. *In addition to the assumptions so far made, we suppose* *that the functional* $a(\cdot,\cdot)$ *satisfies Assumption (II) and for some* $v_0 \in K(\Gamma_0, \psi; F, w)$ *with* $\Phi(\gamma v_0) < \infty$

$$\frac{a(v, v-v_0)+\Phi(\gamma v)}{\|v\|_{1,p}} \longrightarrow \infty \quad \text{as } \|v\|_{1,p}\to\infty,\ v\in K(\Gamma_0, \psi; F, w). \tag{3.7}$$

Then the problem $P_1[\Phi, w, \psi, \psi^*, f]$ *admits a solution.*

PROOF. We define an operator $T: W^{1,p}(\Omega)\to(W^{1,p}(\Omega))^*$ by

$$\langle Tv, z\rangle = a(v, z)-\int_\Omega fzdx-\langle\psi^*, \gamma z\rangle_\Gamma .$$

Since $a(\cdot,\cdot)$ satisfies Assumption (II), we see that T is bounded and pseudo-monotone. A function $\tilde{\Phi}$ on $K(\Gamma_0, \psi; F, w)$ given by $\tilde{\Phi}(v)=\Phi(\gamma v)$ is convex, $\not\equiv\infty$, and lower semicontinuous on $K(\Gamma_0, \psi; F, w)$. Besides, it follows from (3.7) that for the same v_0 as in (3.7)

$$\frac{\langle Tv, v-v_0\rangle+\tilde{\Phi}(v)}{\|v\|_{1,p}} \longrightarrow \infty \quad \text{as } \|v\|_{1,p}\to\infty,\ v\in K(\Gamma_0, \psi; F, w).$$

Therefore, by Theorem A, the variational inequality

$$\langle Tu, u-v\rangle \leqq \tilde{\Phi}(v)-\tilde{\Phi}(u) \quad \text{for all } v\in K(\Gamma_0, \psi; F, w),$$

or equivalently, $V_1[\Phi, w, \psi, \psi^*, f]$ has a solution u. By Proposition 3.1, this function u is also a solution of $P_1[\Phi, w, \psi, \psi^*, f]$. q. e. d.

3.2. Special cases.

We now state some special cases of Theorem 3.2.

CASE 1. In case $\Gamma_0=\Gamma$ and $\psi=0$, note that $\hat{K}(\Gamma, 0; F, w)=\{0\}$ and $\partial\Phi(0) = W^{-1/p',p'}(\Gamma)$. Hence, Theorem 3.2 shows that

$$\begin{cases} -\sum_{k=1}^{N}\dfrac{\partial}{\partial x_k}A_k(x, u, \nabla u)+A_0(x, u, \nabla u)=f & \text{in } \Omega\setminus F, \\ \gamma u=0 & \text{a. e. on } \Gamma, \\ u=w & \text{on } F \text{ in the sense of } W^{1,p}(\Omega) \end{cases}$$

has a solution u in $W^{1,p}(\Omega)$, provided that for some $v_0\in K(\Gamma, 0; F, w)$

$$\frac{a(v, v-v_0)}{\|v\|_{1,p}} \longrightarrow \infty \quad \text{as } \|v\|_{1,p}\to\infty,\ v\in K(\Gamma, 0; F, w).$$

CASE 2. We consider the case in which $F=\emptyset$ and $\Gamma_0=\Gamma$. In this case,

$$K(\Gamma, \psi; \emptyset, w)=K(\Gamma, \psi)=\{v\in W^{1,p}(\Omega);\ \gamma v=\psi \text{ a. e. on } \Gamma\}$$

and $\partial\Phi(\psi)=W^{-1/p',p'}(\Gamma)$. Suppose that

$$\frac{a(v, v-v_0)}{\|v\|_{1,p}} \longrightarrow \infty \quad \text{as } \|v\|_{1,p}\to\infty,\ v\in K(\Gamma, \psi).$$

Then, applying Theorem 3.2, we see that the Dirichlet problem

$$\begin{cases} -\sum_{k=1}^{N} \frac{\partial}{\partial x_k} A_k(x, u, \nabla u) + A_0(x, u, \nabla u) = f & \text{in } \Omega, \\ \gamma u = \psi \quad \text{a.e. on } \Gamma \end{cases}$$

has a solution in $W^{1,p}(\Omega)$.

CASE 3. Take $\Gamma_0 = \emptyset$ and $F = \emptyset$ in Theorem 3.2. Then, clearly, $K(\emptyset, \psi; \emptyset, w) = W^{1,p}(\Omega)$. If $\Phi \equiv 0$ on $W^{1/p',p}(\Gamma)$, then $\partial\Phi(\hat{v}) = \{0\}$ for every $\hat{v} \in W^{1/p',p}(\Gamma)$. Hence, under the assumption that

$$\frac{a(v, v)}{\|v\|_{1,p}} \longrightarrow \infty \quad \text{as } \|v\|_{1,p} \to \infty,$$

the problem of the Neumann type:

$$\begin{cases} -\sum_{k=1}^{N} \frac{\partial}{\partial x_k} A_k(x, u, \nabla u) + A_0(x, u, \nabla u) = f & \text{in } \Omega \\ \boldsymbol{Ba}(u) = \psi^* \end{cases}$$

has a solution in $W^{1,p}(\Omega)$.

CASE 4. Let $j(r)$ be a continuous and convex function on R such that $j \circ \hat{v} \in L^1(\Gamma)$ for every $\hat{v} \in L^p(\Gamma)$. Suppose that j is bounded below and the function $\tilde{\Phi}$ on $L^p(\Gamma)$ given by

$$\tilde{\Phi}(\hat{v}) = \int_{\Gamma} j(\hat{v}) d\Gamma \quad \text{for } \hat{v} \in L^p(\Gamma)$$

is continuous. Then, the subdifferential l of j is given by

$$l(r) = [j'_-(r), j'_+(r)] \quad \text{for every } r \in R$$

(cf. Rockafellar [**20**; § 24]), where j'_- and j'_+ denote the left and right derivatives of j, respectively. It is clear that $\tilde{\Phi}$ is convex and bounded below on $L^p(\Gamma)$. Moreover, $\hat{v}^* \in \partial\tilde{\Phi}(\hat{v})$ if and only if $\hat{v}^* \in L^{p'}(\Gamma)$ and $\hat{v}^* \in l \circ \hat{v}$ a.e. on Γ (see Brezis [**2**; Appendix I]). Now, we take $F = \emptyset$, $\Gamma_0 = \emptyset$ and $\Phi = \tilde{\Phi}|_{W^{1/p',p}(\Gamma)}$ in Theorem 3.2. In this case, (3.3) is written in the following form:

$$-\boldsymbol{Ba}(u) + \psi^* \in l(\gamma u) \quad \text{a.e. on } \Gamma. \tag{3.8}$$

In fact, by (3.3),

$$\langle -\boldsymbol{Ba}(u) + \psi^*, \hat{v} - \gamma u \rangle_\Gamma \leqq \Phi(\hat{v}) - \Phi(\gamma u) \quad \text{for all } \hat{v} \in W^{1/p',p}(\Gamma).$$

Substituting $\gamma u \pm \hat{w}$ with $\hat{w} \in W^{1/p',p}(\Gamma)$ for $\hat{v}$ in the above inequality, we have

$$\begin{aligned} \tilde{\Phi}(\gamma u) - \tilde{\Phi}(\gamma u - \hat{w}) &\leqq \langle -\boldsymbol{Ba}(u) + \psi^*, \hat{w} \rangle_\Gamma \\ &\leqq \tilde{\Phi}(\gamma u + \hat{w}) - \tilde{\Phi}(\gamma u) \quad \text{for all } \hat{w} \in W^{1/p',p}(\Gamma). \end{aligned}$$

Since $W^{1/p',p}(\Gamma)$ is dense in $L^p(\Gamma)$, the continuity of $\tilde{\Phi}$ on $L^p(\Gamma)$ and the above inequality imply that

$$-\boldsymbol{Ba}(u)+\psi^* \in L^{p'}(\Gamma)$$

and

$$\int_\Gamma(-\boldsymbol{Ba}(u)+\psi^*)(\hat{v}-\gamma u)d\Gamma \leqq \tilde{\Phi}(\hat{v})-\tilde{\Phi}(\gamma u) \qquad \text{for all} \quad \hat{v}\in L^p(\Gamma),$$

that is,

$$-\boldsymbol{Ba}(u)+\psi^* \in \partial\tilde{\Phi}(\gamma u),$$

and so (3.8) holds.

For example, if $j(r)=|r|$, then

$$l(r)=\begin{cases} 1 & \text{if} \quad r>0, \\ [-1, 1] & \text{if} \quad r=0, \\ -1 & \text{if} \quad r<0. \end{cases}$$

In this case, (3.8) is of the form

$$\begin{cases} -\boldsymbol{Ba}(u)+\psi^*=1 & \text{a. e. on} \quad \{x'\in\Gamma\,;\,\gamma u(x')>0\}, \\ -1\leqq -\boldsymbol{Ba}(u)+\psi^*\leqq 1 & \text{a. e. on} \quad \{x'\in\Gamma\,;\,\gamma u(x')=0\}, \\ -\boldsymbol{Ba}(u)+\psi^*=-1 & \text{a. e. on} \quad \{x'\in\Gamma\,;\,\gamma u(x')<0\} \end{cases}$$

(cf. [**2**; § 1.2]).

CASE 5. Let $2\leqq p<\infty$, $g\in L^\infty(\Gamma)$ with $g\geqq 0$ a. e. on Γ. Define $\tilde{\Phi}$ by

$$\tilde{\Phi}(\hat{v})=\frac{1}{p}\int_\Gamma g|\hat{v}|^p d\Gamma \qquad \text{for} \quad \hat{v}\in L^p(\Gamma).$$

Then it is easy to see that $\tilde{\Phi}$ is everywhere differentiable in the sense of Fréchet and the derivative coincides with the subdifferential $\partial\tilde{\Phi}$ of $\tilde{\Phi}$, that is,

$$\partial\tilde{\Phi}(\hat{v})=g|\hat{v}|^{p-2}\hat{v} \qquad \text{for each} \quad \hat{v}\in L^p(\Gamma).$$

In Theorem 3.2, let $F=\emptyset$. Then

$$K(\Gamma_0, \psi\,;\,\emptyset, w)=K(\Gamma_0, \psi)=\{v\in W^{1,p}(\Omega)\,;\,\gamma v=\psi \text{ a. e. on } \Gamma_0\}.$$

If we put

$$\Phi(\hat{v})=\begin{cases} \tilde{\Phi}(\hat{v}) & \text{if} \quad \hat{v}\in\hat{K}(\Gamma_0, \psi)=\{\hat{v}=\gamma v\,;\,v\in K(\Gamma_0, \psi)\}, \\ \infty & \text{otherwise}, \end{cases} \tag{3.9}$$

then (3.3) is written in the following form:

$$-\boldsymbol{Ba}(u)+\psi^*=g|\gamma u|^{p-2}(\gamma u) \qquad \text{on} \quad \Gamma\setminus\Gamma_0 \tag{3.10}$$

(in the distribution sense).

In fact, by (3.3),

$$\langle \boldsymbol{Ba}(u)-\psi^*, \gamma u-\hat{v}\rangle_\Gamma \leqq \Phi(\hat{v})-\Phi(\gamma u) \qquad \text{for all} \quad \hat{v}\in\hat{K}(\Gamma_0, \psi).$$

Taking $\hat{v}=\gamma u+t\hat{w}$ with $0<t<1$ and $\hat{w}\in\mathscr{D}(\Gamma\setminus\Gamma_0)$ in the above inequality, we obtain

$$\langle -\boldsymbol{Ba}(u)+\psi^*,\ \hat{w}\rangle_\Gamma \leqq \frac{1}{t}(\tilde{\Phi}(\gamma u+t\hat{w})-\tilde{\Phi}(\gamma u)) .$$

Hence we have by letting $t\downarrow 0$

$$\langle -\boldsymbol{Ba}(u)+\psi^*,\ \hat{w}\rangle_\Gamma \leqq \int_\Gamma g|\gamma u|^{p-2}(\gamma u)\hat{w}d\Gamma .$$

This implies that

$$\langle -\boldsymbol{Ba}(u)+\psi^*,\ \hat{w}\rangle_\Gamma = \int_\Gamma g|\gamma u|^{p-2}(\gamma u)\hat{w}d\Gamma \qquad \text{for all} \quad \hat{w}\in\mathcal{D}(\Gamma\diagdown\Gamma_0) .$$

Thus we have (3.10).

§ 4. Unilateral boundary value problems.

Let $1<p<\infty$ and $1/p+1/p'=1$. Let Γ_0 be a compact subset of Γ, F be a compact subset of Ω, $w\in W^{1,p}(\Omega)$ and $\psi\in W^{1/p',p}(\Gamma)$. Then we put

$$C(\Gamma_0, \psi\,;F, w)=\{v\in W^{1,p}(\Omega)\,;\gamma v\geqq\psi \text{ on } \Gamma_0 \text{ in the sense of } W^{1/p',p}(\Gamma),\ v=w \text{ on } F \text{ in the sense of } W^{1,p}(\Omega)\}$$

and

$$\hat{C}(\Gamma_0, \psi\,;F, w)=\{\hat{v}=\gamma v\,;\ v\in C(\Gamma_0, \psi\,;F, w)\} .$$

Obviously, $C(\Gamma_0, \psi\,;F, w)$ (resp. $\hat{C}(\Gamma_0, \psi\,;F, w)$) is closed and convex in $W^{1,p}(\Omega)$ (resp. $W^{1/p',p}(\Gamma)$).

Let $A_j(x, \zeta, \xi)$, $j=0, 1, \cdots, N$, be integral operators satisfying Assumption (I). Then, given $\psi\in W^{1/p',p}(\Gamma)$, $\psi^*\in W^{-1/p',p'}(\Gamma)$, $w\in W^{1,p}(\Omega)$ and $f\in L^{p'}(\Omega)$, we pose the following problem $P_2[w, \psi, \psi^*, f]$: Find $u\in W^{1,p}(\Omega)$ such that

(4.1) $$-\sum_{k=1}^N \frac{\partial}{\partial x_k}A_k(x, u, \nabla u)+A_0(x, u, \nabla u)=f \quad \text{in} \quad \Omega\diagdown F$$
(in the distribution sense),

(4.2) $$\begin{cases} \gamma u\geqq\psi & \text{on } \Gamma_0 \text{ in the sense of } W^{1/p',p}(\Gamma), \\ u=w & \text{on } F \text{ in the sense of } W^{1,p}(\Omega), \end{cases}$$

(4.3) $$\boldsymbol{Ba}(u)\geqq\psi^* \quad \text{on } \Gamma \text{ (in the distribution sense)},$$

(4.4) $$\boldsymbol{Ba}(u)=\psi^* \quad \text{on } \Gamma\diagdown\Gamma_0 \text{ (in the distribution sense)},$$

(4.5) $$\langle \boldsymbol{Ba}(u)-\psi^*,\ \gamma u-\psi\rangle_\Gamma=0 ,$$

where $\boldsymbol{a}(u)$ is as in the previous section. Again $\boldsymbol{Ba}(u)$ makes sense, since $\boldsymbol{a}(u)\in[L^{p'}(\Omega)]^N$ and div $\boldsymbol{a}(u)\in L^{p'}(\Omega\diagdown F)$ by Assumption (I) and (4.1).

REMARK 1. The set of conditions (4.3), (4.4) and (4.5) may be formally written in the following form

$$\begin{cases} \sum_{k=1}^{N} A_k(x, u, \nabla u)\nu_k \geqq \psi^* & \text{on } \Gamma, \\ \sum_{k=1}^{N} A_k(x, u, \nabla u)\nu_k = \psi^* & \text{on } \Gamma\setminus\Gamma_0, \\ (\sum_{k=1}^{N} A_k(x, u, \nabla u)\nu_k - \psi^*)(u-\psi) = 0 & \text{on } \Gamma. \end{cases}$$

Let $a(\cdot, \cdot)$ be the functional on $W^{1,p}(\Omega)\times W^{1,p}(\Omega)$ given by (2.1). Then the variational inequality $V_2[w, \psi, \psi^*, f]$ associated with the above problem is of the following form: Find $u\in W^{1,p}(\Omega)$ such that

$$u\in C(\Gamma_0, \psi; F, w), \tag{4.6}$$

$$a(u, u-v)-\int_{\Omega} f(u-v)dx \leqq \langle\psi^*, \gamma u-\gamma v\rangle_{\Gamma} \quad \text{for all } v\in C(\Gamma_0, \psi; F, w). \tag{4.7}$$

Now, we show

PROPOSITION 4.1. *Problems $P_2[w, \psi, \psi^*, f]$ and $V_2[w, \psi, \psi^*, f]$ are equivalent to each other.*

PROOF. We first prove that a solution u of $V_2[w, \psi, \psi^*, f]$ satisfies (4.1) $\sim$(4.5). In fact, substituting $u\pm\phi$ with $\phi\in\mathcal{D}(\Omega\setminus F)$ for v in (4.7), we obtain (4.1), and hence we see that

$$\operatorname{div} \boldsymbol{a}(u)\in L^{p'}(\Omega\setminus F).$$

Therefore we can use the formula (1.7) with $\boldsymbol{v}=\boldsymbol{a}(u)$ and $v\in K(F, 0)$. Now, choose $\tilde{v}\in C(\Gamma_0, \psi; F, w)$ with $\gamma\tilde{v}=\psi$. Substituting $2u-\tilde{v}$ for v in (4.7), we have

$$a(u, u-\tilde{v})\geqq \int_{\Omega} f(u-\tilde{v})dx+\langle\psi^*, \gamma u-\gamma\tilde{v}\rangle_{\Gamma}.$$

Combining (4.7) with $v=\tilde{v}$, we have

$$a(u, u-\tilde{v})=\int_{\Omega} f(u-\tilde{v})dx+\langle\psi^*, \gamma u-\gamma\tilde{v}\rangle_{\Gamma}. \tag{4.8}$$

On the other hand, by the formula (1.7) and the equation (4.1),

$$\langle \boldsymbol{Ba}(u), \gamma v\rangle_{\Gamma}=a(u, v)-\int_{\Omega} fv dx \quad \text{for all } v\in K(F, 0). \tag{4.9}$$

Letting $v=u-\tilde{v}$ in (4.9) and making use of (4.8), we have

$$\langle \boldsymbol{Ba}(u), \gamma u-\gamma\tilde{v}\rangle_{\Gamma}=\langle\psi^*, \gamma u-\gamma\tilde{v}\rangle_{\Gamma}.$$

Hence (4.5) holds. Next, let $\hat{\phi}$ be any function in $\mathcal{D}(\Gamma)$ and ϕ be a function in $\mathcal{D}(\bar{\Omega})$ such that $\gamma\phi=\hat{\phi}$ and $\phi=0$ on a neighborhood of F. If $\hat{\phi}\geqq 0$ on Γ, then $u+\phi\in C(\Gamma_0, \psi; F, w)$ and so it follows from (4.7) and (4.9) that $\langle \boldsymbol{Ba}(u)$

$-\psi^*, \hat{\phi}\rangle_\Gamma \geqq 0$. Hence we have (4.3). If $\hat{\phi} \in \mathcal{D}(\Gamma \setminus \Gamma_0)$, then $u \pm \phi \in C(\Gamma_0, \psi; F, w)$ and so it follows from (4.7) and (4.9) again that

$$\langle \boldsymbol{Ba}(u) - \psi^*, \hat{\phi}\rangle_\Gamma = 0 .$$

Thus (4.4) is proved. Finally, (4.2) follows directly from (4.6).

Conversely, assume that u is a solution of $P_2[w, \psi, \psi^*, f]$. It is enough to prove only (4.7). We first observe that

$$\text{(4.10)} \qquad \langle \boldsymbol{Ba}(u) - \psi^*, \psi - \gamma v\rangle_\Gamma \leqq 0 \qquad \text{for all} \quad v \in C(\Gamma_0, \psi; F, w) .$$

Indeed, for each $v \in C(\Gamma_0, \psi; F, w)$ there is a sequence $\{\phi_k\} \subset \mathcal{D}(\bar{\Omega})$ such that $\gamma\phi_k \geqq 0$ on a neighborhood of Γ_0 and $\gamma\phi_k \overset{s}{\to} \gamma v - \psi$ in $W^{1/p',p}(\Gamma)$ as $k \to \infty$. Hence, noting that

$$\langle \boldsymbol{Ba}(u) - \psi^*, \gamma\phi_k\rangle_\Gamma \geqq 0 \qquad \text{for all} \quad k$$

by (4.3) and (4.4), we get

$$\langle \boldsymbol{Ba}(u) - \psi^*, \gamma v - \psi\rangle_\Gamma = \lim_{k\to\infty} \langle \boldsymbol{Ba}(u) - \psi^*, \gamma\phi_k\rangle_\Gamma \geqq 0 .$$

Thus we have (4.10). Next, take $\tilde{v} \in C(\Gamma_0, \psi; F, w)$ with $\gamma\tilde{v} = \psi$. Then, since (4.9) holds, we see from (4.5) that

$$\text{(4.11)} \qquad \begin{aligned} 0 &= \langle \boldsymbol{Ba}(u) - \psi^*, \gamma u - \psi\rangle_\Gamma \\ &= a(u, u - \tilde{v}) - \int_\Omega f(u - \tilde{v})dx - \langle \psi^*, \gamma u - \psi\rangle_\Gamma . \end{aligned}$$

Hence, by (4.10) and using (4.9) again, we have for any $v \in C(\Gamma_0, \psi; F, w)$

$$\text{(4.12)} \qquad \begin{aligned} 0 &\geqq \langle \boldsymbol{Ba}(u) - \psi^*, \psi - \gamma v\rangle_\Gamma \\ &= a(u, \tilde{v} - v) - \int_\Omega f(\tilde{v} - v)dx - \langle \psi^*, \psi - \gamma v\rangle_\Gamma . \end{aligned}$$

Combining (4.11) and (4.12), we obtain (4.7). q. e. d.

From Theorem A and the above proposition we get the following existence theorem for the problem $P_2[w, \psi, \psi^*, f]$.

THEOREM 4.2. *If, in addition to the above assumptions, we suppose that the functional* $a(\cdot, \cdot)$ *satisfies Assumption (II) and that for some* $v_0 \in C(\Gamma_0, \psi; F, w)$

$$\frac{a(v, v - v_0)}{\|v\|_{1,p}} \longrightarrow \infty \qquad \textit{as} \quad \|v\|_{1,p} \to \infty, \ v \in C(\Gamma_0, \psi; F, w) ,$$

then $P_2[w, \psi, \psi^*, f]$ *admits a solution.*

In fact, applying Theorem A for $K = C(\Gamma_0, \psi; F, w)$, $\Phi \equiv 0$ and T given by $\langle Tu, v\rangle = a(u, v) - \int_\Omega fv dx - \langle \psi^*, \gamma v\rangle_\Gamma$, we see that $V_2[w, \psi, \psi^*, f]$ has a solution, and hence so does $P_2[w, \psi, \psi^*, f]$.

REMARK 2. In the parabolic case, analogues of Theorems 3.2 and 4.2 are valid under further restrictions on $A_j(x, \zeta, \xi)$; certain nonlinear parabolic partial differential equations of the form

$$\frac{\partial u}{\partial t}-\sum_{k=1}^{N}\frac{\partial}{\partial x_k}A_k(x, u, \nabla u)+A_0(x, u, \nabla u)=f \quad \text{in} \quad (0, T)\times\Omega$$

with initial condition and with boundary conditions of the same types as in the problems treated above are equivalent to initial-value problems for the parabolic variational inequalities associated with them (see [**9**], [**10**]).

EXAMPLE 4.3. Let $2\leqq p<\infty$ and let us consider the case where $F=\emptyset$, $\Gamma_0=\Gamma$, $\phi=0$ and $\phi^*=0$. Then, in view of the above theorem, given $f\in L^{p'}(\Omega)$, the problem

$$\begin{cases} -\sum_{k=1}^{N}\frac{\partial}{\partial x_k}\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^{p-2}\frac{\partial u}{\partial x_k}\Big)+|u|^{p-2}u=f & \text{in } \Omega\,, \\ \gamma u\geqq 0 & \text{a. e. on } \Gamma\,, \\ \boldsymbol{Ba}(u)\geqq 0 & \text{on } \Gamma\,, \\ \langle \boldsymbol{Ba}(u), \gamma u\rangle_{\Gamma}=0 & \end{cases}$$

has a solution in $W^{1,p}(\Omega)$ (cf. [**15**; Chapter 1]), where

$$\boldsymbol{a}(u)=\Big(\Big|\frac{\partial u}{\partial x_1}\Big|^{p-2}\frac{\partial u}{\partial x_1}, \Big|\frac{\partial u}{\partial x_2}\Big|^{p-2}\frac{\partial u}{\partial x_2}, \cdots, \Big|\frac{\partial u}{\partial x_N}\Big|^{p-2}\frac{\partial u}{\partial x_N}\Big).$$

§ 5. Some results on convergence of sets and of functions.

In this section, let $1<p<\infty$ and $1/p+1/p'=1$. We use the same notations as in the preceding sections; for a compact subset Γ_0 of Γ, a compact subset F of Ω, $\phi\in W^{1/p',p}(\Gamma)$ and $w\in W^{1,p}(\Omega)$ we set

$$K(\Gamma_0, \phi)=\{v\in W^{1,p}(\Omega)\,;\ \gamma v=\phi \text{ a. e. on } \Gamma_0\}\,,$$

$$C(\Gamma_0, \phi)=\{v\in W^{1,p}(\Omega)\,;\ \gamma v\geqq\phi \text{ on } \Gamma_0 \text{ in the sense of } W^{1/p',p}(\Gamma)\}\,,$$

$$K(F, w)=\{v\in W^{1,p}(\Omega)\,;\ v=w \text{ on } F \text{ in the sense of } W^{1,p}(\Omega)\}\,,$$

$$K(\Gamma_0, \phi\,; F, w)=K(\Gamma_0, \phi)\cap K(F, w)\,,$$

$$C(\Gamma_0, \phi\,; F, w)=C(\Gamma_0, \phi)\cap K(F, w)\,.$$

5.1. Convergence of subsets and of functions.

The following notion of convergence of sets is due to Mosco [**19**]. Let X be a real reflexive Banach space. For a sequence $\{S_k\}$ of subsets of X, we define

$$s\text{-Liminf}\, S_k = \{v \in X\,;\ \text{there is a sequence } \{v_k\} \text{ with } v_k \in S_k \text{ for all } k \text{ such that } v_k \xrightarrow{s} v \text{ in } X \text{ as } k \to \infty\}$$

and

$$w\text{-Limsup}\, S_k = \{v \in X\,;\ \text{there is a sequence } \{v_j\} \text{ with } v_j \in S_{k_j} \text{ for all } j \text{ such that } \{S_{k_j}\} \text{ is a subsequence of } \{S_k\} \text{ and } v_j \xrightarrow{w} v \text{ in } X \text{ as } j \to \infty\}\,.$$

We say that $\{S_k\}$ converges to a subset S in X, if

$$S = s\text{-Liminf}\, S_k = w\text{-Limsup}\, S_k\,.$$

We then write either $S_k \to S$ in X or $S = \mathrm{Lim}\, S_k$ in X.

Let Φ be a function on X with values in $[-\infty, \infty]$. The epigraph epi (Φ) of Φ is the set $\{(v, r) \in X \times R\,;\ \Phi(v) \leqq r\}$. Let $\{\Phi_k\}$ be a sequence of functions on X with values in $[-\infty, \infty]$. Then, by "$\Phi = \mathrm{Lim}\, \Phi_k$ in X" we mean that epi $(\Phi_k) = \mathrm{Lim}$ epi (Φ_k) in $X \times R$.

The following lemma due to Mosco [**19**; Lemma 1.10] gives a characterization of convergence of functions in the above sense.

LEMMA 5.1. *Let $\{\Phi_k\}$ be a sequence of functions on X. Then $\Phi = \mathrm{Lim}\, \Phi_k$ in X if and only if* (a) *and* (b) *below are satisfied:*

(a) *For each $v \in X$ there is a sequence $\{v_k\} \subset X$ such that $v_k \xrightarrow{s} v$ in X and*

$$\limsup_{k\to\infty} \Phi_k(v_k) \leqq \Phi(v)\,.$$

(b) *If $\{\Phi_{k_j}\}$ is a subsequence of $\{\Phi_k\}$ and $\{v_j\}$ is a sequence in X weakly convergent to $v \in X$, then*

$$\liminf_{k\to\infty} \Phi_{k_j}(v_j) \geqq \Phi(v)\,.$$

Here, liminf and limsup are taken in $[-\infty, \infty]$.

5.2. Some results on convergence of convex sets.

PROPOSITION 5.2. *Let $\{\psi_k\} \subset W^{1/p',p}(\Gamma)$ and $\{w_k\} \subset W^{1,p}(\Omega)$ be sequences such that*

$$\psi_k \xrightarrow{s} \psi \ \textit{in}\ W^{1/p',p}(\Gamma) \ \textit{and}\ w_k \xrightarrow{s} w \ \textit{in}\ W^{1,p}(\Omega) \ \textit{as}\ k \to \infty\,.$$

Then,

$$(1)\qquad K(F, w) = \mathrm{Lim}\, K(F, w_k) \qquad in\ W^{1,p}(\Omega)\,,$$

$$(2)\qquad K(\Gamma_0, \psi) = \mathrm{Lim}\, K(\Gamma_0, \psi_k) \qquad in\ W^{1,p}(\Omega)\,,$$

$$(3)\qquad C(\Gamma_0, \psi) = \mathrm{Lim}\, C(\Gamma_0, \psi_k) \qquad in\ W^{1,p}(\Omega)\,,$$

$$(4)\qquad K(\Gamma_0, \psi\,; F, w) = \mathrm{Lim}\, K(\Gamma_0, \psi_k\,; F, w_k) \qquad in\ W^{1,p}(\Omega)\,,$$

(5) $C(\Gamma_0, \psi; F, w) = \mathrm{Lim}\, C(\Gamma_0, \psi_k; F, w_k)$ *in* $W^{1,p}(\Omega)$.

PROOF. We first show (1). Let $v \in K(F, w)$ be any function. Then $v_k = w_k + (v-w) \in K(F, w_k)$ for all k and $\|v_k - v\|_{1,p} = \|w_k - w\|_{1,p} \to 0$ as $k \to \infty$. Therefore,

$$K(F, w) \subset s\text{-}\mathrm{Liminf}\, K(F, w_k). \tag{5.1}$$

Next, let $\{v_j\}$ be a sequence with $v_j \in K(F, w_{k_j})$ for all j such that $\{w_{k_j}\}$ is a subsequence of $\{w_k\}$ and $v_j \xrightarrow{w} v$ in $W^{1,p}(\Omega)$. Then $v_j - w_{k_j} \in K(F, 0)$ and $v_j - w_{k_j} \xrightarrow{w} v - w$ in $W^{1,p}(\Omega)$. Since $K(F, 0)$ is convex and closed in $W^{1,p}(\Omega)$, we have $v - w \in K(F, 0)$. Thus $v \in K(F, w)$. Hence

$$w\text{-}\mathrm{Limsup}\, K(F, w_k) \subset K(F, w). \tag{5.2}$$

Since it is trivial that

$$s\text{-}\mathrm{Liminf}\, K(F, w_k) \subset w\text{-}\mathrm{Limsup}\, K(F, w_k),$$

it follows from (5.1) and (5.2) that $K(F, w) = \mathrm{Lim}\, K(F, w_k)$ in $W^{1,p}(\Omega)$.

To show (2) we take a function $\tilde{v} \in W^{1,p}(\Omega)$ and a sequence $\{\tilde{v}_k\} \subset W^{1,p}(\Omega)$ such that $\gamma\tilde{v} = \psi$, $\gamma\tilde{v}_k = \psi_k$ for all k and $\tilde{v}_k \xrightarrow{s} \tilde{v}$ in $W^{1,p}(\Omega)$. Let $v \in K(\Gamma_0, \psi)$ be any function and set $v_k = \tilde{v}_k + v - \tilde{v}$ for each k. Then $v_k \in K(\Gamma_0, \psi_k)$ for all k and $v_k \xrightarrow{s} v$ in $W^{1,p}(\Omega)$. Hence,

$$K(\Gamma_0, \psi) \subset s\text{-}\mathrm{Liminf}\, K(\Gamma_0, \psi_k).$$

Furthermore, just as in the case of (1), we can prove that

$$w\text{-}\mathrm{Limsup}\, K(\Gamma_0, \psi_k) \subset K(\Gamma_0, \psi).$$

Thus (2) holds.

(3) is also proved just as (2).

Next, using (1) and (2), we shall show (4). Let v be any function in $K(\Gamma_0, \psi; F, w)$ and ρ be a function in $\mathcal{D}(\Omega)$ such that $\rho = 1$ on a neighborhood of F. Since $K(\Gamma_0, \psi; F, w) = K(\Gamma_0, \psi) \cap K(F, w)$, by (1) and (2) there are sequences $\{v_k\}$ with $v_k \in K(F, w_k)$ and $\{\tilde{v}_k\}$ with $\tilde{v}_k \in K(\Gamma_0, \psi_k)$ for each k such that $v_k \xrightarrow{s} v$ in $W^{1,p}(\Omega)$ as $k \to \infty$ and $\tilde{v}_k \xrightarrow{s} v$ in $W^{1,p}(\Omega)$ as $k \to \infty$. Therefore, if we put $u_k = \rho v_k + (1-\rho)\tilde{v}_k$ for each k, then $u_k \in K(\Gamma_0, \psi_k; F, w_k)$ for all k and $u_k \xrightarrow{s} v$ in $W^{1,p}(\Omega)$, so that

$$s\text{-}\mathrm{Liminf}\, K(\Gamma_0, \psi_k; F, w_k) \supset K(\Gamma_0, \psi; F, w).$$

Since it is clear by (1) and (2) that

$$w\text{-}\mathrm{Limsup}\, K(\Gamma_0, \psi_k; F, w_k) \subset K(\Gamma_0, \psi; F, w),$$

we have (4).

Finally, (5) follows from (1) and (3) by the same method as above.

q. e. d.

§ 6. Convergence of solutions.

We investigate convergence properties of solutions of problems treated in Sections 3 and 4. In this section, let $1<p<\infty$ and $1/p+1/p'=1$.

6.1. Convergence theorem.

Let $a(\cdot, \cdot)$ be a functional on $W^{1,p}(\Omega)\times W^{1,p}(\Omega)$ given by (2.1) satisfying Assumption (I), K be a closed convex subset of $W^{1,p}(\Omega)$ and Ψ be a lower semicontinuous convex function on K such that $\Psi(v)\in(-\infty, \infty]$ for all $v\in K$ and $\Psi\not\equiv\infty$ on K. Suppose that there are sequences $\{a^{(n)}(\cdot, \cdot)\}$, $\{K_n\}$ and $\{\Psi_n\}$ with the following properties:

(i) Each $a^{(n)}(\cdot, \cdot)$ is a functional on $W^{1,p}(\Omega)\times W^{1,p}(\Omega)$ given by the form (2.1) satisfying Assumptions (I) and (II) in Section 2. We require

(A$_1$) $\{a^{(n)}(\cdot, \cdot)\}$ is uniformly bounded on bounded subsets of $W^{1,p}(\Omega)$ $\times W^{1,p}(\Omega)$, i. e., for each bounded subset B of $W^{1,p}(\Omega)$, there is a positive constant M such that

$$|a^{(n)}(v, w)|\leqq M \qquad \text{for all } n \text{ and all } v,\ w\in B.$$

(A$_2$) If $\{a^{(n_k)}(\cdot, \cdot)\}$ is a subsequence of $\{a^{(n)}(\cdot, \cdot)\}$ and $\{v_k\}$ is a sequence in $W^{1,p}(\Omega)$ weakly convergent to $v\in W^{1,p}(\Omega)$ and if

$$\limsup_{k\to\infty} a^{(n_k)}(v_k, v_k-v)\leqq 0,$$

then

$$\liminf_{k\to\infty} a^{(n_k)}(v_k, v_k-w)\geqq a(v, v-w)$$

$$\text{for every } w\in W^{1,p}(\Omega).$$

(ii) Each K_n is a closed convex subset of $W^{1,p}(\Omega)$ and

$$K=\mathrm{Lim}\, K_n \qquad \text{in} \quad W^{1,p}(\Omega).$$

(iii) Each Ψ_n is a lower semicontinuous convex function on $W^{1,p}(\Omega)$ such that $\Psi_n\not\equiv\infty$ on K_n, $\Psi_n(v)\in(-\infty, \infty]$ for all $v\in K_n$,

$$\{v\in W^{1,p}(\Omega)\,;\ \Psi_n(v)<\infty\}\subset K_n,$$

and

$$\Psi=\mathrm{Lim}\,\Psi_n \qquad \text{in } W^{1,p}(\Omega).$$

(iv) There is a bounded sequence $\{a_n\}\subset W^{1,p}(\Omega)$ with $a_n\in K_n$ for all n such that

(B$_1$) $$\sup_n \Psi_n(a_n)<\infty,$$

(B$_2$) for each n,

$$\frac{a^{(n)}(v, v-a_n)+\Psi_n(v)}{\|v\|_{1,p}} \longrightarrow \infty \qquad \text{as } \|v\|_{1,p}\to\infty,\ v\in K_n\,,$$

(B_3) for any sequence $\{v_n\}$ with $v_n\in K_n$, $\|v_n\|_{1,p}\to\infty$ implies that

$$\frac{a^{(n)}(v_n, v_n-a_n)+\Psi_n(v_n)}{\|v_n\|_{1,p}} \longrightarrow \infty\,.$$

Under these hypotheses (i)-(iv) we have the following convergence theorem.

THEOREM 6.1. *Let $f\in L^{p'}(\Omega)$ and $\{f_n\}\subset L^{p'}(\Omega)$. Suppose that $f_n\overset{s}{\to}f$ in $L^{p'}(\Omega)$ as $n\to\infty$. If we denote by S the set of all solutions of the variational inequality:*

$$\text{(V)}\qquad \begin{cases} u\in K\,, \\ a(u, u-v)-\int_\Omega f(u-v)dx\leqq\Psi(v)-\Psi(u) \qquad \textit{for all } v\in K\,, \end{cases}$$

and, for each n, denote by S_n the set of all solutions of the variational inequality:

$$(\text{V}_n)\qquad \begin{cases} u_n\in K_n\,, \\ a^{(n)}(u_n, u_n-v)-\int_\Omega f_n(u_n-v)dx\leqq\Psi_n(v)-\Psi_n(u_n) \qquad \textit{for all } v\in K_n\,, \end{cases}$$

then we have:

(a) $\qquad w\text{-Limsup}\, S_n\neq\emptyset$ *and* $w\text{-Limsup}\, S_n\subset S$.

(b) *Let $\{u_k\}$ be a sequence such that $u_k\in S_{n_k}$ for a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ and $u_k\overset{w}{\to}u$ in $W^{1,p}(\Omega)$ as $k\to\infty$ for some $u\in W^{1,p}(\Omega)$ (hence $u\in S$ by* (a)). *Then*

$$\lim_{k\to\infty} a^{(n_k)}(u_k, u_k)=a(u, u)$$

and

$$\lim_{k\to\infty}\Psi_{n_k}(u_k)=\Psi(u)\,.$$

For a proof of this theorem, see [8; Theorem 4.2].

REMARK. We set

$$a(v, w)=\sum_{j=1}^N\int_\Omega A_j(x, v, \nabla v)\frac{\partial w}{\partial x_j}dx+\int_\Omega A_0(x, v, \nabla v)wdx$$

and for each n,

$$a^{(n)}(v, w)=\sum_{j=1}^N\int_\Omega A_j^{(n)}(x, v, \nabla v)\frac{\partial w}{\partial x_j}dx+\int_\Omega A_0^{(n)}(x, v, \nabla v)wdx\,.$$

Suppose that $a(\cdot,\cdot)$ and every $a^{(n)}(\cdot,\cdot)$ satisfy Assumptions (I) and (II) in § 2, and that there are sequences $\{c_n\}$ of positive numbers and $\{h_n\}$ of functions in $L^{p'}(\Omega)$ such that for each n

$$(6.1)\qquad |A_k^{(n)}(x, \zeta, \xi_1, \xi_2, \cdots, \xi_N)-A_k(x, \zeta, \xi_1, \xi_2, \cdots, \xi_N)|$$
$$\leqq c_n(|\zeta|^{p-1}+\sum_{j=1}^{N}|\xi_j|^{p-1})+h_n(x), \qquad k=0, 1, \cdots, N,$$

and

$$c_n \longrightarrow 0,$$

$$h_n \overset{s}{\longrightarrow} 0 \quad \text{in } L^{p'}(\Omega)$$

as $n\to\infty$. Then $a(\cdot, \cdot)$ and $\{a^{(n)}(\cdot, \cdot)\}$ satisfy conditions (A_1) and (A_2). In fact, it is easy to see that the sequence $\{a^{(n)}(\cdot, \cdot)\}$ satisfies condition (A_1). Let $\{v_k\}$ be a sequence in $W^{1,p}(\Omega)$ weakly convergent to $v\in W^{1,p}(\Omega)$ and $\{a^{(n_k)}(\cdot, \cdot)\}$ be a subsequence of $\{a^{(n)}(\cdot, \cdot)\}$ such that

$$(6.2)\qquad \limsup_{k\to\infty} a^{(n_k)}(v_k, v_k-v)\leqq 0.$$

Then, by (6.1),

$$|a^{(n_k)}(v_k, v_k)-a(v_k, v_k)| \longrightarrow 0$$

and

$$|a^{(n_k)}(v_k, v)-a(v_k, v)| \longrightarrow 0$$

as $k\to\infty$. Since

$$a^{(n_k)}(v_k, v_k)-a^{(n_k)}(v_k, v)$$
$$\geqq -|a^{(n_k)}(v_k, v_k)-a(v_k, v_k)|-|a(v_k, v)-a^{(n_k)}(v_k, v)|$$
$$+a(v_k, v_k)-a(v_k, v)$$

we have by (6.2)

$$\limsup_{k\to\infty} a(v_k, v_k-v)\leqq 0.$$

Therefore, Assumption (II) for $a(\cdot, \cdot)$ implies that

$$\liminf_{k\to\infty} a(v_k, v_k-w)\geqq a(v, v-w) \qquad \text{for every } w\in W^{1,p}(\Omega).$$

Hence, using (6.1) again, we obtain

$$\liminf_{k\to\infty} a^{(n_k)}(v_k, v_k-w)\geqq a(v, v-w) \qquad \text{for every } w\in W^{1,p}(\Omega).$$

Thus condition (A_2) is verified.

6.2. Examples.

We suppose that $2\leqq p<\infty$. Let us consider

$$a^{(n)}(v, w)=\sum_{j=1}^{N}\int_\Omega \alpha^{(n)}|\nabla v|^{p-2}\frac{\partial v}{\partial x_j}\frac{\partial w}{\partial x_j}dx+\int_\Omega \beta^{(n)}|v|^{p-2}vwdx,$$

$n=1, 2, \cdots$, and

$$a(v, w)=\sum_{j=1}^{N}\int_\Omega \alpha|\nabla v|^{p-2}\frac{\partial v}{\partial x_j}\frac{\partial w}{\partial x_j}dx+\int_\Omega \beta|v|^{p-2}vwdx.$$

Assume:

(1) For a positive constant c,

$$\alpha^{(n)}(x) \geqq c, \quad \beta^{(n)}(x) \geqq c \qquad \text{a. e. on } \Omega \text{ for } n=1, 2, \cdots,$$

(2) α, β, $\alpha^{(n)}$, $\beta^{(n)} \in L^\infty(\Omega)$ and

$$\alpha^{(n)} \xrightarrow{s} \alpha, \quad \beta^{(n)} \xrightarrow{s} \beta \qquad \text{in } L^\infty(\Omega) \text{ as } n \to \infty .$$

Then, clearly, $\{a^{(n)}(\cdot, \cdot)\}$ and $a(\cdot, \cdot)$ satisfy condition (i) in the previous paragraph. As is easily seen, for each bounded subset B of $W^{1,p}(\Omega)$ there is a constant $\tilde{c} > 0$ such that

$$\inf_{w \in B} \frac{a^{(n)}(v, v-w)}{\|v\|_{1,p}} \geqq \tilde{c}\|v\|_{1,p}^{p-1} - \tilde{c}\|v\|_{1,p}^{p-3} \qquad \text{for all } v \in W^{1,p}(\Omega) \text{ with } v \neq 0 .$$

EXAMPLE 6.2. Let F be a compact subset of Ω and Γ_0 be a closed subset of Γ, and let $\{\phi_n\} \subset W^{1/p',p}(\Gamma)$, $\{\phi_n^*\} \subset W^{-1/p',p'}(\Gamma)$, $\{f_n\} \subset L^{p'}(\Omega)$ and $\{g_n\} \subset L^\infty(\Gamma)$ with $g_n \geqq 0$ a. e. on Γ and $\{w_n\} \subset W^{1,p}(\Omega)$ be sequences such that

$$\begin{aligned} w_n &\xrightarrow{s} w && \text{in } W^{1,p}(\Omega), \\ \phi_n &\xrightarrow{s} \phi && \text{in } W^{1/p',p}(\Gamma), \\ \phi_n^* &\xrightarrow{s} \phi^* && \text{in } W^{-1/p',p'}(\Gamma), \\ f_n &\xrightarrow{s} f && \text{in } L^{p'}(\Omega), \\ g_n &\xrightarrow{s} g && \text{in } L^\infty(\Gamma), \end{aligned}$$

as $n \to \infty$. Then we consider the problems

$$\text{(P)} \qquad \begin{cases} -\sum_{j=1}^{N} \dfrac{\partial}{\partial x_j}\left(\alpha|\nabla u|^{p-2}\dfrac{\partial u}{\partial x_j}\right) + \beta|u|^{p-2}u = f & \text{in } \Omega \setminus F, \\ \gamma u = \phi \quad \text{a. e. on } \Gamma_0, \\ u = w \quad \text{on } F \text{ in the sense of } W^{1,p}(\Omega), \\ -\boldsymbol{Ba}(u) + \phi^* = g|\gamma u|^{p-2}(\gamma u) & \text{on } \Gamma \setminus \Gamma_0, \end{cases}$$

and for each n

$$\text{(P}_n\text{)} \qquad \begin{cases} -\sum_{j=1}^{N} \dfrac{\partial}{\partial x_j}\left(\alpha^{(n)}|\nabla u_n|^{p-2}\dfrac{\partial u_n}{\partial x_j}\right) + \beta^{(n)}|u_n|^{p-2}u_n = f_n & \text{in } \Omega \setminus F, \\ \gamma u_n = \phi_n \quad \text{a. e. on } \Gamma_0, \\ u_n = w_n \quad \text{on } F \text{ in the sense of } W^{1,p}(\Omega), \\ -\boldsymbol{Ba}^{(n)}(u_n) + \phi_n^* = g_n|\gamma u_n|^{p-2}(\gamma u_n) & \text{on } \Gamma \setminus \Gamma_0, \end{cases}$$

where

$$\boldsymbol{a}(u)=\left(\alpha|\nabla u|^{p-2}\frac{\partial u}{\partial x_1}, \cdots, \alpha|\nabla u|^{p-2}\frac{\partial u}{\partial x_N}\right)$$

and

$$\boldsymbol{a}^{(n)}(u_n)=\left(\alpha^{(n)}|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_1}, \cdots, \alpha^{(n)}|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_N}\right).$$

We now show that every (P_n) admits a unique solution as well as (P) and that the solution u_n of (P_n) converges strongly to the solution u of (P) in $W^{1,p}(\Omega)$. In fact, as was seen in Case 5 in Section 3, these problems (P) and (P_n) are equivalent to the associated variational inequalities

$$(V)\qquad \begin{cases} u\in K(\Gamma_0, \psi\,; F, w)\,, \\ a(u, u-v)-\int_\Omega f(u-v)dx \leqq \Phi(\gamma v)-\Phi(\gamma u) \\ \qquad\qquad\qquad\qquad \text{for all } v\in K(\Gamma_0, \psi\,; F, w) \end{cases}$$

where

$$\Phi(\hat{v})=\begin{cases} \frac{1}{p}\int_\Omega g|\hat{v}|^p d\Gamma+\langle\psi^*, \hat{v}\rangle_\Gamma\,, & \text{if } \hat{v}\in\hat{K}(\Gamma_0, \psi\,; F, w)\,, \\ \infty & \text{otherwise}\,, \end{cases}$$

and

$$(V_n)\qquad \begin{cases} u_n\in K(\Gamma_0, \psi_n\,; F, w_n)\,, \\ a^{(n)}(u_n, u_n-v)-\int_\Omega f_n(u_n-v)dx \leqq \Phi_n(\gamma v)-\Phi_n(\gamma u_n) \\ \qquad\qquad\qquad\qquad \text{for all } v\in K(\Gamma_0, \psi_n\,; F, w_n) \end{cases}$$

where

$$\Phi_n(\hat{v})=\begin{cases} \frac{1}{p}\int_\Gamma g_n|\hat{v}|^p d\Gamma+\langle\psi_n^*, \hat{v}\rangle_\Gamma & \text{if } \hat{v}\in\hat{K}(\Gamma_0, \psi_n\,; F, w_n)\,, \\ \infty & \text{otherwise}\,, \end{cases}$$

respectively. We note that by Theorem A the above problem (V) has a unique solution as well as the problem (V_n), since the operators from $W^{1,p}(\Omega)$ into $(W^{1,p}(\Omega))^*$ associated with $a(\cdot,\cdot)$ and $a^{(n)}(\cdot,\cdot)$ are strictly monotone. Under our assumptions, we have by (4) of Proposition 5.2, $K(\Gamma_0, \psi\,; F, w)=\mathrm{Lim}\,K(\Gamma_0, \psi_n\,; F, w_n)$ in $W^{1,p}(\Omega)$ and $\Phi\circ\gamma=\mathrm{Lim}\,\Phi_n\circ\gamma$ in $W^{1,p}(\Omega)$ by Lemma 5.1. Hence conditions (ii) and (iii) for $K=K(\Gamma_0, \psi\,; F, w)$, $K_n=K(\Gamma_0, \psi_n\,; F, w_n)$, $\Psi=\Phi\circ\gamma$ and $\Psi_n=\Phi_n\circ\gamma$ in the previous paragraph are satisfied. It is also easy to check condition (iv) for $K=K(\Gamma_0, \psi\,; F, w)$, $K_n=K(\Gamma_0, \psi_n\,; F, w_n)$, $\Psi=\Phi\circ\gamma$ and $\Psi_n=\Phi_n\circ\gamma$. Therefore, applying (a) of Theorem 6.1, we obtain that the solution u_n of (P_n) converges weakly to the solution u of (P) in $W^{1,p}(\Omega)$, so that $u_n\overset{s}{\to}u$ in $L^p(\Omega)$ by the compactness of the natural injection from $W^{1,p}(\Omega)$ into $L^p(\Omega)$. Moreover, by (b) of Theorem 6.1, $\lim_{n\to\infty}\int_\Omega \alpha^{(n)}|\nabla u_n|^p dx$

$=\int_{\Omega}\alpha|\nabla u|^p dx$, and so $\lim_{n\to\infty}\int_{\Omega}|\nabla u_n|^p dx=\int_{\Omega}|\nabla u|^p dx$. Hence $\|u_n\|_{1,p}\to\|u\|_{1,p}$. From this together with the uniform convexity of $W^{1,p}(\Omega)$ we conclude that $u_n \overset{s}{\to} u$ in $W^{1,p}(\Omega)$ as $n\to\infty$.

EXAMPLE 6.3. Under the same assumptions as in Example 6.2, we consider problems of another type:

$$
(\mathrm{P})' \quad \begin{cases} -\sum_{j=1}^{N}\dfrac{\partial}{\partial x_j}\left(\alpha|\nabla u|^{p-2}\dfrac{\partial u}{\partial x_j}\right)+\beta|u|^{p-2}u=f & \text{in } \Omega\setminus F, \\ \gamma u\geqq\psi & \text{on } \Gamma_0 \text{ in the sense of } W^{1/p',p}(\Gamma), \\ u=w & \text{on } F \text{ in the sense of } W^{1,p}(\Omega), \\ \boldsymbol{Ba}(u)\geqq\psi^* & \text{on } \Gamma, \\ \boldsymbol{Ba}(u)=\psi^* & \text{on } \Gamma\setminus\Gamma_0, \\ \langle \boldsymbol{Ba}(u)-\psi^*, \gamma u-\psi\rangle_\Gamma=0, \end{cases}
$$

and

$$
(\mathrm{P}_n)' \quad \begin{cases} -\sum_{j=1}^{N}\dfrac{\partial}{\partial x_j}\left(\alpha^{(n)}|\nabla u_n|^{p-2}\dfrac{\partial u_n}{\partial x_j}\right)+\beta^{(n)}|u_n|^{p-2}u_n=f_n & \text{in } \Omega\setminus F, \\ \gamma u_n\geqq\psi_n & \text{on } \Gamma_0 \text{ in the sense of } W^{1/p',p}(\Gamma), \\ u_n=w_n & \text{on } F \text{ in the sense of } W^{1,p}(\Omega), \\ \boldsymbol{Ba}^{(n)}(u_n)\geqq\psi_n^* & \text{on } \Gamma, \\ \boldsymbol{Ba}^{(n)}(u_n)=\psi_n^* & \text{on } \Gamma\setminus\Gamma_0, \\ \langle \boldsymbol{B}\alpha^{(n)}(u_n)-\psi_n^*, \gamma u_n-\psi_n\rangle_\Gamma=0. \end{cases}
$$

Applying Theorem 6.1 and using the fact ((5) of Proposition 5.2) that

$$C(\Gamma_0, \psi; F, w)=\mathrm{Lim}\, C(\Gamma_0, \psi_n; F, w_n) \qquad \text{in } W^{1,p}(\Omega),$$

we can prove, just as in Example 6.2, that the solution u_n of $(\mathrm{P}_n)'$ strongly converges to the solution u of $(\mathrm{P})'$ as $n\to\infty$.

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