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Manifolds with vanishing Weyl or Bochner curvature tensor

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§1. Introduction.

Let M be a Riemannian manifold of dimension $n>3$ and denote by g_{ji}, $K_{kji}{}^{h}$, K_{ji} and K the metric tensor, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively.

If M is locally conformal to a Euclidean space then M is said to be conformally flat. For a conformally flat M, the Weyl conformal curvature tensor given by

$$C_{kji}{}^{h}=K_{kji}{}^{h}+\delta_{k}^{h}C_{ji}-\delta_{j}^{h}C_{ki}+C_{k}{}^{h}g_{ji}-C_{j}{}^{h}g_{ki} \tag{1.1}$$

vanishes identically, where

$$C_{ji}=-\frac{1}{n-2}K_{ji}+\frac{1}{2(n-1)(n-2)}Kg_{ji}\,,\qquad C_{k}{}^{h}=C_{kt}g^{th}\,, \tag{1.2}$$

g^{th} being contravariant components of the metric tensor. Conversely if $C_{kji}{}^{h}$ vanishes identically, then M is conformally flat [3], [6].

One of the purposes of the present paper is to prove the following:

THEOREM 1. *In order that a Riemannian manifold of dimension $n>3$ is conformally flat, it is necessary and sufficient that there exists a (unique) quadratic form Q on the manifold such that the sectional curvature $K(\sigma)$ with respect to a section σ is the trace of the restriction of Q to σ, i.e. $K(\sigma)=$ traceQ/σ, the metric being also restricted to σ.*

Let M be an n-dimensional Kaehlerian manifold and denote by g_{ji}, $F_{i}{}^{h}$, $K_{kji}{}^{h}$, K_{ji} and K the metric tensor, the complex structure tensor, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively. Bochner [1] (see also [4], [9]) introduced a curvature tensor given by

$$\begin{aligned} B_{kji}{}^{h}=&K_{kji}{}^{h}+\delta_{k}^{h}L_{ji}-\delta_{j}^{h}L_{ki}+L_{k}{}^{h}g_{ji}-L_{j}{}^{h}g_{ki}\\ &+F_{k}{}^{h}M_{ji}-F_{j}{}^{h}M_{ki}+M_{k}{}^{h}F_{ji}-M_{j}{}^{h}F_{ki}-2(M_{kj}F_{i}{}^{h}+F_{kj}M_{i}{}^{h})\,, \end{aligned} \tag{1.3}$$

where

$$L_{ji}=-\frac{1}{n+4}K_{ji}+\frac{1}{2(n+2)(n+4)}Kg_{ji}\,,\qquad L_{k}{}^{h}=L_{kt}g^{th}\,,$$

$$M_{ji}=-L_{jt}F_{i}{}^{t}\,,\qquad M_{k}{}^{h}=M_{kt}g^{th}$$

and $F_{ji}=F_j{}^t g_{ti}$, as a curvature tensor which corresponds to the Weyl conformal curvature tensor in a Riemannian manifold (see also [4], [5], [8], [10]).

Another purpose of the present paper is to prove the following:

THEOREM 2. *In order that the Bochner curvature tensor of a Kaehlerian manifold vanishes, it is necessary and sufficient that there exists a (unique) hybrid quadratic form Q such that the sectional curvature $K(\sigma)$ with respect to a holomorphic section σ is the trace of the restriction of Q to σ, i.e. $K(\sigma)=\mathrm{trace}\, Q/\sigma$, the metric being also restricted to σ.*

§ 2. Riemannian manifolds with vanishing Weyl conformal curvature tensor.

Suppose that M is a conformally flat Riemannian manifold of dimension $n>3$, then we have

$$C_{kji}{}^h=0\,, \tag{2.1}$$

that is

$$K_{kjih}=-g_{kh}C_{ji}+g_{jh}C_{ki}-C_{kh}g_{ji}+C_{jh}g_{ki}\,, \tag{2.2}$$

where $K_{kjih}=K_{kji}{}^t g_{th}$ and consequently the sectional curvature $K(\sigma)$ with respect to a section σ spanned by vectors X and Y is given by

$$\begin{aligned} K(\sigma)&=\frac{K_{kjih}X^kY^jX^iY^h}{(g_{kh}g_{ji}-g_{jh}g_{ki})X^kY^jX^iY^h} \\ &=\frac{1}{(X,Y)^2-(X,X)(Y,Y)}[(Y,Y)C_{ji}X^jX^i \\ &\qquad -2(X,Y)C_{ji}X^jY^i+(X,X)C_{ji}Y^jY^i]\,, \end{aligned} \tag{2.3}$$

where (X, Y) denotes the inner product of X and Y. Thus if X and Y are mutually orthogonal unit vectors, then we have

$$\begin{aligned} K(\sigma)&=-K_{kjih}X^kY^jX^iY^h \\ &=-C_{ji}X^jX^i-C_{ji}Y^jY^i\,, \end{aligned} \tag{2.4}$$

that is, the sectional curvature $K(\sigma)$ with respect to σ is given by the trace of the restriction of $Q(X, X)=-C(X, X)$ to σ.

Conversely, suppose that the sectional curvature $K(\sigma)$ of a Riemannian manifold M with respect to a section σ spanned by two vectors X and Y is given by

$$\begin{aligned} K(\sigma)&=\frac{K_{kjih}X^kY^jX^iY^h}{(g_{kh}g_{ji}-g_{jh}g_{ki})X^kY^jX^iY^h} \\ &=-C_{ji}U^jU^i-C_{ji}V^jV^i\,, \end{aligned} \tag{2.5}$$

where $C_{ji}U^jU^i=C(U, U)$ is a certain quadratic form and U and V are mutually orthogonal unit vectors spanning the section σ.

The expression $-C(U, U)-C(V, V)$ being independent of the choice of mutually orthogonal unit vectors U and V in the section σ, we put

$$U^h=\frac{X^h}{\sqrt{(X, X)}},\tag{2.6}$$
$$V^h=\frac{(X, X)Y^h-(X, Y)X^h}{\sqrt{(X, X)}\sqrt{(X, X)(Y, Y)-(X, Y)^2}}.$$

Then (2.5) becomes

$$\frac{K_{kjih}X^kY^jX^iY^h}{(g_{kh}g_{ji}-g_{jh}g_{ki})X^kY^jX^iY^h}$$
$$=-\frac{1}{(X, X)(Y, Y)-(X, Y)^2}[(Y, Y)C_{ji}X^jX^i$$
$$-2(X, Y)C_{ji}X^jY^i+(X, X)C_{ji}Y^jY^i],$$

that is,

$$K_{kjih}X^kY^jX^iY^h \tag{2.7}$$
$$=g_{jh}Y^jY^hC_{ki}X^kX^i-2g_{kh}X^kY^hC_{ji}X^iY^j+g_{ki}X^kX^iC_{jh}Y^jY^h.$$

Since X's are arbitrary, we have, from (2.7),

$$K_{kjih}Y^jY^h+K_{ijkh}Y^jY^h$$
$$=2g_{jh}Y^jY^hC_{ki}-2g_{kh}Y^hC_{ji}Y^j-2g_{ih}Y^hC_{jk}Y^j+2g_{ki}C_{jh}Y^jY^h,$$

from which, Y's being arbitrary,

$$K_{kjih}+K_{khij}+K_{ijkh}+K_{ihkj}$$
$$=4g_{jh}C_{ki}-2g_{kh}C_{ji}-2g_{kj}C_{hi}-2g_{ih}C_{jk}-2g_{ij}C_{hk}+4g_{ki}C_{jh}.$$

Taking the skew-symmetric part with respect to k and j of this equation, we find

$$2K_{kjih}+K_{khij}-K_{jhik}+K_{ijkh}-K_{ikjh}+2K_{ihkj}$$
$$=4g_{jh}C_{ki}-4g_{kh}C_{ji}-2g_{kh}C_{ji}+2g_{jh}C_{ki}$$
$$-2g_{ij}C_{hk}+2g_{ik}C_{hj}+4g_{ki}C_{jh}-4g_{ji}C_{kh},$$

that is, using the first Bianchi identity,

$$K_{kjih}=-g_{kh}C_{ji}+g_{jh}C_{ki}-C_{kh}g_{ji}+C_{jh}g_{ki}.\tag{2.8}$$

From (2.8), we have, by transvection with g^{kh},

$$K_{ji}=-(n-2)C_{ji}-g^{kh}C_{kh}g_{ji},\tag{2.9}$$

from which, by transvection with g^{ji},

$$K=-2(n-1)g^{kh}C_{kh},$$

that is,

$$g^{kh}C_{kh}=-\frac{1}{2(n-1)}K. \tag{2.10}$$

Substituting (2.10) into (2.9), we obtain

$$C_{ji}=-\frac{1}{n-2}K_{ji}+\frac{1}{2(n-1)(n-2)}Kg_{ji}. \tag{2.11}$$

Thus (2.8) is equivalent to

$$C_{kji}{}^{h}=0,$$

which shows that the Riemannian manifold is conformally flat. Thus Theorem 1 is proved.

REMARK. In [2], Kulkarni also obtained a characterization of a conformally flat space in terms of sectional curvature different from ours.

§ 3. Kaehlerian manifolds with vanishing Bochner curvature tensor.

Suppose that the Bochner curvature tensor of a Kaehlerian manifold vanishes:

$$B_{kji}{}^{h}=0, \tag{3.1}$$

then we have

$$\begin{aligned} K_{kjih}=&-g_{kh}L_{ji}+g_{jh}L_{ki}-L_{kh}g_{ji}+L_{jh}g_{ki}\\ &-F_{kh}M_{ji}+F_{jh}M_{ki}-M_{kh}F_{ji}+M_{jh}F_{ki}+2(M_{kj}F_{ih}+F_{kj}M_{ih}), \end{aligned} \tag{3.2}$$

and consequently the holomorphic sectional curvature $K(\sigma)$ with respect to a holomorphic section σ spanned by X and FX is given by

$$\begin{aligned} K(X)&=-\frac{1}{(X,X)^2}K_{ktis}X^kF_j{}^tX^jX^iF_h{}^sX^h\\ &=-\frac{8}{(X,X)}L_{ji}X^jX^i, \end{aligned} \tag{3.3}$$

where we have used

$$M_{ji}=-L_{jt}F_i{}^t, \qquad L_{ji}=M_{jt}F_i{}^t$$

and

$$L_{qp}F_j{}^qF_i{}^p=L_{ji}, \tag{3.4}$$

that is, L_{ji} are components of a hybrid tensor of type (0, 2), (see Yano [7], Chapter IV).

Thus for the sectional curvature $K(\sigma)$ with respect to a holomorphic section σ spanned by a unit vector U and its transform FU by F, we have

$$K(U)=-8L_{ji}U^jU^i=-4L_{ji}U^jU^i-4L_{ts}F_j{}^tU^jF_i{}^sU^i. \tag{3.5}$$

Thus the holomorphic sectional curvature $K(\sigma)$ is the trace of the restriction of $Q(X, X)=-4L(X, X)$ to σ.

Conversely suppose that the sectional curvature $K(\sigma)$ of a Kaehlerian manifold with respect to a holomorphic section σ spanned by X and FX is given by

$$\begin{aligned} K(X) &= -\frac{K_{ktis}X^kF_j{}^tX^jX^iF_h{}^sX^h}{(X, X)^2} \\ &= -8L_{ji}U^jU^i, \end{aligned} \tag{3.6}$$

where $L_{ji}U^jU^i$ is a certain quadratic form whose coefficients satisfy (3.4) and

$$U^h=\frac{X^h}{\sqrt{(X, X)}}. \tag{3.7}$$

Then from (3.6) and (3.7), we have

$$K_{ktis}F_j{}^tF_h{}^sX^kX^jX^iX^h=8g_{kj}L_{ih}X^kX^jX^iX^h. \tag{3.8}$$

Since X's are arbitrary and

$$K_{ktis}F_j{}^t=K_{jtis}F_k{}^t, \qquad K_{ktis}F_h{}^s=K_{kths}F_i{}^s,$$
$$K_{ktis}F_j{}^tF_h{}^s=K_{itks}F_h{}^tF_j{}^s,$$

(see Yano [7], Chapter IV), we have, from (3.8),

$$\begin{aligned} &K_{ktis}F_j{}^tF_h{}^s+K_{kths}F_i{}^tF_j{}^s+K_{ktjs}F_h{}^tF_i{}^s \\ &=4[g_{kj}L_{ih}+g_{ki}L_{hj}+g_{kh}L_{ji}+L_{kj}g_{ih}+L_{ki}g_{hj}+L_{kh}g_{ji}], \end{aligned}$$

or

$$\begin{aligned} &K_{ktis}F_q{}^tF_p{}^s+K_{ktps}F_i{}^tF_q{}^s+K_{ktqs}F_p{}^tF_i{}^s \\ &=4[g_{kq}L_{ip}+g_{ki}L_{pq}+g_{kp}L_{qi}+L_{kq}g_{ip}+L_{ki}g_{pq}+L_{kp}g_{qi}]. \end{aligned}$$

Transvecting this with $F_j{}^qF_h{}^p$, we find

$$\begin{aligned} &K_{kjih}-K_{ktpj}F_i{}^tF_h{}^p-K_{khqs}F_j{}^qF_i{}^s \\ &=4[-F_{jk}M_{ih}+g_{ki}L_{jh}-F_{hk}M_{ij}-M_{kj}F_{hi}+L_{ki}g_{jh}-M_{kh}F_{ji}], \end{aligned}$$

where

$$M_{ji}=-L_{jt}F_i{}^t, \tag{3.9}$$

or

$$\begin{aligned} &K_{kjih}-K_{ktpj}F_i{}^tF_h{}^p-K_{khji} \\ &=4[F_{kj}M_{ih}+g_{ki}L_{jh}-F_{kh}M_{ji}+M_{kj}F_{ih}+L_{ki}g_{jh}-M_{kh}F_{ji}], \end{aligned}$$

because of $K_{khqs}F_j{}^qF_i{}^s=K_{khji}$.

Taking the skew-symmetric part of this equation with respect to k and j and taking account of

$$K_{ktpj}F_i{}^tF_h{}^p-K_{jtpk}F_i{}^tF_h{}^p=(K_{ktpj}-K_{jtpk})F_i{}^tF_h{}^p$$
$$=-K_{kjtp}F_i{}^tF_h{}^p=-K_{kjih},$$

we find

$$2K_{kjih}+K_{kjih}-K_{khji}+K_{jhki}$$
$$=4[2F_{kj}M_{ih}+g_{ki}L_{jh}-g_{ji}L_{kh}-F_{kh}M_{ji}+F_{jh}M_{ki}$$
$$+2M_{kj}F_{ih}+L_{ki}g_{jh}-L_{ji}g_{kh}-M_{kh}F_{ji}+M_{jh}F_{ki}],$$

or

$$K_{kjih}=-[g_{kh}L_{ji}-g_{jh}L_{ki}+L_{kh}g_{ji}-L_{jh}g_{ki}$$
$$+F_{kh}M_{ji}-F_{jh}M_{ki}+M_{kh}F_{ji}-M_{jh}F_{ki}-2(M_{kj}F_{ih}+F_{kj}M_{ih})]. \tag{3.10}$$

Transvecting (3.10) with g^{kh}, we find

$$K_{ji}=-[(n+4)L_{ji}+g^{kh}L_{kh}g_{ji}], \tag{3.11}$$

from which, by transvection with g^{ji},

$$K=-2(n+2)g^{kh}L_{kh},$$

or

$$g^{kh}L_{kh}=-\frac{1}{2(n+2)}K. \tag{3.12}$$

Substituting (3.12) into (3.11), we obtain

$$K_{ji}=-\left[(n+4)L_{ji}-\frac{1}{2(n+2)}Kg_{ji}\right],$$

that is,

$$L_{ji}=-\frac{1}{n+4}K_{ji}+\frac{1}{2(n+2)(n+4)}Kg_{ji}. \tag{3.13}$$

Thus (3.10) gives

$$B_{kji}{}^h=0, \tag{3.14}$$

and consequently Theorem 2 is proved.

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