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Eigenfunctions of the laplacian on a real hyperbolic space

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The present paper deals with the Poisson integral representation of eigenfunctions of the laplacian on a real hyperbolic space. Let G be a generalized Lorentz group $SO_0(n, 1)$ and $G=KAN$ be an Iwasawa decomposition of G, where K is a maximal compact subgroup of G. The associated riemannian symmetric space $X=G/K$ is called a real hyperbolic space. We denote by Δ the laplacian on X corresponding to the G-invariant riemannian metric induced by the Killing form of the Lie algebra of G. Let M be the centralizer of A in K and put $B=K/M$. Then for every complex number s a real analytic function $P_s(z, b)$ on $X\times B$, called the Poisson kernel, is defined (§ 2). Let $C(B)$ denote the space of continuous functions on B. The Poisson transform $\mathcal{P}_s(\phi)$ of $\phi\in C(B)$ is defined by

$$\mathcal{P}_s(\phi)(z)=\int_B P_s(z, b)\phi(b)db\,,$$

where db denotes the normalized K-invariant measure on B. Although the functions $\mathcal{P}_s(\phi)(\phi\in C(B))$ are eigenfunctions of Δ, they do not exhaust all of the eigenfunctions of Δ on X. It is our problem to specify the space whose image under the Poisson transform exhausts the eigenfunctions of Δ. The Corollary to Theorem 5.5 in § 5 answers this problem. Namely, it states that any eigenfunction of the laplacian on X can be represented as the Poisson transform $\mathcal{P}_s(T)$ of a Sato's hyperfunction T on B with some complex number s. In the case of the unit disc, S. Helgason proved in [5] that any eigenfunction of the laplacian (with respect to the Poincaré metric) can be given also as the Poisson transform of a hyperfunction on the unit circle.

The contents of this paper are as follows. From § 1 to § 3 we assume that G is a connected real semisimple Lie group of real rank one with finite center. In § 2 we define the Poisson transform of a continuous function on B and show that any eigenfunction of the laplacian on X can be expanded in an absolutely convergent series of the Poisson transforms of K-finite functions on B. In § 3 we prove a Fatou type theorem which will be used in § 4.

From § 4 to the last we assume furthermore that X is a real hyperbolic space. In § 4, by using the Fatou type theorem proved in § 3, we determine the Poisson transform of a K-finite function explicitly. The final section is devoted to proving the main result. First we define the Poisson transform of a hyperfunction on B. Then by using the explicit form of the Poisson transform of a K-finite function we prove that any eigenfunction can be given as the Poisson transform of a hyperfunction.

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§ 1. Notation and preliminaries.

Throughout this paper we assume that G is a connected real semisimple Lie group of real rank one with finite center. Let $\mathfrak{g}_0$ be the Lie algebra of G, $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$, $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}_0$ and $\mathfrak{a}_+$ a maximal abelian subspace of $\mathfrak{p}_0$. Let $\mathfrak{a}_0$ be a maximal abelian subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{a}_+$ and put $\mathfrak{a}_-=\mathfrak{a}_0\cap\mathfrak{k}_0$. Then $\mathfrak{a}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, $\mathfrak{a}_0=\mathfrak{a}_++\mathfrak{a}_-$ (direct sum) and $\mathfrak{a}_+=\mathfrak{a}_0\cap\mathfrak{p}_0$. We complexify $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{a}_0$, $\mathfrak{a}_+$ and $\mathfrak{a}_-$ to $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{t}$ in $\mathfrak{g}$ respectively. We denote by $\langle\,,\rangle$ the Killing form of $\mathfrak{g}$. For $\lambda\in\mathfrak{a}^*$, we denote by $\bar{\lambda}$ the restriction of λ to $\mathfrak{a}_\mathfrak{p}$ and let H_λ denote the element in $\mathfrak{a}$ determined by $\langle H_\lambda, H\rangle=\lambda(H)$ for $H\in\mathfrak{a}$. For λ, $\mu\in\mathfrak{a}^*$, put $\langle\lambda, \mu\rangle=\langle H_\lambda, H_\mu\rangle$. We introduce compatible orders in the spaces of real-valued linear forms on $\mathfrak{a}_++(-1)^{1/2}\mathfrak{a}_-$ and $\mathfrak{a}_+$. Let P denote the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$ under this ordering, P_+ the set of $\alpha\in P$ such that $\bar{\alpha}\neq 0$ and Σ_+ the set of $\bar{\alpha}$ with $\alpha\in P_+$. Since $\mathfrak{a}_+$ is one-dimensional, we can select $\mu_0\in\Sigma_+$ such that $2\mu_0$ is the only other possible element in Σ_+. Put

$$P_{\mu_0}=\{\alpha\in P_+\mid \bar{\alpha}=\mu_0\}\,,$$
$$P_{2\mu_0}=\{\alpha\in P_+\mid \bar{\alpha}=2\mu_0\}\,,$$

and let p (resp. q) denote the number of roots in P_{μ_0} (resp. $P_{2\mu_0}$). We put

$$\rho=\frac{1}{2}\sum_{\alpha\in P_+}\bar{\alpha}\,,$$
$$\mathfrak{n}=\sum_{\alpha\in P_+}\mathfrak{g}^\alpha\,,\qquad \mathfrak{n}_0=\mathfrak{n}\cap\mathfrak{g}_0\,,$$

where $\mathfrak{g}^\alpha$ is the root subspace of $\mathfrak{g}$ corresponding to α. Let K, A, N denote the analytic subgroups of G with Lie algebras $\mathfrak{k}_0$, $\mathfrak{a}_+$, $\mathfrak{n}_0$ respectively. Then K is a maximal compact subgroup and $G=KAN$ is an Iwasawa decomposition of G. For $x\in G$, we can define an element $H(x)$ in $\mathfrak{a}_+$ by $x\in K\exp(H(x))N$. Put $X=G/K$ and $B=K/M$ where M is the centralizer of A in K. We denote by dk (resp. db) the K-invariant normalized measure on K (resp. B).

We shall use the standard notation $\boldsymbol{N}$, $\boldsymbol{R}$, $\boldsymbol{C}$ for the set of natural numbers, the field of real numbers and the field of complex numbers respectively; N^0 is the set of non-negative integers. If E is a differentiable manifold, $C(E)$ (resp. $C^\infty(E)$) denotes the space of all continuous (resp. infinitely differentiable) functions on E.

§ 2. Poisson transform of a continuous function on B.

In this section, we define the Poisson transform of a continuous function on $B=K/M$ and study its image.

We identify C with $\mathfrak{a}_\mathfrak{p}^*$ by

$$\lambda=-(-1)^{1/2}s\rho\,,\qquad \lambda\in\mathfrak{a}_\mathfrak{p}^*\,,\qquad s\in\boldsymbol{C}\,.$$

For each $s\in\boldsymbol{C}$, we define a real analytic function $P_s(z, b)$ on $X\times B$, called the Poisson kernel, by

$$P_s(xK, kM)=\exp\{-(1+s)\rho(H(x^{-1}k))\}\,.$$

DEFINITION. For every continuous function ϕ on B, we define a function $\mathcal{P}_s(\phi)$ on X, called the Poisson transform of ϕ, by

$$\mathcal{P}_s(\phi)(z)=\int_B P_s(z, b)\phi(b)db\,,\qquad z\in X\,,$$

where db is the normalized K-invariant measure on B.

Let R denote the set of equivalence classes of irreducible unitary representations of K and R^0 denote the subset of those classes which are of class one with respect to the subgroup M of K. For each $\gamma\in R$, we take and fix a representative $(\tau^\gamma, W^\gamma)\in\gamma$ and choose an orthonormal base $\{w_1^\gamma, \cdots, w_{d(\gamma)}^\gamma\}$ of W^γ with respect to the unitary inner product $(\,,\,)$ of W^γ so that w_1^γ is an M-fixed vector if $\gamma\in R^0$, where $d(\gamma)$ is the dimension of W^γ. We identify the functions on B with those on K which are right M-invariant. Let π be the left regular representation of K on $C^\infty(K)$, $C^\infty(B)$ and $C^\infty(X)$, and put

$$V^\gamma=\{\phi\in C^\infty(K)\mid \phi \text{ transforms according to } \gamma \text{ under } \pi\}\,,$$

$$\tau_{ij}^\gamma(k)=(\tau^\gamma(k)w_j^\gamma, w_i^\gamma)\,,$$

$$\phi_{ij}^\gamma(k)=d(\gamma)^{1/2}\bar{\tau}_{ij}^\gamma(k)\,,$$

$$\phi_i^\gamma(k)=\phi_{i1}^\gamma(k)\,,$$

for $\gamma\in R$, $1\leqq i, j\leqq d(\gamma)$. From the Peter-Weyl theory,

$$\{\phi_{ij}^\gamma\mid 1\leqq i, j\leqq d(\gamma)\}$$

is an orthonormal base of V^γ $(\gamma\in R)$ and

$$\{\phi_{ij}\mid \gamma\in R,\ 1\leqq i, j\leqq d(\gamma)\}$$

is a complete orthonormal base of $L^2(K)$. Let W_M^γ be the subspace of M-fixed vectors in W^γ. B. Kostant showed in [**9**, Theorem 6] that $\dim W_M^\gamma=1$ when G is of real rank one. Therefore in our case, W_M^γ is spanned by w_1^γ and

$$\{\phi_i^\gamma \mid \gamma\in R^0,\ 1\leqq i\leqq d(\gamma)\}$$

is a complete orthonormal base of $L^2(B)$.

Let $\varDelta$ be the laplacian corresponding to the G-invariant riemannian metric on X induced by the Killing form of $\mathfrak{g}_0$. We identify the functions on X with those on G which are right K-invariant. For each $s\in \boldsymbol{C}$, put

$$\mathcal{H}_s(X)=\{f\in C^\infty(X) \mid \varDelta f=(s^2-1)<\rho,\ \rho>f\}\,,$$

$$\mathcal{H}_s^\gamma(X)=\{f\in \mathcal{H}_s(X) \mid f \text{ transforms according to } \gamma \text{ under } \pi\}\,,$$

and define a holomorphic function $e(s)$ on $\boldsymbol{C}$ by

$$e(s)=\Gamma\Big(\frac{1}{2}\Big(\frac{p}{2}+1+\Big(\frac{p}{2}+q\Big)s\Big)\Big)^{-1}\Gamma\Big(\frac{1}{2}\Big(\frac{p}{2}+q+\Big(\frac{p}{2}+q\Big)s\Big)\Big)^{-1},$$

where Γ denotes the gamma function. Then we have the following proposition due to S. Helgason:

PROPOSITION 2.1.

(1) *$\mathcal{P}_s$ maps $C(B)$ into $\mathcal{H}_s(X)$ and V^γ into $\mathcal{H}_s^\gamma(X)$.*

(2) *$\mathcal{P}_s$ is injective on $C(B)$ if and only if $e(s)\neq 0$.*

(3) *If $\mathcal{H}_s^\gamma(X)\neq\{0\}$, then γ belongs to R^0.*

(4) *If $\mathcal{P}_s$ is injective, then $\mathcal{P}_s$ maps V^γ onto $\mathcal{H}_s^\gamma(X)$.*

For the proof, see Theorem 1.1 and Theorem 1.4 in Chap. IV of [**5**].

For $\gamma\in R^0$, we put $f_{si}^\gamma=\mathcal{P}_s(\phi_i^\gamma)$ and $f_s^\gamma=f_{s1}^\gamma$. Then we have

PROPOSITION 2.2. *Suppose that $e(s)\neq 0$ and let f be a function in $\mathcal{H}_s(X)$.*

(1) *There exist unique complex numbers $a_i^\gamma\,(\gamma\in R^0,\ 1\leqq i\leqq d(\gamma))$ such that*

$$f(z)=\sum_{\gamma\in R^0}\sum_{i=1}^{d(\gamma)} a_i^\gamma f_{si}^\gamma(z)\,.$$

The series converges absolutely for any z in X.

(2) *Let $\phi_f^z(k)$ be a function on K defined by $\phi_f^z(k)=f(kz)$ $(k\in K)$. Then*

$$\phi_f^z=\sum_{\gamma\in R^0} d(\gamma)^{-1/2}\sum_{i,j=1}^{d(\gamma)} a_i^\gamma f_{sj}^\gamma(z)\phi_{ij}^\gamma\,.$$

The series converges absolutely and uniformly on K.

(3) *Let $\|\ \|$ denote the norm of $L^2(K)$. Then*

$$\|\phi_f^z\|^2=\sum_{\gamma\in R^0} d(\gamma)^{-1}\Big(\sum_{i=1}^{d(\gamma)}|a_i^\gamma|^2\Big)\Big(\sum_{j=1}^{d(\gamma)}|f_{sj}^\gamma(z)|^2\Big)\,.$$

PROOF. By the theory of Fourier expansion of C^∞-functions on compact

Lie groups (cf. [**19**]), ϕ_f^z can be expanded in an absolutely and uniformly convergent series on K:

$$\phi_f^z = \sum_{\gamma\in R} \sum_{i,j=1}^{d(\gamma)} b_{ij}^\gamma(z)\phi_{ij}^\gamma , \tag{2.1}$$

where b_{ij}^γ is given by

$$b_{ij}^\gamma(z) = \int_K f(kz)\bar{\phi}_{ij}^\gamma(k)dk \tag{2.2}$$

and dk is the normalized Haar measure on K. Since Δ is G-invariant and f is a function in $\mathcal{H}_s(X)$, b_{ij}^γ lies in $\mathcal{H}_s(X)$. Furthermore we have from (2.2) that

$$\pi(k)b_{ij}^\gamma = \sum_{l=1}^{d(\gamma)} \tau_{lj}^\gamma(k) b_{il}^\gamma .$$

Therefore b_{ij}^γ lies in $\mathcal{H}_s^\gamma(X)$ for $1\leqq i,\ j\leqq d(\gamma)$. Putting $k=e$ (the identity in K) in (2.1), we have an absolutely convergent expansion of f:

$$f(z) = \sum_{\gamma\in R} d(\gamma)^{1/2} \sum_{i=1}^{d(\gamma)} b_{ii}^\gamma(z) , \tag{2.3}$$

since $\phi_{ij}^\gamma(e) = d(\gamma)^{1/2}\delta_{ij}$. If $\sum_{i=1}^{d(\gamma)} b_{ii}^\gamma(z) \neq 0$, from Proposition 2.1, we can conclude that γ belongs to R^0 and that there exist unique complex numbers a_i^γ ($\gamma\in R^0$, $1\leqq i\leqq d(\gamma)$) such that

$$d(\gamma)^{1/2} \sum_{i=1}^{d(\gamma)} b_{ii}^\gamma = \sum_{i=1}^{d(\gamma)} a_i^\gamma f_{si}^\gamma . \tag{2.4}$$

Since z is arbitrary, replacing z by kz in (2.4), we have

$$\begin{aligned} d(\gamma)^{1/2} \sum_{i=1}^{d(\gamma)} b_{ii}^\gamma(kz) &= d(\gamma)^{1/2} \sum_{i,j=1}^{d(\gamma)} \tau_{ji}^\gamma(k^{-1}) b_{ij}^\gamma(z) \\ &= \sum_{i,j=1}^{d(\gamma)} b_{ij}^\gamma(z)\phi_{ij}^\gamma(k) , \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} f_{si}^\gamma(kz) &= \int_B P_s(kz,\ b)\phi_i^\gamma(b)db \\ &= \int_B P_s(z,\ k^{-1}b)\phi_i^\gamma(b)db \\ &= \int_B P_s(z,\ b)\phi_i^\gamma(kb)db \\ &= d(\gamma)^{-1/2} \sum_{j=1}^{d(\gamma)} \int_B P_s(z,\ b)\phi_{ij}^\gamma(k)\phi_j^\gamma(b)db \\ &= d(\gamma)^{-1/2} \sum_{j=1}^{d(\gamma)} f_{sj}^\gamma(z)\phi_{ij}^\gamma(k) \end{aligned} \tag{2.6}$$

for $1\leqq i\leqq d(\gamma)$. From (2.4), (2.5) and (2.6) we have

$$\sum_{i,j=1}^{d(\gamma)} b_{ij}^{\gamma}(z)\phi_{ij}^{\gamma}=d(\gamma)^{-1/2}\sum_{i,j=1}^{d(\gamma)} a_i^{\gamma} f_{sj}^{\gamma}\phi_{ij}^{\gamma}.$$

Since ϕ_{ij}^{γ} are linearly independent, we can deduce that

$$b_{ij}^{\gamma}=d(\gamma)^{-1/2}a_i^{\gamma}f_{sj}^{\gamma}$$

for $1\leqq i,\ j\leqq d(\gamma)$. Putting $i=l$ in the above equality, we obtain from (2.3) an absolutely convergent expansion of f:

$$f(z)=\sum_{\gamma\in R^0}\sum_{i=1}^{d(\gamma)} a_i^{\gamma} f_{si}^{\gamma}(z),$$

which proves (1) in the proposition.

Next, from (1) and (2.6) we have

$$\begin{aligned}\phi_{f}^{z}(k)=f(kz)&=\sum_{\gamma\in R^0}\sum_{i=1}^{d(\gamma)} a_i^{\gamma} f_{si}^{\gamma}(kz)\\&=\sum_{\gamma\in R^0} d(\gamma)^{-1/2}\sum_{i,j=1}^{d(\gamma)} a_i^{\gamma} f_{sj}^{\gamma}(z)\phi_{ij}^{\gamma}(k),\end{aligned}$$

which proves (2) and (3) immediately. This completes the proof.

We denote by $\mathfrak{B}$ the universal enveloping algebra of $\mathfrak{g}$ and regard the elements of $\mathfrak{B}$ as left G-invariant differential operators on G. Let Ω be the Casimir element of $\mathfrak{B}$. Then, as is well-known,

$$(\Delta f)(xK)=(\Omega f)(x)$$

for $f\in C^{\infty}(X)$, $x\in G$. It is easy to see that $uf=0$ for $f\in C^{\infty}(X)$ and $u\in\mathfrak{B}\mathfrak{k}$. Therefore we may transform Ω module $\mathfrak{B}\mathfrak{k}$. Let L be the differential of the left regular representation of G on $C^{\infty}(X)$ and extend it to the representation of $\mathfrak{B}$. For every root α, we select $X_{\alpha}\in\mathfrak{g}^{\alpha}$ so that $\langle X_{\alpha}, X_{-\alpha}\rangle=1$, and choose bases $\{H_1\}$ and $\{H_2,\cdots,H_m\}$ of $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}_{t}$ respectively so that $\langle H_i, H_j\rangle=\delta_{ij}$ $(1\leqq i,\ j\leqq m)$. Then $H_1,\cdots,H_m$ together with $X_{\alpha}, X_{-\alpha}$ $(\alpha\in P)$ form a base of $\mathfrak{g}$. For $\alpha\in\pm P$, let $X_{\alpha}=Z_{\alpha}+Y_{\alpha}$ where $Z_{\alpha}\in\mathfrak{k}$ and $Y_{\alpha}\in\mathfrak{p}$. Then from (2.14) in [**15**], we have

$$\begin{aligned}(\Omega f)(a)=[\{H_1^2&+\sum_{\alpha\in P_+}(\coth\alpha(H))H_{\bar{\alpha}}\\&-\sum_{\alpha\in P_+}(\sinh\alpha(H))^{-2}L(Z_{\alpha}Z_{-\alpha}+Z_{-\alpha}Z_{\alpha})\}f](a)\end{aligned}\tag{2.7}$$

for any $f\in C^{\infty}(X)$ and $a=\exp H$ $(H\in\mathfrak{a}_{+}-\{0\})$.

Let H_0 be the element of $\mathfrak{a}_{+}$ such that $\mu_0(H_0)=1$. Then $\langle H_0, H_0\rangle=2p+8q$, $H_{\mu_0}=(2p+8q)^{-1}H_0$, $\langle\rho,\rho\rangle=\rho(H_0)^2\langle H_0, H_0\rangle^{-1}$ and we may put $H_1=(2p+8q)^{-1/2}H_0$. For $t\in\boldsymbol{R}$, we put $a_t=\exp tH_0$. Then t can be regarded as a coordinate function on the one-dimensional Lie group A. We put

$$\omega_{\mu_0}=\sum_{\alpha\in P_{\mu_0}}(Z_{\alpha}Z_{-\alpha}+Z_{-\alpha}Z_{\alpha}),$$

$$\omega_{2\mu_0}=\sum_{\alpha\in P_{2\mu_0}}(Z_\alpha Z_{-\alpha}+Z_{-\alpha}Z_\alpha).$$

Then from (2.7) we have immediately the following

PROPOSITION 2.3. *Let f be a function in $\mathcal{H}_s(X)$. Then f satisfies the differential equation*

$$\frac{d^2}{dt^2}f(a_t)+(p\coth t+2q\coth 2t)\frac{d}{dt}f(a_t)-\frac{2p+8q}{(\sinh t)^2}\{L(\omega_{\mu_0})f\}(a_t)$$
$$-\frac{2p+8q}{(\sinh 2t)^2}\{L(\omega_{2\mu_0})f\}(a_t)+(1-s^2)\Big(\frac{p}{2}+q\Big)^2 f(a_t)=0.$$

§ 3. Fatou type theorem for symmetric spaces of rank one.

In this section we shall prove a Fatou type theorem for a symmetric space of rank one (Theorem 3.1) which will be used in § 4 for determination of the Poisson transform of a K-finite function. Put $f_s=\mathcal{P}_s(1_B)$, where 1_B denotes the constant function identically equal to 1 on B. We remark that f_s coincides with Harish-Chandra's spherical function ϕ_λ ($\lambda=-(-1)^{1/2}s\rho$).

THEOREM 3.1. *Let s be a complex number and assume that* $\mathrm{Re}(s)>0$. *Then $f_s(aK)$ ($a\in A$) is not equal to zero when $\rho(H(a))$ is sufficiently large, and for any continuous function ϕ on B*

$$\lim_{\rho(H(a))\to\infty}\frac{1}{f_s(aK)}\mathcal{P}_s(\phi)(kaK)=\phi(kM)$$

uniformly on B.

For the proof of the theorem, we need several lemmas. We use the parameter t on A introduced in § 2.

LEMMA 3.2. *Let s be a complex number and put $\sigma=p/2+q$. Then*

$$f_s(a_tK)=(\cosh t)^{(s-1)\sigma}F\Big(\frac{1-s}{2}\sigma,\frac{1-s}{2}\sigma+\frac{1-q}{2},\frac{p+q+1}{2};(\tanh t)^2\Big),$$

where F denotes the hypergeometric function.

PROOF. We consider the differential equation in Proposition 2.3. Since f_s is K-invariant, we have

$$L(\omega_{\mu_0})f_s=L(\omega_{2\mu_0})f_s=0,$$

and therefore we obtain a differential equation

$$\frac{d^2f_s}{dt^2}+(p\coth t+2q\coth 2t)\frac{df_s}{dt}+(1-s^2)\sigma^2f_s=0.$$

We put $z=(\tanh t)^2$. Then the above differential equation becomes

$$4z(1-z)^2\frac{d^2f_s}{dz^2}+2(1-z)\{(p+q+1)+(q-3)z\}\frac{df_s}{dz}+(1-s^2)\sigma^2f_s=0,$$

whose fundamental system of solutions is given by

$$(1-z)^{(1-s)\sigma/2}F\Big(\frac{1-s}{2}\sigma,\ \frac{1-s}{2}\sigma+\frac{1-q}{2},\ \frac{p+q+1}{2}\ ;\ z\Big),$$

$$z^{-(p+q-1)/2}(1-z)^{(1-s)\sigma/2}F\Big(-\frac{1+s}{2}\sigma+1,\ \frac{1-s}{2}\sigma-\frac{p+q-1}{2},\ -\frac{p+q-3}{2}\ ;\ z\Big).$$

Observing that f_s is a C^∞-function in t, we can find a constant c such that

$$f_s(a_tK)=c(\cosh t)^{(s-1)\sigma}F\Big(\frac{1-s}{2}\sigma,\ \frac{1-s}{2}\sigma+\frac{1-q}{2},\ \frac{p+q+1}{2}\ ;\ (\tanh t)^2\Big).$$

From the definition of f_s, $f_s(eK)=1$. Then it is easy to see that $c=1$, which completes the proof.

LEMMA 3.3. *Suppose that $\xi=\mathrm{Re}\,(s)>0$. Then there exists a $\delta>0$ such that*

$$2\delta(\cosh t)^{(\xi-1)\sigma}\geqq|f_s(a_tK)|\geqq\delta(\cosh t)^{(\xi-1)\sigma}$$

for sufficiently large t.

PROOF. As is well-known (cf. [**11**, p. 244]), for a hypergeometric function $F(\alpha,\beta,\gamma\,;z)$,

$$\lim_{z\to1-0}F(\alpha,\beta,\gamma\,;z)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

provided that $\mathrm{Re}\,(\gamma)>0$ and $\mathrm{Re}\,(\gamma-\alpha-\beta)>0$. If we put $\alpha=(1-s)\sigma/2$, $\beta=(1-s)\sigma/2+(1-q)/2$ and $\gamma=(p+q+1)/2$, then $\mathrm{Re}(\gamma)>0$ and $\mathrm{Re}(\gamma-\alpha-\beta)=\mathrm{Re}\,(s\sigma)=\xi\sigma>0$. Hence

$$\lim_{t\to\infty}F\Big(\frac{1-s}{2}\sigma,\ \frac{1-s}{2}\sigma+\frac{1-q}{2},\ \frac{p+q+1}{2}\ ;\ (\tanh t)^2\Big)$$

$$=\frac{\Gamma\Big(\frac{p+q+1}{2}\Big)\Gamma(s\sigma)}{\Gamma\Big(\frac{1+s}{2}\sigma\Big)\Gamma\Big(\frac{1+s}{2}\sigma+\frac{1-q}{2}\Big)}.$$

Now we put

$$\delta=\frac{2}{3}\left|\frac{\Gamma\Big(\frac{p+q+1}{2}\Big)\Gamma(s\sigma)}{\Gamma\Big(\frac{1+s}{2}\sigma\Big)\Gamma\Big(\frac{1+s}{2}\sigma+\frac{1-q}{2}\Big)}\right|.$$

Then δ is finite and positive, and

$$2\delta\geqq\left|F\Big(\frac{1-s}{2}\sigma,\ \frac{1-s}{2}\sigma+\frac{1-q}{2},\ \frac{p+q+1}{2}\ ;\ (\tanh t)^2\Big)\right|\geqq\delta$$

for sufficiently large t. Taking into account that

$$|(\cosh t)^{(s-1)\sigma}|=(\cosh t)^{(\xi-1)\sigma},$$

we obtain the required inequality from Lemma 3.2, which finishes the proof.

From Lemma 3.3, we have immediately the following

COROLLARY 3.4. *Suppose that* $\xi=\operatorname{Re}(s)>0$. *Then there exists an* $\eta>0$ *such that*

$$\frac{f_\xi(a_tK)}{|f_s(a_tK)|}\leqq\eta$$

for sufficiently large t.

LEMMA 3.5. *Assume that* $\operatorname{Re}(s)>0$. *Then for any neighborhood* U *of* eM *in* B,

$$\lim_{t\to\infty}\sup_{b\in B-U}\left|\frac{P_s(a_tK, b)}{f_s(a_tK)}\right|=0 .$$

PROOF. Put $\xi=\operatorname{Re}(s)$. From Theorem 3.1 in [8], there exists a positive continuous function F on $B^*=B-\{eM\}$ such that

$$\exp\{-2\rho(H(a^{-1}k))\}\leqq F(kM)\exp\{-2\rho(H(a))\}$$

for $kM\in B^*$ and $a\in A$ with $\rho(H(a))>0$. Consequently

$$\begin{aligned}|P_s(a_tK, kM)|&=P_\xi(a_tK, kM)\\&=\exp\{-(1+\xi)\rho(H(a_t^{-1}k))\}\\&\leqq\psi(kM)\exp\{-(1+\xi)\sigma t\}\ ,\end{aligned}$$

where we put

$$\psi(kM)=F(kM)^{(1+\xi)/2} .$$

From Lemma 3.3, there exists a $\delta>0$ such that

$$|f_s(a_tK)|\geqq\delta(\cosh t)^{(\xi-1)\sigma}$$

for sufficiently large t. Hence

$$\begin{aligned}\left|\frac{P_s(a_tK, b)}{f_s(a_tK)}\right|&\leqq\delta^{-1}\psi(b)(\cosh t)^{(1-\xi)\sigma}\exp\{-(1+\xi)\sigma t\}\\&\leqq\delta^{-1}\psi(b)\exp\{(|1-\xi|-(1+\xi))\sigma t\}\end{aligned}$$

for such t, because

$$(\cosh t)^{(1-\xi)\sigma}\leqq(\cosh t)^{|1-\xi|\sigma}\leqq\exp(|1-\xi|\sigma|t|) .$$

Since $|1-\xi|-(1+\xi)<0$ and ψ is bounded on $B-U$, we can see that

$$\lim_{t\to\infty}\sup_{b\in B-U}\left|\frac{P_s(a_tK, b)}{f_s(a_tK)}\right|=0 ,$$

which completes the proof.

PROOF OF THEOREM 3.1. We put $a=a_t$ and notice that $\rho(H(a))\to\infty$ is equivalent to $t\to\infty$. Then the first assertion of the theorem is a consequence of Lemma 3.3. Since

$$\mathcal{P}_s(\phi)(ka_tK)=\int_B P_s(ka_tK,\,b)\phi(b)db$$

$$=\int_B P_s(a_tK,\,b)\phi(kb)db$$

and

$$f_s(a_tK)=\mathcal{P}_s(1_B)(a_t)=\int_B P_s(a_tK,\,b)db\,,$$

we have

$$\left|\frac{1}{f_s(a_tK)}\,\mathcal{P}_s(\phi)(ka_tK)-\phi(kM)\right|$$

$$=\left|\frac{1}{f_s(a_tK)}\int_B P_s(a_tK,\,b)(\phi(kb)-\phi(kM))db\right|$$

$$\leqq\int_B\left|\frac{P_s(a_tK,\,b)}{f_s(a_tK)}\right||\phi(kb)-\phi(kM)|\,db\,. \tag{3.1}$$

On the other hand, since $\phi\in C(B)$, for any $\varepsilon>0$ we can find a neighbourhood U of eM in B such that

$$|\phi(kb)-\phi(kM)|<\varepsilon$$

for any $k\in K$ and any $b\in U$. Putting

$$m=2\sup_{b\in B}|\phi(b)|,$$

we have

$$\int_B\left|\frac{P_s(a_tK,\,b)}{f_s(a_tK)}\right||\phi(kb)-\phi(kM)|\,db$$

$$\leqq\varepsilon\int_U\left|\frac{P_s(a_tK,\,b)}{f_s(a_tK)}\right|db+m\sup_{b\in B-U}\left|\frac{P_s(a_tK,\,b)}{f_s(a_tK)}\right|. \tag{3.2}$$

Moreover

$$\int_U\left|\frac{P_s(a_tK,\,b)}{f_s(a_tK)}\right|db=\int_U\frac{P_\xi(a_tK,\,b)}{|f_s(a_tK)|}\,db$$

$$=\frac{f_\xi(a_tK)}{|f_s(a_tK)|}\int_U\frac{P_\xi(a_tK,\,b)}{f_\xi(a_tK)}\,db$$

$$\leqq\frac{f_\xi(a_tK)}{|f_s(a_tK)|}\int_B\frac{P_\xi(a_tK,\,b)}{f_\xi(a_tK)}\,db$$

$$=\frac{f_\xi(a_tK)}{|f_s(a_tK)|}\,. \tag{3.3}$$

Hence from (3.1), (3.2) and (3.3) we have

$$\left|\frac{1}{f_s(a_tK)}\,\mathcal{P}_s(\phi)(ka_tK)-\phi(kM)\right|$$

$$\leqq\varepsilon\frac{f_\xi(a_tK)}{|f_s(a_tK)|}+m\sup_{b\in B-U}\left|\frac{P_s(a_tK,\,b)}{f_s(a_tK)}\right|.$$

This inequality together with Corollary 3.4 and Lemma 3.5 leads to our theorem.

§ 4. K-finite eigenfunctions of Δ on a real hyperbolic space.

From now on, we assume moreover that $G=SO_0(n, 1)$ $(n\geqq 3)$, a generalized Lorentz group. The associated symmetric space $X=G/K$ is called a real hyperbolic space. We notice that for $G=SO_0(n, 1)$, we have $p=n-1$, $q=0$, $P_{\mu_0}=P_+$, $P_{2\mu_0}=\emptyset$ and $\sigma=p/2+q=(n-1)/2$.

In this section we determine the Poisson transform of a K-finite function on B, by using the Fatou type theorem. The maximal compact subgroup K is isomorphic to $SO(n)$ and M is isomorphic to $SO(n-1)$. Therefore $B=K/M$ is (real-analytically) isomorphic to S^{n-1} ($(n-1)$-dimensional sphere). Le ω_K be the Casimir operator of K and let R^0 be the set of equivalence classes of irreducible unitary representations of K of class one with respect to M. Since the elements of V^γ transform according to γ under the representation π of K, $\pi(\omega_K)$ is a scalar operator on V^γ, where π denotes also the differential of π. We denote this scalar by $\lambda(\gamma)$. In this case there exists a bijection Λ of R^0 onto $\boldsymbol{N}^0$ such that

$$\lambda(\gamma)=\frac{l(l+n-2)}{2(n-2)}, \tag{4.1}$$

where $l=\Lambda(\gamma)$ (for details, see [**15**, § 3]). By this bijection Λ, we identify R^0 with $\boldsymbol{N}^0$ and write τ^l, V^l, $\lambda(l)$, $d(l)$, $\mathcal{H}_s^l$, ϕ_i^l and f_{si}^l instead of τ^γ, V^γ, $\lambda(\gamma)$, $d(\gamma)$, $\mathcal{H}_s^\gamma$, ϕ_i^γ and f_{si}^γ respectively.

Let $\mathfrak{m}_0$ be the Lie algebra of M and $\mathfrak{m}$ be its complexification in $\mathfrak{g}$. Then from Lemma 3.2 in [**15**],

$$\omega_{\mu_0}\equiv\frac{n-2}{n-1}\,\omega_K \bmod \mathfrak{m}_0\mathfrak{B}. \tag{4.2}$$

By the way, since M centralizes A,

$$f(\exp tY)a)=f(a\exp tY)$$

for $a\in A$, $Y\in\mathfrak{m}_0$ and $t\in\boldsymbol{R}$. Hence we have

$$(L(u)f)(aK)=0 \tag{4.3}$$

for $f\in C^\infty(X)$, $a\in A$ and $u\in\mathfrak{m}\mathfrak{B}$.

LEMMA 4.1. *Let s be a complex number and l be a non-negative integer. Then for each function f in $\mathcal{H}_s^l$,*

$$(L(\omega_{\mu_0})f)(aK)=\frac{l(l+n-2)}{2(n-1)}f(aK) \qquad (a\in A).$$

PROOF. It is clear that $L(\omega_K)$ is a scalar operator on $\mathcal{H}_s^l$ and the scalar is equal to $\lambda(l)$. From (4.1), (4.2) and (4.3)

$$(L(\omega_{\mu_0})f)(aK)=\frac{n-2}{n-1}(L(\omega_K)f)(aK)$$
$$=-\frac{l(l+n-2)}{2(n-1)}f(aK),$$

which completes the proof.

PROPOSITION 4.2. *Let s and l be as in Lemma 4.1. Then for each function f in $\mathcal{H}_s^l$, there exists a constant $c\in \boldsymbol{C}$ such that*

$$f(a_{2t}K)=c(\tanh t)^l(\cosh t)^{2(s-1)\sigma}$$
$$\times F\Big(l+(1-s)\sigma,\ -s\sigma+\frac{1}{2},\ l+\sigma+\frac{1}{2};\ (\tanh t)^2\Big),$$

where $\sigma=(n-1)/2$.

PROOF. From Proposition 2.3 and Lemma 4.1 it follows that f satisfies the differential equation

$$\frac{d^2f}{dt^2}+(n-1)\coth t\frac{df}{dt}-\frac{l(l+n-2)}{(\sinh t)^2}f+(1-s^2)\sigma^2f=0.$$

We introduce a parameter $z=(\tanh(t/2))^2$. Then the above differential equation turns into

$$z(1-z)^2\frac{d^2f}{dz^2}+\frac{1}{2}(1-z)(nz-4z+n)\frac{df}{dz}-l(l+n-2)\frac{(1-z)^2}{4z}f+(1-s^2)\sigma^2f=0.$$

A fundamental system of solutions of this differential equation is given by

$$z^{l/2}(1-z)^{(1-s)\sigma}F\Big(l+(1-s)\sigma,\ -s\sigma+\frac{1}{2},\ l+\sigma+\frac{1}{2};\ z\Big),$$
$$z^{-l/2-\sigma+1/2}(1-z)^{(1-s)\sigma}F\Big(-l-(1+s)\sigma+1,\ -s\sigma+\frac{1}{2},\ -l-\sigma+\frac{3}{2};\ z\Big).$$

Since $f(a_{2t}K)$ is a C^∞-function in t and $1-z=(\cosh t)^{-2}$, there exists a constant $c\in \boldsymbol{C}$ such that

$$f(a_{2t}K)=c(\tanh t)^l(\cosh t)^{2(s-1)\sigma}$$
$$\times F\Big(l+(1-s)\sigma,\ -s\sigma+\frac{1}{2},\ l+\sigma+\frac{1}{2};\ (\tanh t)^2\Big),$$

which completes the proof.

Now, using Proposition 4.2, we shall determine the Poisson transform of a K-finite function on B explicitly.

PROPOSITION 4.3. *Let s and l be as in Lemma 4.1, ϕ be a function in V^l, and put $f=\mathcal{P}_s(\phi)$. Then*

$$f(a_{2t}K)=\phi(eM)\frac{\Gamma\Big(\sigma+\frac{1}{2}\Big)\Gamma(l+(1+s)\sigma)}{\Gamma\Big(l+\sigma+\frac{1}{2}\Big)\Gamma((1+s)\sigma)}(\tanh t)^l(\cosh t)^{2(s-1)\sigma}$$

$$\times F\Big(l+(1-s)\sigma,\ -s\sigma+\frac{1}{2},\ l+\sigma+\frac{1}{2}\ ;\ (\tanh t)^2\Big).$$

PROOF. From Proposition 4.2, it follows that there exists a constant c such that

$$f(a_{2t}K)=c(\tanh t)^l(\cosh t)^{2(s-1)\sigma}$$
$$\times F\Big(l+(1-s)\sigma,\ -s\sigma+\frac{1}{2},\ l+\sigma+\frac{1}{2}\ ;\ (\tanh t)^2\Big). \tag{4.4}$$

We notice that f_s^0 coincides with $f_s=\mathcal{P}_s(1_B)$ defined in § 3, since $f_s^0=\mathcal{P}_s(\phi_1^0)$ and $\phi_1^0=1_B$. Observing that $f_s\in\mathcal{H}_s^0$ and $f_s(eK)=1$, we have

$$f_s(a_{2t}K)=(\cosh t)^{2(s-1)\sigma}F\Big((1-s)\sigma,\ -s\sigma+\frac{1}{2},\ \sigma+\frac{1}{2}\ ;\ (\tanh t)^2\Big). \tag{4.5}$$

Now we assume that $\mathrm{Re}(s)>0$. Then from Theorem 3.1,

$$\lim_{t\to\infty}\frac{f(a_{2t}K)}{f_s(a_{2t}K)}=\lim_{t\to\infty}\frac{\mathcal{P}_s(\phi)(a_{2t}K)}{f_s(a_{2t}K)}=\phi(eM). \tag{4.6}$$

On the other hand, from (4.4) and (4.5) we have

$$\lim_{t\to\infty}\frac{f(a_{2t}K)}{f_s(a_{2t}K)}=c\frac{\Gamma\Big(l+\sigma+\frac{1}{2}\Big)\Gamma(2s\sigma)}{\Gamma\Big(s\sigma+\frac{1}{2}\Big)\Gamma(l+(1+s)\sigma)}\cdot\frac{\Gamma\Big(s\sigma+\frac{1}{2}\Big)\Gamma((1+s)\sigma)}{\Gamma\Big(\sigma+\frac{1}{2}\Big)\Gamma(2s\sigma)}$$
$$=c\frac{\Gamma\Big(l+\sigma+\frac{1}{2}\Big)\Gamma((1+s)\sigma)}{\Gamma\Big(\sigma+\frac{1}{2}\Big)\Gamma(l+(1+s)\sigma)}, \tag{4.7}$$

since

$$\lim_{t\to\infty}F\Big(l+(1-s)\sigma,\ -s\sigma+\frac{1}{2},\ l+\sigma+\frac{1}{2}\ ;\ (\tanh t)^2\Big)$$
$$=\frac{\Gamma\Big(l+\sigma+\frac{1}{2}\Big)\Gamma(2s\sigma)}{\Gamma\Big(s\sigma+\frac{1}{2}\Big)\Gamma(l+(1+s)\sigma)}$$

for $\mathrm{Re}(s)>0$ (cf. [**11**, p. 244]). Therefore it follows from (4.6) and (4.7) that

$$c=\phi(eM)\frac{\Gamma\Big(\sigma+\frac{1}{2}\Big)\Gamma(l+(1+s)\sigma)}{\Gamma\Big(l+\sigma+\frac{1}{2}\Big)\Gamma((1+s)\sigma)}.$$

Hence from (4.4) we obtain

$$f(a_{2t}K)=\phi(eM)\frac{\Gamma\Big(\sigma+\frac{1}{2}\Big)\Gamma(l+(1+s)\sigma)}{\Gamma\Big(l+\sigma+\frac{1}{2}\Big)\Gamma((1+s)\sigma)}(\tanh t)^l(\cosh t)^{2(s-1)\sigma}$$

$$\times F\Big(l+(1-s)\sigma,\ -s\sigma+\frac{1}{2},\ l+\sigma+\frac{1}{2};\ (\tanh t)^2\Big) \tag{4.8}$$

for $\mathrm{Re}(s)>0$. We fix t and l in (4.8). Then both sides of (4.8) are holomorphic in s. Therefore (4.8) is valid for any $s\in \boldsymbol{C}$ from the uniqueness of analytic continuation, which finishes the proof.

We recall that $f_s^l=f_{s1}^l$.

COROLLARY 4.4. *Let s and l be as in Lemma 4.1. Then*

$$f_s^l(a_{2t}K)=d(l)^{1/2}\frac{\Gamma\Big(\sigma+\frac{1}{2}\Big)\Gamma(l+(1+s)\sigma)}{\Gamma\Big(l+\sigma+\frac{1}{2}\Big)\Gamma((1+s)\sigma)}(\tanh t)^l(\cosh t)^{2(s-1)\sigma}$$

$$\times F\Big(l+(1-s)\sigma,\ -s\sigma+\frac{1}{2},\ l+\sigma+\frac{1}{2};\ (\tanh t)^2\Big),$$

$$f_{si}^l(a_{2t}K)=0 \qquad (2\leqq i\leqq d(l)),$$

where

$$d(l)=\frac{2l+n-2}{n-2}\cdot\frac{\Gamma(l+n-2)}{\Gamma(l+1)\Gamma(n-2)}.$$

PROOF. For $K=SO(n)$, $d(l)$ is given by ([**17**, p. 68])

$$d(l)=\frac{2l+n-2}{n-2}\cdot\frac{\Gamma(l+n-2)}{\Gamma(l+1)\Gamma(n-2)}.$$

Since $\phi_i^l(eM)=d(l)^{1/2}\delta_{il}$, applying Proposition 4.3 for $\phi=\phi_i^l$ we have this corollary.

REMARK. Put

$$e(l,s)=\frac{\Gamma\Big(\sigma+\frac{1}{2}\Big)\Gamma(l+(1+s)\sigma)}{\Gamma\Big(l+\sigma+\frac{1}{2}\Big)\Gamma((1+s)\sigma)}.$$

From Corollary 4.4, we can conclude that $\mathcal{P}_s$ is injective on V^l if and only if $e(l,s)\neq 0$. On the other hand for $G=SO_0(n,1)$

$$e(s)=\frac{1}{\Gamma\Big(\frac{1}{2}(1+s)\sigma+\frac{1}{2}\Big)\Gamma\Big(\frac{1}{2}(1+s)\sigma\Big)}$$

$$=\frac{2^{(1+s)\sigma}}{2\pi^{1/2}\Gamma((1+s)\sigma)}.$$

From Proposition 2.1 $e(s)\neq 0$ if and only if $\mathcal{P}_s$ is injective on $C(B)$. But we can obtain a more precise information on the injectivity of $\mathcal{P}_s$ on V^l by $e(l,s)$. In fact, assume that $e(s)=0$. Then $-(1+s)\sigma\in \boldsymbol{N}^0$, and for $l>-(1+s)\sigma$, $e(l,s)=0$. Hence $\mathcal{P}_s(V^l)=\{0\}$ for $l>-(1+s)\sigma$.

§ 5. Poisson transform of a hyperfunction on B.

In this section we define the Poisson transform $\mathcal{P}_s(T)$ on a real hyperbolic space X of a hyperfunction T on $B=K/M$ and prove Theorem 5.5 which asserts that if $e(s)\neq 0$, $\mathcal{P}_s$ is an isomorphism of $\mathcal{B}(B)$ onto $\mathcal{H}_s(X)$, where $\mathcal{B}(B)$ is the space of Sato's hyperfunctions ([18]) and $\mathcal{H}_s(X)$ is the space of eigenfunctions of Δ on X with eigenvalue $(s^2-1)\langle\rho, \rho\rangle$.

At first we review the topology of the space of real analytic functions on a real analytic manifold. Let F be a paracompact connected real analytic manifold of dimension m. Then there exists a paracompact complex m-dimensional manifold W which contains F as a real analytic closed submanifold (see [1]). For an open subset U of W, we denote by $H(U)$ the space of holomorphic functions on U topologized by uniform convergence on compact subsets. Then $\mathcal{A}(F)$, the space of real analytic functions on F, is topologized by

$$\mathcal{A}(F)=\varinjlim_{U\supset F} H(U)\,,$$

where $\varinjlim H(U)$ denotes the inductive limit of the topological space $H(U)$ (cf. [12], [14]). We denote by $\mathcal{A}'(F)$ the space of continuous linear functions of $\mathcal{A}(F)$ into $\boldsymbol{C}$. The elements of $\mathcal{A}'(F)$ are called analytic functionals on F. If, in particular, F is an oriented compact connected real analytic manifold, by Sato [18] $\mathcal{A}'(F)$ is canonically isomorphic to $\mathcal{B}(F)$, the space of hyperfunctions on F. (According to [14], $\mathcal{A}'(F)$ is isomorphic to $\mathcal{B}(F)$ even if F is not oriented.)

On the other hand, if F is a compact connected real analytic riemannian manifold, $\mathcal{A}(F)$ can be characterized as follows (for details see [15, § 1] or [4, § 1]). Let ω be the laplacian on F and $L^2(F)$ be the space of square-integrable functions on F with respect to the measure induced by the riemannian metric on F. We denote the unitary inner product and the norm of $L^2(F)$ by $(\ ,\ )$ and $\|\ \|$ respectively. As is well-known, the eigenvalues of ω are non-negative and countable, and the space of eigenfunctions of each eigenvalue is finite-dimensional. We denote the eigenvalues of ω by λ_n $(n\in\boldsymbol{N}^0)$ and order them so that $\lambda_n<\lambda_m$ if $n<m$. Let E_n be the space of eigenfunctions of ω with eigenvalue λ_n and $d(n)$ be the dimension of E_n. Then as an orthonormal base of E_n, we can choose analytic functions ϕ_i^n on F $(n\in\boldsymbol{N}^0,\ 1\leqq i\leqq d(n))$, and

$$\{\phi_i^n \mid n\in\boldsymbol{N}^0,\ 1\leqq i\leqq d(n)\}$$

is a complete orthonormal base of $L^2(F)$. For each $\phi\in C^\infty(F)$ we define $\omega^{1/2}\phi\in C^\infty(F)$ by

$$\omega^{1/2}\phi=\sum_{n\in N0}\lambda_n^{1/2}\sum_{i=1}^{d(n)}a_i^n\phi_i^n\,,$$

where $a_i^n=(\phi,\phi_i^n)$. We define seminorms $\|\ \|_h$ $(h>0)$ on $C^\infty(F)$ and subspaces $\mathcal{A}_h(F)$ of $C^\infty(F)$ by

$$\|\phi\|_h=\sup_{m\in N0}\frac{\|\omega^{m/2}\phi\|}{m!\,h^m}\,,$$

$$\mathcal{A}_h(F)=\{\phi\in C^\infty(F)\mid\|\phi\|_h<\infty\}\,.$$

Then $\mathcal{A}_h(F)$ is a Banach space with the norm $\|\ \|_h$. From Proposition 1.6 in [**15**], $\mathcal{A}(F)$ coincides with the inductive limit of $\mathcal{A}_h(F)$ as a topological space. That is,

$$\mathcal{A}(F)=\varinjlim_{h\to\infty}\mathcal{A}_h(F)\,.$$

From now on, we put $F=B\,(=K/M)$. Since $K\cong SO(n)$ and $M\cong SO(n-1)$, B is real-analytically isomorphic to the $(n-1)$-dimensional sphere S^{n-1}. Therefore $\mathcal{A}'(B)$ is isomorphic to $\mathcal{B}(B)$ by the above arguments. Henceforth we write $\mathcal{B}(B)$ for $\mathcal{A}'(B)$ and call the elements of $\mathcal{A}'(B)$ hyperfunctions on B. We denote the value of $T\in\mathcal{B}(B)$ at $\phi\in\mathcal{A}(B)$ by

$$\int_B\phi(b)dT(b)\,.$$

As in [**15**], we take the Casimir operator ω_K as the laplacian on B. Then we can take λ_l, V^l, $d(l)$ which are introduced in §2 and §4 as λ_n, E_n, $d(n)$ respectively. Put

$$\mathcal{F}_b(B)=\{(a_i^l)_{1\leq i\leq d(l)}^{l\in N0}\mid a_i^l\in\boldsymbol{C},\ \sum_{l\in N0}\sum_{i=1}^{d(l)}|a_i^l|\exp(-t\lambda_l^{1/2})<\infty\quad\text{for any } t>0\}$$

and define a mapping Ψ of $\mathcal{B}(B)$ into $\boldsymbol{C}^N$ by

$$\Psi(T)=(a_i^l)\,,\qquad a_i^l=\int_B\bar{\phi}_i^l(b)dT(b)\,,$$

for $T\in\mathcal{B}(B)$. Then by Theorem 1.8 and the remark in [**15**, §1], Ψ is an isomorphism of $\mathcal{B}(B)$ onto $\mathcal{F}_b(B)$ and $\mathcal{F}_b(B)$ is also given by

$$\mathcal{F}_b(B)=\{(a_i^l)_{1\leq i\leq d(l)}^{l\in N0}\mid a_i^l\in\boldsymbol{C},\ \sum_{l\in N0}\sum_{i=1}^{d(l)}|a_i^l|^2\exp(-t\lambda_l^{1/2})<\infty\quad\text{for any } t>0\}\,.\tag{5.1}$$

Now, we define the Poisson transform $\mathcal{P}_s(T)$ on the real hyperbolic space X of a hyperfunction T on B. Since the Poisson kernel $P_s(z,b)$ is real analytic in b, we can operate $T\in\mathcal{B}(B)$ on $P_s(z,b)$. Thus we put

$$\mathcal{P}_s(T)(z)=\int_B P_s(z,b)dT(b)\,.$$

PROPOSITION 5.1. *Let T be a hyperfunction on B and put $\Psi(T)=(a_i^l)$. Then*

$$\mathcal{P}_s(T)(z)=\sum_{l\in N^0}\sum_{i=1}^{d(l)} a_i^l f_{si}^l(z)$$

for any z in X, where $f_{si}^l=\mathcal{P}_s(\phi_i^l)$.

PROOF. Fix an arbitrary z in X. Then $P_{\bar{s}}(z, b)$ can be expanded in an absolutely and uniformly convergent Fourier series

$$P_{\bar{s}}(z, b)=\sum_{l\in N^0}\sum_{i=1}^{d(l)}\phi_i^l(b)\int_B P_{\bar{s}}(z, b)\bar{\phi}_i^l(b)db\,,$$

which converges also in $\mathcal{A}(B)$ by Corollary 1 to Proposition 1.7 in [15]. Taking complex conjugate of the above equality, we have

$$P_s(z, b)=\sum_{l\in N^0}\sum_{i=1}^{d(l)}\bar{\phi}_i^l(b)\int_B P_s(z, b)\phi_i^l(b)db\,,$$

which also converges in $\mathcal{A}(B)$. From the continuity of T on $\mathcal{A}(B)$ we have

$$\mathcal{P}_s(T)(z)=\sum_{l\in N^0}\sum_{i=1}^{d(l)}\int_B\bar{\phi}_i^l(b)dT(b)\int_B P_s(z, b)\phi_i^l(b)db\,.$$

Since

$$a_i^l=\int_B\bar{\phi}_i^l(b)dT(b)$$

and

$$f_{si}^l(z)=\int_B P_s(z, b)\phi_i^l(b)db\,,$$

we obtain

$$\mathcal{P}_s(T)(z)=\sum_{l\in N^0}\sum_{i=1}^{d(l)} a_i^l f_{si}^l(z)\,,$$

which completes the proof.

PROPOSITION 5.2. (1) *For any complex number s and any sequence (a_i^l) in $\mathcal{F}_b(B)$, the series*

$$\sum_{l\in N^0}\sum_{i=1}^{d(l)} a_i^l f_{si}^l(z)$$

converges absolutely and uniformly on every compact subset in X.

(1) *Suppose furthermore that $e(s)\neq 0$. Let f be a function in $\mathcal{H}_s(X)$ and expand f as*

$$f=\sum_{l\in N^0}\sum_{i=1}^{d(l)} a_i^l f_{si}^l$$

by Proposition 2.2. Then the sequence (a_i^l) lies in $\mathcal{F}_b(B)$.

For the proof of the proposition, we need the following

LEMMA 5.3. *Let α and β be complex numbers and γ be a positive number. Then for any h with $0<h<1$, there exists an integer $l_0\in N^0$ such that for any integer $l\geqq l_0$,*

(1) $|F(l+\alpha, \beta, l+\gamma\,; z)|\leqq(1-h)^{-|\beta|}$ *for* $|z|\leqq h^2$,

(2) $|F(l+\alpha, \beta, l+\gamma; h)| \geqq \frac{1}{2}(1-h)^{-\mathrm{Re}(\beta)}$.

PROOF. We notice that ([**11**, p. 258])

$$F(1, \beta, 1; z)=(1-z)^{-\beta},$$

and that

$$\lim_{m\to\infty}\frac{m+|\alpha|}{m+\gamma}=\lim_{m\to\infty}\frac{m+\alpha}{m+\gamma}=1. \tag{5.2}$$

(1) Assume that $|z| \leqq h^2$. Then

$$\begin{aligned}|F(l+\alpha, \beta, l+\gamma; z)| &= \left|\sum_{n=0}^{\infty}\frac{(l+\alpha)_n(\beta)_n}{(l+\gamma)_n}\frac{z^n}{n!}\right| \\ &\leqq \sum_{n=0}^{\infty}\frac{(l+|\alpha|)_n}{(l+\gamma)_n}(|\beta|)_n\frac{h^{2n}}{n!}.\end{aligned}$$

Since $1/h>1$, by (5.2) we can choose a non-negative integer l_0 so that

$$0 \leqq \frac{m+|\alpha|}{m+\gamma} \leqq \frac{1}{h}$$

for $m \geqq l_0$. Then for $l \geqq l_0$ and $n \geqq 0$,

$$\frac{(l+|\alpha|)_n}{(l+\gamma)_n} \leqq \frac{1}{h^n}.$$

Therefore we have

$$\begin{aligned}|F(l+\alpha, \beta, l+\gamma; z)| &\leqq \sum_{n=0}^{\infty}\frac{1}{h^n}(|\beta|)_n\frac{h^{2n}}{n!}=\sum_{n=0}^{\infty}(|\beta|)_n\frac{h^n}{n!} \\ &= F(1, |\beta|, 1; h)=(1-h)^{-|\beta|}.\end{aligned}$$

(2) Since $(1-x)^{-|\beta|}$ is continuous in $-1<x<1$, we can choose an $\varepsilon>1$ such that $\varepsilon h<1$ and

$$0<(1-\varepsilon h)^{-|\beta|}-(1-h)^{-|\beta|} \leqq \frac{1}{2}(1-h)^{-\mathrm{Re}(\beta)}. \tag{5.3}$$

On the other hand, since

$$\begin{aligned}F(l+\alpha, \beta, l+\gamma; h)-(1-h)^{-\beta} &= \sum_{n=0}^{\infty}\frac{(l+\alpha)_n(\beta)_n}{(l+\gamma)_n}\frac{h^n}{n!}-\sum_{n=0}^{\infty}(\beta)_n\frac{h^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{(l+\alpha)_n}{(l+\gamma)_n}-1\right)(\beta)_n\frac{h^n}{n!},\end{aligned}$$

we have

$$|F(l+\alpha, \beta, l+\gamma; h)-(1-h)^{-\beta}| \leqq \sum_{n=0}^{\infty}\left|\frac{(l+\alpha)_n}{(l+\gamma)_n}-1\right|(|\beta|)_n\frac{h^n}{n!}.$$

We can choose an integer l_0 by (5.2) such that for $m \geqq l_0$

$$\left|\frac{m+\alpha}{m+\gamma}-1\right| \leqq \varepsilon-1.$$

By the way, it can be easily shown that $|z_i-1| \leqq \varepsilon-1$ $(i=1, \cdots, k)$ implies

$|z_1 \cdots z_k - 1| \leqq \varepsilon^k - 1$. Therefore we have

$$\left|\frac{(l+\alpha)_n}{(l+\gamma)_n} - 1\right| \leqq \varepsilon^n - 1$$

for $l \geqq l_0$ and $n \geqq 0$. Hence we obtain

$$\begin{aligned} |F(l+\alpha, \beta, l+\gamma; h) - (1-h)^{-\beta}| &\leqq \sum_{n=0}^{\infty} (\varepsilon^n - 1)(|\beta|)_n \frac{h^n}{n!} \\ &\leqq \sum_{n=0}^{\infty} (|\beta|)_n \frac{(\varepsilon h)^n}{n!} - \sum_{n=0}^{\infty} (|\beta|)_n \frac{h^n}{n!} \\ &= F(1, |\beta|, 1; \varepsilon h) - F(1, |\beta|, 1; h) \\ &= (1-\varepsilon h)^{-|\beta|} - (1-h)^{-|\beta|}. \end{aligned} \tag{5.4}$$

On the other hand, we have

$$\begin{aligned} |F(l+\alpha, \beta, l+\gamma; h) - (1-h)^{-\beta}| &\geqq |(1-h)^{-\beta}| - |F(l+\alpha, \beta, l+\gamma; h)| \\ &= (1-h)^{-\mathrm{Re}(\beta)} - |F(l+\alpha, \beta, l+\gamma; h)|. \end{aligned} \tag{5.5}$$

From (5.3), (5.4) and (5.5) it follows immediately that

$$|F(l+\alpha, \beta, l+\gamma; h)| \geqq \frac{1}{2}(1-h)^{-\mathrm{Re}(\beta)}$$

for any $l \geqq l_0$, which completes the proof of the lemma.

PROOF OF PROPOSITION 5.2. First we notice (Corollary 4.4) that

$$\begin{aligned} f_s^l(a_{2t}K) = f_{s1}^l(a_{2t}K) &= d(l)^{1/2} e(l, s)(\tanh t)^l (\cosh t)^{2(s-1)\sigma} \\ &\times F\left(l+(1-s)\sigma, -s\sigma+\frac{1}{2}, l+\sigma+\frac{1}{2}; (\tanh t)^2\right), \\ f_{si}^l(a_{2t}K) &= 0 \qquad (2 \leqq i \leqq d(l)), \end{aligned} \tag{5.6}$$

where

$$d(l) = \frac{2l+n-2}{n-2} \cdot \frac{\Gamma(l+n-2)}{\Gamma(l+1)\Gamma(n-2)},$$

$$e(l, s) = \frac{\Gamma\left(\sigma+\frac{1}{2}\right)\Gamma(l+(1+s)\sigma)}{\Gamma\left(l+\sigma+\frac{1}{2}\right)\Gamma((1+s)\sigma)},$$

$$\sigma = \frac{p}{2} + q = \frac{n-1}{2}.$$

We put $\xi = \mathrm{Re}(s)$ and $\nu = \left|-s\sigma + \frac{1}{2}\right|$.

(1) For every $\eta > 0$ we define a compact subset U_η in X by

$$U_\eta = \{z = k a_{2t} K \mid |\tanh t| \leqq \exp(-2\eta),\ k \in K\}.$$

Put

$$S_m(z)=\sum_{l\geqq m}\sum_{i=1}^{d(l)}|a_i^l|\,|f_{si}^l(z)|\,.$$

Then for the proof of (1), we have only to show the uniform convergence of the series S_0 on every U_η $(\eta>0)$. Fix an arbitrary $\eta>0$. It is easy to see that

$$\lim_{l\to\infty}|d(l)^{1/2}e(l,\,s)|^{1/l}\leqq 1\,. \tag{5.7}$$

Since $\exp\eta>1$ and $0<\exp(-2\eta)<1$, from (5.7) and Lemma 5.3 we can choose an $l_0\in \boldsymbol{N}^0$ such that for $l\geqq l_0$,

$$|d(l)^{1/2}e(l,\,s)|^{1/l}\leqq \exp\eta\,,$$

$$\left|F\Big(l+(1-s)\sigma,\,-s\sigma+\frac{1}{2},\,l+\sigma+\frac{1}{2}\,;\,z\Big)\right|$$
$$\leqq(1-\exp(-2\eta))^{-\nu}\qquad \text{for}\quad |z|\leqq\exp(-4\eta)\,. \tag{5.8}$$

Since $|\bar{\tau}_{ij}^l(k)|\leqq 1$ $(k\in K)$ and from (2.6)

$$f_{si}^l(kz)=\sum_{j=1}^{d(l)}f_{sj}^l(z)\bar{\tau}_{ij}(k)\,,$$

we have

$$S_{l_0}(ka_{2t}K)\leqq\sum_{l\geqq l_0}\sum_{i,j=1}^{d(l)}|a_i^l|\,|f_{sj}^l(a_{2t}K)|\,|\bar{\tau}_{ij}^l(k)|$$
$$\leqq\sum_{l\geqq l_0}\sum_{i,j=1}^{d(l)}|a_i^l|\,|f_{sj}^l(a_{2t}K)|\,.$$

Therefore, from (5.6) and (5.8), putting $r=|\tanh t|$ we have for $z=ka_{2t}K\in U_\eta$

$$S_{l_0}(z)\leqq\sum_{l\geqq l_0}\sum_{i=1}^{d(l)}|a_i^l|\,|f_s^l(a_{2t}K)|$$
$$\leqq(\cosh t)^{2(\xi-1)\sigma}(1-\exp(-2\eta))^{-\nu}\sum_{l\geqq l_0}\sum_{i=1}^{d(l)}|a_i^l|(r|d(l)^{1/2}e(l,\,s)|^{1/l})^l\,.$$

We put

$$M_\eta=(1-\exp(-2\eta))^{-\nu}\sup_{r\leqq\exp(-2\eta)}(\cosh t)^{2(\xi-1)\sigma}\,.$$

Since

$$r|d(l)^{1/2}e(l,\,s)|^{1/l}\leqq\exp(-2\eta)\exp(\eta)=\exp(-\eta)\,,$$

we have for $l\geqq l_0$ that

$$S_{l_0}(z)\leqq M_\eta\sum_{l\geqq l_0}\sum_{i=1}^{d(l)}|a_i^l|\exp(-\eta l)\,.$$

On the other hand

$$\lambda_l=\frac{l(l+n-2)}{2(n-2)}$$

implies that

$$\frac{l}{\sqrt{2(n-2)}}\leqq\lambda_l^{1/2}\leqq l \tag{5.9}$$

for $l \in \boldsymbol{N}^0$. Hence we have

$$S_{l_0}(z) \leqq M_\eta \sum_{l \geqq l_0} \sum_{i=1}^{d(l)} |a_i^l| \exp(-\eta \lambda_l^{1/2}),$$

which is finite because $(a_i^l) \in \mathcal{F}_b(B)$. Consequently $S_0(z)$ is uniformly convergent in U_η.

(2) Let $\eta > 0$ and choose a $t \in \boldsymbol{R}$ so that

$$r = \tanh t = \exp\left(\frac{-\eta}{4\sqrt{2(n-2)}}\right).$$

From the assumption that $e(s) \neq 0$, it follows that $e(l, s) \neq 0$ for $l \in \boldsymbol{N}^0$ and that

$$\lim_{l \to \infty} |e(l, s)|^{1/l} = 1. \tag{5.10}$$

Therefore by Lemma 5.3 and (5.10) we can find an $l_0 \in \boldsymbol{N}^0$ such that for $l \geqq l_0$

$$|e(l, s)|^{2/l} \geqq \exp\left(\frac{-\eta}{2\sqrt{2(n-2)}}\right),$$

$$\left|F\left(l+(1-s)\sigma, -s\sigma+\frac{1}{2}, l+\sigma+\frac{1}{2}; r^2\right)\right| \geqq \frac{1}{2}(1-r^2)^{\xi\sigma-1/2}.$$

Then, for $z = a_{2t}K$, from Proposition 2.2 we have

$$\begin{aligned}
\|\phi_f^s\|^2 &\geqq \sum_{l \in \boldsymbol{N}0} d(l)^{-1} \left(\sum_{i=1}^{d(l)} |a_i^l|^2\right) \left(\sum_{j=1}^{d(l)} |f_{sj}^l(z)|^2\right) \\
&\geqq \sum_{l \in \boldsymbol{N}0} d(l)^{-1} \left(\sum_{i=1}^{d(l)} |a_i^l|^2\right) |f_s^l(z)|^2 \\
&\geqq \frac{1}{4}(1-r^2)^{2\xi\sigma-1} (\cosh t)^{4(\xi-1)\sigma} \sum_{l \geqq l_0} (|e(l, s)|^{2/l} r^2)^l \left(\sum_{i=1}^{d(l)} |a_i^l|^2\right) \\
&\geqq \frac{1}{4}(1-r^2)^{2\xi\sigma-1} (\cosh t)^{4(\xi-1)\sigma} \sum_{l \geqq l_0} \left(\sum_{i=1}^{d(l)} |a_i^l|^2\right) \exp\left(\frac{-\eta l}{\sqrt{2(n-2)}}\right),
\end{aligned}$$

since

$$\begin{aligned}
r^2 |e(l, s)|^{2/l} &\geqq \exp\left(\frac{-\eta}{2\sqrt{2(n-2)}}\right) \exp\left(\frac{-\eta}{2\sqrt{2(n-2)}}\right) \\
&= \exp\left(\frac{-\eta}{\sqrt{2(n-2)}}\right).
\end{aligned}$$

Therefore from (5.9) we have

$$\|\phi_f^s\|^2 \geqq \frac{1}{4}(1-r^2)^{2\xi\sigma-1} (\cosh t)^{4(\xi-1)\sigma} \sum_{l \geqq l_0} \left(\sum_{i=1}^{d(l)} |a_i^l|^2\right) \exp(-\eta \lambda_l^{1/2}),$$

which implies that

$$\sum_{l \in \boldsymbol{N}0} \left(\sum_{i=1}^{d(l)} |a_i^l|^2\right) \exp(-\eta \lambda_l^{1/2}) < \infty.$$

Since η is arbitrary, we can conclude that (a_i^l) lies in $\mathcal{F}_b(B)$ by (5.1). This finishes the proof.

PROPOSITION 5.4. *Let G be a connected real semisimple Lie group of real rank one with finite center, K be a maximal compact subgroup of G and Δ be the laplacian on $X=G/K$ corresponding to the riemannian metric induced by the Killing form of the Lie algebra of G. Suppose that f_n $(n\in \boldsymbol{N}^0)$ are eigenfunctions of Δ with eigenvalue μ and that $\sum_{n\in N^0} f_n$ is absolutely and uniformly convergent on every compact subset in X. Then $\sum_{n\in N^0} f_n$ is also an eigenfunction of Δ with the same eigenvalue μ.*

PROOF. For $x\in G$, we define an operator M^x on $C(X)$ by

$$(M^x f)(gK)=\int_K f(gkxK)dk \qquad (g\in G)$$

for $f\in C(X)$. Then by [**6**, Chap. X, Lemma 7.1 and Theorem 7.2] there exists a C^∞-function λ on X such that

$$(M^x f_n)(z)=\lambda(xK)f_n(z) \qquad (z\in X),$$
$$(\Delta\lambda)(eK)=\mu .$$

Since Σf_n is absolutely and uniformly convergent on every compact subset in X, we have

$$\begin{aligned}[M^x(\sum_{n\in N^0} f_n)](z) &= \sum_{n\in N^0}(M^x f_n)(z)\\ &= \sum_{n\in N^0}\lambda(xK)f_n(z)\\ &= \lambda(xK)(\sum_{n\in N^0} f_n)(z).\end{aligned}$$

Therefore by [**6**, Chap. X, Theorem 7.2], $\sum_{n\in N^0} f_n$ is an eigenfunction of Δ with eigenvalue $(\Delta\lambda)(eK)=\mu$, which completes the proof.

Now we can state the main theorem. For the notation, see § 2.

THEOREM 5.5. *Let $X=G/K$ be a real hyperbolic space.*

(1) *The Poisson transform $\mathcal{P}_s$ maps $\mathcal{B}(B)$ into $\mathcal{H}_s(X)$.*

(2) *If $e(s)\neq 0$, $\mathcal{P}_s$ is an isomorphism of $\mathcal{B}(B)$ onto $\mathcal{H}_s(X)$.*

COROLLARY 5.6. *For a real hyperbolic space, any eigenfunction f of Δ can be represented as*

$$f(z)=\int_B P_s(z, b)dT(b)$$

with some complex number s and some hyperfunction T on B.

PROOF. Let $\Delta f=\mu f$. We can choose an $s\in \boldsymbol{C}$ such that $\mu=(s^2-1)\langle\rho, \rho\rangle$ and $\mathrm{Re}(s)\geqq 0$. Then $e(s)\neq 0$ and we have only to apply Theorem 5.5 to f.

PROOF OF THEOREM 5.5. (1) Let T be a hyperfunction on B and put

$\Psi(T)=(a_i^l)$. By Proposition 5.1,

$$\mathcal{P}_s(T)(z)=\sum_{l\in N^0}\sum_{i=1}^{d(l)} a_i^l f_{si}^l(z)\,,$$

which is absolutely and uniformly convergent on every compact subset in X by Proposition 5.2. Since f_{si}^l lies in $\mathcal{H}_s(X)$, we can conclude from Proposition 5.4 that $\mathcal{P}_s(T)$ also lies in $\mathcal{H}_s(X)$.

(2) The surjectivity of $\mathcal{P}_s$ is clear from (2) in Proposition 5.2. Assume that $\mathcal{P}_s(T)=0$ $(T\in\mathcal{B}(B))$. Then, putting $\Psi(T)=(a_i^l)$, we have

$$\sum_{l\in N^0}\sum_{i=1}^{d(l)} a_i^l f_{si}^l(z)=0 \qquad (z\in X)\,.$$

Replacing z by kz, from (2.6) we have

$$\sum_{l\in N_0} d(l)^{-1/2}\sum_{i,j=1}^{d(l)} a_i^l f_{sj}^l(z)\phi_{ij}^l(k)=0\,.$$

Since ϕ_{ij}^l are linearly independent on K, we obtain

$$a_i^l f_{si}^l(z)=0 \qquad (z\in X)$$

for l and $1\leqq i\leqq d(l)$. The condition $e(s)\neq 0$ implies $f_{si}^l\not\equiv 0$ on X. Therefore we have $a_i^l=0$, which completes the proof.

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