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Complex submanifolds with constant scalar curvature in a Kaehler manifold

By Masahiro KON

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Introduction.

In [8] Smyth showed that an Einstein complex hypersurface in a complex space form is locally symmetric, and he proved the classification theorem of it and Chern [1] proved the corresponding local theorem. And moreover Takahashi [9] showed that the condition that a hypersurface is Einstein can be relaxed to the condition that the Ricci tensor is parallel. These results were studied also by Nomizu-Smyth [4]. And by the method of algebraic geometry Kobayashi [2] proved that $P^n(C)$ and the complex quadric Q^n are the only compact complex hypersurfaces *imbedded* in $P^{n+1}(C)$ which have constant scalar curvature. On the other hand, Ogiue [6] studied a non-singular algebraic variety from the differential geometric point of view and gave sufficient conditions for a complex submanifold to be totally geodesic.

In this note we shall give a condition for a compact complex submanifold *immersed* in a projective space to be Einstein. From this, we shall prove that a compact complex hypersurface *immersed* in $P^{n+1}(C)$ with constant scalar curvature is either a hyperplane or a hyperquadric.

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§ 1. Preliminaries.

Let $\bar{M}$ be a Kaehler manifold of complex dimension $n+p$ with structure tensor field J and the Kaehler metric $\langle , \rangle$, and let M be an n-dimensional complex submanifold of $\bar{M}$. The Riemannian metric induced on M is a Kaehler metric, which is denoted by the same $\langle , \rangle$ and all metric properties of M refer to this metric. The complex structure of M is denoted by the same J as in $\bar{M}$. By $\bar{\nabla}$, we denote the covariant differentiation in $\bar{M}$ and by ∇ the one in M determined by the induced metric. For any tangent vector fields X, Y and normal vector field N on M, the Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \qquad \bar{\nabla}_X N = -A^N(X) + D_X N,$$

where $\langle B(X, Y), N\rangle = \langle A^N(X), Y\rangle$ and D is the linear connection in the normal bundle $T(M)^\perp$. Both A and B are called the second fundamental form of M. Let $\bar{R}$ and R denote the curvature tensors of $\bar{M}$ and M respectively. If we assume that $\bar{M}$ is of constant holomorphic sectional curvature c, then the curvature tensor $\bar{R}$ of $\bar{M}$ is represented by the following:

$$\text{(1.1)} \qquad \bar{R}_{X,Y}Z = \frac{1}{4}c(\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle Z, JY\rangle JX - \langle Z, JX\rangle JY + 2\langle X, JY\rangle JZ),$$

$$\text{(1.2)} \qquad \bar{R}_{X,Y}Z = R_{X,Y}Z - A^{B(Y,Z)}(X) + A^{B(X,Z)}(Y).$$

Let $v_1, \cdots, v_{2p}$ be a frame for $T_m(M)^\perp$, and let $x, y \in T_m(M)$. Then the Ricci tensor S of M is given by

$$\text{(1.3)} \qquad S(x, y) = \frac{1}{2}(n+1)c\langle x, y\rangle - \sum_{j=1}^{2p} \langle A^j A^j(x), y\rangle.$$

Here we write A^j instead of A^{v_j} to simplify the presentation. We denote by Q the Ricci operator of M defined by setting $S(x, y) = \langle Qx, y\rangle$. From (1.3), the scalar curvature K of M is given by

$$\text{(1.4)} \qquad K = n(n+1)c - \|A\|^2,$$

where $\|A\|$ denotes the length of the second fundamental form.

On the other hand, we have the relations between the second fundamental form A and the complex structure J:

$$\text{(1.5)} \qquad A^N J + J A^N = 0, \qquad A^{JN} - J A^N = 0.$$

§ 2. Complex submanifolds with constant scalar curvature.

First we prepare two lemmas for a Kaehler manifold M of complex dimension n. Let $e_1, \cdots, e_{2n}$ be a frame for $T_m(M)$, and let $E_1, \cdots, E_{2n}$ be local, orthonormal vector fields on M which extend $e_1, \cdots, e_{2n}$, and which are covariant constant with respect to ∇ at $m \in M$. Let $x, y, z \in T_m(M)$. Extend x, y, z to X, Y, Z, local vector fields on M such that all are covariant constant at $m \in M$ with respect to ∇. Then using the standard facts about the covariant differentiation, we obtain the following:

LEMMA 1. *The Ricci tensor S of a Kaehler manifold M satisfies the following*

$$\nabla_z(S)(x, y) = \nabla_x(S)(y, z) + \nabla_{Jy}(S)(Jx, z).$$

PROOF. The curvature tensor R and the Ricci tensor S of M possess the properties (cf. [3], p. 149)

$$S(Jx, Jy)=S(x, y) \quad \text{and} \quad S(x, y)=\frac{1}{2}(\text{Trace of } J\circ R_{x,Jy}).$$

From this and Bianchi's identity, we have

$$\begin{aligned}\nabla_z(S)(x, y)&=\nabla_z(S(X, Y))=\nabla_z\Big(\frac{1}{2}\sum_{i=1}^{2n}\langle JR_{X,JY}E_i, E_i\rangle\Big)\\&=\frac{1}{2}\sum_{i=1}^{2n}(\langle J\nabla_x(R)_{z,Jy}e_i, e_i\rangle+\langle J\nabla_{Jy}(R)_{x,z}e_i, e_i\rangle)\\&=\nabla_x(S)(y, z)+\nabla_{Jy}(S)(Jx, z).\end{aligned}$$

Now we define the "restricted" Laplacian of a tensor field T of type (r, s) on M. First we set

$$\nabla_{X,Z}T=\nabla_X(\nabla_Y T)-\nabla_{\nabla_X Y}T,$$

where X and Y are vector fields on M. Then the "restricted" Laplacian $\nabla^2 T$ is defined by

$$\nabla^2(T)(m)=\sum_{i=1}^{2n}\nabla_{E_i}\nabla_{E_i}T(m).$$

This is independent of the choice of an orthonormal basis.

LEMMA 2. *If a Kaehler manifold M has the constant scalar curvature, then we have*

$$\nabla^2(S)(x, y)=2\sum_{i=1}^{2n}R_{e_i,x}(S)(e_i, y).$$

PROOF. Since M has the constant scalar curvature, the Ricci tensor S of M satisfies $\sum_{i=1}^{2n}\nabla_{e_i}(S)(e_i, x)=0$ for any vector $x\in T_m(M)$. Thus Lemma 1 implies

$$\begin{aligned}\nabla^2(S)(x, y)&=\sum_{i=1}^{2n}\nabla_{E_i}\nabla_{E_i}(S)(x, y)=\sum_{i=1}^{2n}\nabla_{E_i}(\nabla_{E_i}(S)(X, Y))\\&=\sum_{i=1}^{2n}(\nabla_{E_i}(\nabla_X(S)(E_i, Y))+\nabla_{E_i}(\nabla_{JY}(S)(E_i, JX)))\\&=\sum_{i=1}^{2n}(R_{e_i,x}(S)(e_i, y)+R_{e_i,Jy}(S)(e_i, Jx))\\&=2\sum_{i=1}^{2n}R_{e_i,x}(S)(e_i, y).\end{aligned}$$

REMARK 1. Let M be a compact Kaehler manifold with constant scalar curvature. If $R_{X,Y}(R)=0$, we can see that the Ricci tensor of M is parallel, by using Lemma 1 and Lemma 2. And from the integral formula of A. Lichnérowicz (Géométrie des groupes de transformations, p. 10), M is locally symmetric. This result has been proved by Ogawa [**5**].

In the following, let $\bar{M}$ be a Kaehler manifold of complex dimension $n+p$

and constant holomorphic sectional curvature c, and let M be an n-dimensional complex submanifold of $\bar{M}$ with constant scalar curvature K. Hereafter we take a frame $e_1, \cdots, e_{2n}$ in $T_m(M)$ such that $e_{n+i}=Je_i$ $(i=1, \cdots, n)$ and a frame $v_1, \cdots, v_{2p}$ for $T_m(M)^\perp$ such that $v_{p+j}=Jv_j$ $(j=1, \cdots, p)$. Let $x, y \in T_m(M)$. We calculate $\nabla^2(S)(x, y)$ in the following way. Since M is minimal in $\bar{M}$, we obtain, by (1.2),

$$2\sum_{i=1}^{2n} R_{e_i,x}(S)(e_i, y) = -2\sum_{i=1}^{2n} \{S(\bar{R}_{e_i,x}e_i, y)+S(\bar{R}_{e_i,x}y, e_i) + S(A^{B(x,e_i)}(e_i), y) - S(A^{B(e_i,y)}(x), e_i) + S(A^{B(x,y)}(e_i), e_i)\}\,.$$

From (1.1), we have

$$-2\sum_{i=1}^{2n} (S(\bar{R}_{e_i,x}e_i, y)+S(\bar{R}_{e_i,x}y, e_i)) = nc\Big(S(x, y) - \frac{1}{2n}K\langle x, y\rangle\Big)\,.$$

The Ricci tensor S of M has the property $S(Jx, Jy)=S(x, y)$, and hence (1.5) implies that $\sum_{i=1}^{2n} S(A^{B(x,y)}(e_i), e_i)=0$. And we have also

$$\begin{aligned} &-2\sum_{i=1}^{2n} (S(A^{B(x,e_i)}(e_i), y) - S(A^{B(e_i,y)}(x), e_i)) \\ &= -2\sum_{i=1}^{2n}\sum_{j=1}^{2p} (\langle A^j(e_i), Qy\rangle\langle A^j(x), e_i\rangle - \langle A^j(x), Qe_i\rangle\langle A^j(y), e_i\rangle) \\ &= -2\sum_{j=1}^{2p} (\langle QA^jA^j(x), y\rangle - \langle A^jQA^j(x), y\rangle)\,. \end{aligned}$$

Consequently we have

$$2\sum_{i=1}^{2n} R_{e_i,x}(S)(e_i, y) = c\Big(nS(x, y) - \frac{1}{2}K\langle x, y\rangle\Big) - 2\sum_{j=1}^{2p} (QA^jA^j(x), y\rangle - \langle A^jQA^j(x), y\rangle)\,.$$

Therefore Lemma 2 implies the following

$$\langle\nabla^2(Q), Q\rangle = c\Big(n\|Q\|^2 - \frac{1}{2}K^2\Big) - \sum_{j=1}^{2p} \|[Q, A^j]\|^2 \tag{2.1}$$

because $\nabla^2(S)(x, y) = \langle\nabla^2(Q)(x), y\rangle$, where $\|Q\|$ denotes the length of the Ricci operator Q and $[Q, A^j]=QA^j-A^jQ$. If M is an Einstein manifold, then we have always $[Q, A^j]=0$.

Next we consider the application of the equation (2.1) for a complex submanifold. First we obtain obviously

PROPOSITION 1. *Let $\bar{M}$ be a Kaehler manifold of constant holomorphic sectional curvature $c<0$, and let M be a complex submanifold of $\bar{M}$. If the Ricci* *tensor of M is parallel, then M is an Einstein manifold.*

In the following, we take the complex projective space $P^{n+p}(C)$ as an ambient space. Then we have

PROPOSITION 2. *Let M be an n-dimensional compact complex submanifold immersed in $P^{n+p}(C)$ with constant scalar curvature. If $QA^j=A^jQ$ $(j=1, \cdots, p)$, then M is an Einstein manifold.*

PROOF. By the assumption and (2.1) we have the following inequality

$$0 \leqq \int_M \langle \nabla Q, \nabla Q \rangle = -\int_M \langle \nabla^2(Q), Q \rangle = \int_M \left(\frac{1}{2} K^2 - n\|Q\|^2 \right).$$

But we have always $K^2 \leqq 2n\|Q\|^2$, hence we obtain $\nabla Q=0$. Consequently we get $K^2=2n\|Q\|^2$, which shows that M is an Einstein manifold.

THEOREM 1. *Let M be a compact complex hypersurface immersed in $P^{n+1}(C)$. If the scalar curvature of M is constant, then M is either a complex hyperplane $P^n(C)$ or a complex quadric Q^n in $P^{n+1}(C)$.*

PROOF. Let v, Jv be a frame for $T_m(M)^\perp$. Then we have

$$Q = \frac{1}{2}(n+1)I - 2(A^v)^2,$$

by using (1.3) and (1.5). From this we obtain $QA^v=A^vQ$ and M is an Einstein manifold by Proposition 2. Therefore we have our assertion by Theorem 5 of Nomizu-Smyth [4].

REMARK 2. In [2] Kobayashi proved the following: Let M be an n-dimensional compact complex submanifold *imbedded* in $P^{n+p}(C)$. If M is a complete intersection of p non-singular hypersurfaces in $P^{n+p}(C)$ with constant scalar curvature, then M is an Einstein manifold. (See also Ogiue [6].) We have shown that the assumption of this Kobayashi's theorem can be replaced by the condition $QA^j=A^jQ$ $(j=1, \cdots, p)$ which is satisfied always when $p=1$ and our results are obtained also for an *immersed* submanifold.

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Masahiro KON
Department of Mathematics
Science University of Tokyo
Wakamiya, Shinjuku-ku
Tokyo, Japan