J. Math. Soc. Japan
Vol. 27, No. 1, 1975

Some studies on Siegel domains

By Kazufumi NAKAJIMA

(Received June 2, 1973)
(Revised Nov. 7, 1973)

Introduction.

In the present paper, we discuss some geometric properties of Siegel domains without the assumption of homogeneity.

Let D be a Siegel domain of the first or the second kind in $\boldsymbol{C}^{N}$ due to Pyatetski-Shapiro [9]. As is proved in [9] the domain D is holomorphically equivalent to a bounded domain in $\boldsymbol{C}^{N}$. This allows us to define the Bergmann metric in D. We first prove in § 1 the completeness of D with respect to the Bergmann metric, which is an application of a theorem of Kobayashi [6].

Kaup, Matsushima and Ochiai [5] showed that the Lie algebra $\mathfrak{g}(D)$ of all infinitesimal automorphisms of D has the structure of a graded Lie algebra: $\mathfrak{g}(D)=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^{0}+\mathfrak{g}^{1}+\mathfrak{g}^{2}$, and that $\mathfrak{g}^{a}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^{0}$ is the subalgebra of $\mathfrak{g}(D)$ which consists of all infinitesimal affine automorphisms of D. Under the assumption that D is homogeneous, Murakami [8] determined $\mathfrak{g}^{1}$ and $\mathfrak{g}^{2}$ in the space of all polynomial vector fields on D. In § 2, by using the completeness instead of the homogeneity, we characterize $\mathfrak{g}^{1}$ and $\mathfrak{g}^{2}$ by similar conditions of Murakami's. Our characterizations seem to be simpler and more convenient to calculate $\mathfrak{g}(D)$ than those of Kaup [4]. Furthermore in § 3, we shall determine $\mathfrak{g}^{1}$ and $\mathfrak{g}^{2}$ in the algebraic prolongation of $(\mathfrak{g}^{-2}+\mathfrak{g}^{-1}, \mathfrak{g}^{0})$. This was originally introduced by Tanaka [10] in the case where D is homogeneous.

Since the domain D is a complete simply connected Kähler manifold, we can consider the de Rham decomposition of D. Kaneyuki [2] showed that if the domain D is homogeneous then every irreducible component of D is also a Siegel domain. We shall prove in § 4 that if $\mathfrak{g}(D)$ is a direct sum of ideals, then D is decomposed into a product of Siegel domains. And from this fact, we can show the same result as Kaneyuki's without the assumption of the homogeneity of D.

The author should like to thank to Prof. N. Tanaka, who read through the manuscript during the preparation of this paper.

Preliminary remark.

Throughout this paper we use the following notations.

$\boldsymbol{R}$ (resp. $\boldsymbol{C}$) denotes the real (resp. complex) number field. As usual i denotes the element $\sqrt{-1}$ in $\boldsymbol{C}$. For every element z of a complex vector space, we denote by $\mathrm{Re}\, z$ the real part of z, by $\mathrm{Im}\, z$ the imaginary part of z and by $\bar{z}$ the conjugate vector of z. For every vector space W, we denote by $GL(W)$ the general linear group of W and by $\mathfrak{gl}(W)$ the Lie algebra of $GL(W)$.

§1. Siegel domains and the Bergmann metrics.

1.1. Let R be an n-dimensional real vector space. An open set V of R is called a *convex cone* if it satisfies the following conditions:

1) For any $x \in V$ and for any $t>0$, $tx \in V$.

2) For any $x, x' \in V$, $x+x' \in V$.

3) V contains no entire straight lines.

Let W be an m-dimensional complex vector space. We say a mapping F of $W \times W$ into R^c, the complexification of R, is a *V-hermitian form* on W if it satisfies the following conditions:

1) $F(w, w')$ is complex linear in w and $\overline{F(w, w')} = F(w', w)$.

2) $F(w, w) \in \bar{V}$, where $\bar{V}$ is the closure of V in R.

3) $F(w, w)=0$ implies $w=0$.

We define a domain $D(V, F)$ in $R^c \times W$ by

$$D(V, F) = \{(z, w) \in R^c \times W\,;\ \mathrm{Im}\, z - F(w, w) \in V\}\,.$$

The domain $D(V, F)$ is called a *Siegel domain of the second kind.* In the special case where $W=(0)$, the domain $D(V)$ $(=\{z \in R^c\,;\ \mathrm{Im}\, z \in V\})$ is called a *Siegel domain of the first kind.* The following proposition and its proof are well known. But we recall the proof, because we need it in the proof of Theorem 1.3.

PROPOSITION 1.1 (Pyatetski-Shapiro [**9**]). *A Siegel domain $D(V, F)$ is holomorphically equivalent to a bounded domain in $R^c \times W$.*

PROOF. We can assume that there exists a linear coordinate system $y^1, \cdots, y^n$ of R such that V is contained in the cone $y^1>0, \cdots, y^n>0$. Then every component $F^k(w, w')$ of $F(w, w')$ is a positive semi-definite hermitian form on W and we can write

$$F^k(w, w) = |L_1^k(w)|^2 + \cdots + |L_{s_k}^k(w)|^2 \qquad 1 \leqq k \leqq n\,,$$

where each $L_j^k(w)$ is a linear form on W. We choose a subset $\mathcal{L}$ of $\{L_j^k(w)\}$ satisfying the following conditions:

1) $\mathcal{L}$ is linearly independent.

2) Every $L_j^k(w)$ can be written as a linear combination of elements in $\mathcal{L}$. Then we see from the definition of a V-hermitian form that the condition "$L_j^k(w)=0$ for all k, j" implies $w=0$. It follows that the number of elements in $\mathcal{L}$ is equal to $m=\dim W$. We set inductively for every k

$$\tilde{F}^k(w, w)=\sum_j{}' |L_j^k(w)|^2$$

where $\sum'$ indicates that the sum is taken only over the forms which belong to $\mathcal{L}$ and do not appear among $\tilde{F}^1, \cdots, \tilde{F}^{k-1}$. We put

$$\tilde{F}(w, w)=(\tilde{F}^1(w, w), \cdots, \tilde{F}^n(w, w))$$

$$\tilde{D}=\{(z, w)\in R^c\times W\,;\ \operatorname{Im} z^k-\tilde{F}^k(w, w)>0 \quad \text{for all } k\}\,.$$

Then the domain $D(V, F)$ is contained in the domain $\tilde{D}$. Let $\mathcal{L}=\{L_1'(w), \cdots, L_m'(w)\}$. And we write $w'^j=L_j'(w)$. We consider $w'^1, \cdots, w'^m$ as a linear coordinate system of W and with respect to this coordinate system the domain $\tilde{D}$ can be expressed as

$$\operatorname{Im} z^1-(|w'^1|^2+\cdots+|w'^{t_1}|^2)>0$$
$$\cdots\cdots\cdots$$
$$\operatorname{Im} z^n-(|w'^{t_{n-1}+1}|^2+\cdots+|w'^m|^2)>0\,.$$

If we put

$$u_0^k=\frac{z^k-i}{z^k+i}\,,\ u_1^k=\frac{2w'^{t_{k-1}+1}}{z^k+i}\,,\ \cdots,\ u_{t_k}^k=\frac{2w'^{t_k}}{z^k+i} \qquad (t_0=0)\,,$$

then it is known that the domain $\tilde{D}_k$ in $\boldsymbol{C}^{t_k+1}$ defined by $\operatorname{Im} z^k-\sum_{j=1}^{t_k}|w'^j|^2>0$ is holomorphically equivalent to the unit disk $\tilde{B}_k$ defined by $\sum_{j=0}^{t_k}|u_j^k|^2<1$. As a result there exists a bi-holomorphic mapping T of $\tilde{D}=\tilde{D}_1\times\cdots\times\tilde{D}_n$ onto the polydisk $\tilde{B}=\tilde{B}_1\times\cdots\times\tilde{B}_n$. q. e. d.

1.2. By Proposition 1.1 the Siegel domain $D(V, F)$ is holomorphically equivalent to a bounded domain in $R^c\times W$, and hence has the Bergmann metric which is invariant by holomorphic transformations of $D(V, F)$. We use the following theorem.

THEOREM 1.2 (Kobayashi [6]). *Let D be an N-dimensional complex manifold with the Bergmann metric and let K be the Bergmann kernel form on D. Assume that for every sequence S of points of D which has no adherent point in D and for each square integrable holomorphic N-form f, there exists a subsequence S' of S such that*

$$\lim_{S'}\frac{f\wedge\bar{f}}{K}=0\,.$$

Then D is complete with respect to the Bergmann metric.

Now we can prove the following theorem, which plays a fundamental role in this paper.

THEOREM 1.3. *A Siegel domain $D(V, F)$ is complete with respect to the Bergmann metric.*

PROOF. We simply write $D=D(V, F)$. Let S be a sequence of points of D which has no adherent point in D. We shall show that there exist a subsequence $S'=\{p_n\}$ of S and a holomorphic function f_1 on D satisfying the following conditions:

1) $|f_1(p)|<1$ for all $p\in D$.

2) $\lim_{n\to\infty}|f_1(p_n)|=1$.

Let T be the bi-holomorphic mapping of $\tilde{D}$ onto $\tilde{B}$ which is constructed in the proof of Proposition 1.1. We put $B=T(D)$. Since B is relatively compact, we can choose a subsequence $S'=\{p_n\}$ such that the sequence $\{T\{p_n)\}$ converges to a point $q_0\in\partial B$, where ∂B is the boundary of B. We consider the following two cases.

(i) $q_0\in\partial\tilde{B}$. In this case $\sum_{j=0}^{t_k}|u_j^k(q_0)|^2=1$ for some k. We put

$$f_1=\sum_{j=0}^{t_k}\overline{u_j^k(q_0)}u_j^k\circ T\,.$$

Then it is clear that the pair S' and f_1 has the desired properties.

(ii) $q_0\notin\partial\tilde{B}$. Then q_0 is an interior point of $\tilde{B}$. Therefore the sequence $\{p_n\}$ converges to a point $p_0\in\partial D$. We put

$$r_0=\operatorname{Im} z(p_0)-F(w(p_0), w(p_0))\,.$$

Then $r_0\in\partial V$. It is not difficult to observe that there exists a linear coordinate system $y'^1, \cdots, y'^n$ of R satisfying the following conditions:

1) $V\subset\{y'^1>0, \cdots, y'^n>0\}$

2) $y'^1(r_0)=0$.

In this coordinate system, we have

$$\operatorname{Im} z'^1(p_0)-F'^1(w(p_0), w(p_0))=0\,.$$

We can choose $\mathcal{L}$ in the proof of Proposition 1.1 such that $\mathcal{L}$ contains $L_1^1(w), \cdots, L_{s_1}^1(w)$. Then $\tilde{F}'^1(w, w)=F'^1(w, w)$. If we construct a bounded domain B', a polydisk $\tilde{B}'$ and a bi-holomorphic mapping T' by the same method, then we have

$$\lim_{n\to\infty}\sum_{j=0}^{t_1'}|u'^1_j\circ T'(p_n)|^2=1\,.$$

This implies that choosing a subsequence if necessary, the sequence $\{T'(p_n)\}$ converges to a point of the boundary of $\tilde{B}'$. Therefore the case (ii) is reduced to the case (i).

Now by the same argument as in the proof of Theorem 9.2 in [6] we can see that for any square integrable holomorphic N-form f

$$\lim_{S'} \frac{f\wedge\bar{f}}{K}=0 .$$

Thus the domain D satisfies the condition of Theorem 1.2. q. e. d.

§ 2. Infinitesimal automorphisms of a Siegel domain, I.

2.1. Let $D=D(V, F)$ be a Siegel domain, and let $\mathrm{Aut}(D)$ be the automorphism group of D, i.e., the group of all holomorphic transformations of D. Then $\mathrm{Aut}(D)$ is a closed subgroup of the isometry group of the Riemannian manifold D. We identify the algebra of $\mathrm{Aut}(D)$ with the Lie algebra $\mathfrak{g}(D)$ of all infinitesimal automorphisms of D. (We say that a holomorphic vector field on D is an infinitesimal automorphism of D if it generates a global one parameter group of transformations of D.) We denote by $\mathrm{Aff}(D)$ the closed subgroup of the complex affine transformation group $\mathrm{Aff}(R^c\times W)$ of $R^c\times W$ leaving D invariant.

PROPOSITION 2.1 (Pyatetski-Shapiro [9]). *Let* $f\in \mathrm{Aff}(R^c\times W)$. *Then* $f\in \mathrm{Aff}(D)$ *if and only if* f *has the form*

$$f(z, w)=(Az+a+2iF(Bw, c)+iF(c, c),\ Bw+c)$$

where $a\in R$, $c\in W$, $A\in GL(R)$, $B\in GL(W)$, $AV=V$ *and* $AF(w, w')=F(Bw, Bw')$ $(w, w'\in W)$.

Let $\mathfrak{g}^a$ be the subalgebra of $\mathfrak{g}(D)$ corresponding to the subgroup $\mathrm{Aff}(D)$ of $\mathrm{Aut}(D)$. For every $a\in R$ (resp. $c\in W$) we denote by $s(a)$ (resp. $s(c)$) the element of $\mathfrak{g}^a$ induced by the following one parameter group:

$$(z, w) \longrightarrow (z+ta, w) \qquad t\in \boldsymbol{R}$$

$$(\text{resp. } (z, w) \longrightarrow (z+2iF(w, tc)+iF(tc, tc), w+tc) \qquad t\in \boldsymbol{R}) .$$

Then s gives an injective linear mapping of $R+W$ into $\mathfrak{g}^a$ and the following equalities are easily verified:

(2.1) 1) $[s(a), s(b+c)]=0 \qquad a, b\in R$ and $c\in W$

2) $[s(c), s(c')]=-4s(\mathrm{Im}\, F(c, c')) \qquad c, c'\in W$.

We denote by $GL(D)$ the closed subgroup of the general linear group $GL(R^c\times W)$ leaving D invariant. Then by Proposition 2.1 a one parameter group φ_t in $GL(D)$ is of the form:

$$\varphi_t(z, w)=(\exp tAz,\ \exp tBw) \qquad t\in \boldsymbol{R} ,$$

where $A\in \mathfrak{gl}(R)$, $B\in \mathfrak{gl}(W)$, $\exp tAV=V$ and $AF(w, w')=F(Bw, w')+F(w, Bw')$

$(w, w' \in W)$. We denote by $\rho(A, B)$ the holomorphic vector field on D induced by the one parameter group φ_t. Then the following equality holds:

$$(2.2) \qquad [\rho(A, B), s(a+c)] = -s(Aa+Bc) \qquad a \in R \text{ and } c \in W.$$

Let E (resp. I) be the holomorphic vector field on D induced by the following one parameter group in $GL(D)$:

$$(z, w) \longrightarrow (e^{2t}z, e^{t}w) \qquad t \in \boldsymbol{R}$$

$$(\text{resp. } (z, w) \longrightarrow (z, e^{it}w) \qquad t \in \boldsymbol{R}).$$

Clearly E and I are in the center of the subalgebra of $\mathfrak{g}^a$ corresponding to the subgroup $GL(D)$ of Aff (D), and the following equalities are satisfied:

$$(2.3) \qquad 1) \quad [E, s(a)+s(c)] = -2s(a)-s(c)$$

$$2) \quad [I, s(a)+s(c)] = -s(ic) \qquad a \in R,\ c \in W.$$

2.2. Let $e_1, \cdots, e_n$ (resp. $v_1, \cdots, v_m$) be a base of R (resp. of W). We denote by $z^1, \cdots, z^n$ (resp. $w^1, \cdots, w^m$) the linear coordinate system of R^c (resp. of W) corresponding to the base $e_1, \cdots, e_n$ (resp. $v_1, \cdots, v_m$). Then we have the following expression:

$$(2.4) \qquad 1) \quad s(a) = \sum_k a^k \frac{\partial}{\partial z^k} \qquad (a \in R,\ z^k(a) = a^k)$$

$$2) \quad s(c) = \sum_k 2iF^k(w, c)\frac{\partial}{\partial z^k} + \sum_\alpha c^\alpha \frac{\partial}{\partial w^\alpha} \quad (c \in W,\ w^\alpha(c) = c^\alpha)$$

$$3) \quad E = 2\sum_k z^k \frac{\partial}{\partial z^k} + \sum_k w^\alpha \frac{\partial}{\partial w^\alpha}$$

$$4) \quad I = \sum_\alpha iw^\alpha \frac{\partial}{\partial w^\alpha}$$

$$5) \quad \rho(A, B) = \sum_{kj} A^k_j z^j \frac{\partial}{\partial z^k} + \sum_{\alpha\beta} B^\alpha_\beta w^\beta \frac{\partial}{\partial w^\alpha}$$

$$(Ae_j = \sum_k A^k_j e_k,\ Bv_\beta = \sum_\alpha B^\alpha_\beta v_\alpha).$$

We put $\mathfrak{g}^\lambda = \{X \in \mathfrak{g}(D); [E, X] = \lambda X\}$ for $\lambda = -2, -1, 0, 1, 2$. And put $D' = \{(z, 0) \in D\}$, which is equivalent to the domain $D(V)$.

THEOREM 2.2 (Kaup-Matsushima-Ochiai [5]).

1) $\mathfrak{g}(D) = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$ *as a graded Lie algebra.*

2) $\mathfrak{g}^a = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0$, $\mathfrak{g}^{-2} = \{s(a);\ a \in R\}$, $\mathfrak{g}^{-1} = \{s(c);\ c \in W\}$, *and* $\mathfrak{g}^0$ *is the subalgebra corresponding to the subgroup* $GL(D)$.

3) $\mathfrak{g}^{-2} + \mathfrak{g}^0 + \mathfrak{g}^2$ *is the subalgebra corresponding to the subgroup* $\{\sigma \in \text{Aut}(D);$ *σ leaves D' invariant*$\}$.

It is also proved in [5] that every element of $\mathfrak{g}^1$ is necessarily of the form:

$$\sum_k P_{11}^k \frac{\partial}{\partial z^k} + \sum_\alpha P_{10}^\alpha \frac{\partial}{\partial w^\alpha} + \sum_\alpha P_{02}^\alpha \frac{\partial}{\partial w^\alpha} \tag{2.5}$$

and that every element of $\mathfrak{g}^2$ is necessarily of the form:

$$\sum_k P_{20}^k \frac{\partial}{\partial z^k} + \sum_\alpha P_{11}^\alpha \frac{\partial}{\partial w^\alpha} \tag{2.6}$$

where $P_{\mu\nu}^\lambda$ is a polynomial of homogeneous degree μ in z and of homogeneous degree ν in w.

REMARK 1. Let G be the connected component of the identity of $\mathrm{Aut}(D)$. Then $\mathrm{Aut}(D)=G\cdot GL(D)$. This fact is contained in [5].

2.3. In this paragraph we shall show that $\mathfrak{g}^1$ and $\mathfrak{g}^2$ are determined from $\mathfrak{g}^a$. Let K be the Bergmann kernel form on D. Since K is a volume element we can define the function $\mathrm{div}(X)$ for every vector field X by $\mathrm{div}(X)K=L_X K$. Then the following equalities hold:

(2.7) 1) $\mathrm{div}(fX)=Xf+f\,\mathrm{div}(X)$

2) $X\,\mathrm{div}(Y)-Y\,\mathrm{div}(X)=\mathrm{div}([X, Y])$,

where X and Y are vector fields and f is a function.

LEMMA 2.3 (Tanaka [10]). *Let X be a holomorphic vector field on D. Then $X\in\mathfrak{g}(D)$ if and only if* $\mathrm{div}(X+\bar{X})=0$.

REMARK 2. Lemma 2.3 is proved in [10] with the assumption that D is homogeneous, i. e., $\mathrm{Aut}(D)$ is transitive on D. But the proof shows that the transitivity is not needed, since D is complete with respect to the Bergmann metric by Theorem 1.3.

LEMMA 2.4 (cf. [10]). *Let X be a real vector field on D such that $L_X J=0$, where J is the complex structure of D. Suppose that X satisfies the condition $[\mathfrak{g}^{-1}, X-iJX]\subset\mathfrak{g}(D)$. Then we have*

$$\mathrm{div}(X)(z, w)=\mathrm{div}(X)(z-iF(w, w), 0)\,.$$

Furthermore if X satisfies the condition $[\mathfrak{g}^{-2}+\mathfrak{g}^{-1}, X-iJX]\subset\mathfrak{g}(D)$, then we have

$$\mathrm{div}(X)(z, w)=\mathrm{div}(X)(i\,\mathrm{Im}\,z-iF(w, w), 0)\,.$$

PROOF. For every $s(c)\in\mathfrak{g}^{-1}$, we put $Z(c)=s(c)+\overline{s(c)}$. Since $2[Z(c), X]=[s(c), X-iJX]+\overline{[s(c), X-iJX]}$, the condition $[\mathfrak{g}^{-1}, X-iJX]\subset\mathfrak{g}(D)$ implies $\mathrm{div}([Z(c), X])=0$ by Lemma 2.3. Therefore $\mathrm{div}(X)$ is invariant under the transformation of the form: $(z, w)\to(z+2iF(w, c)+iF(c, c), w+c)$ for every $c\in W$. In fact by (2.7) and Lemma 2.3 we get

$$0=\mathrm{div}([Z(c), X])=Z(c)\,\mathrm{div}(X)\,.$$

For every point $(z_0, w_0)\in D$ we choose $c=-w_0$. Then the point (z_0, w_0) is transformed to the point $(z_0 - iF(w_0, w_0), 0)$. Thus we have proved the first assertion. The same argument shows that if X satisfies the condition $[\mathfrak{g}^{-2}, X - iJX] \subset \mathfrak{g}(D)$, then $\operatorname{div}(X)$ is invariant under the transformation of the form: $(z, w) \to (z+a, w)$ for every $a \in R$. Hence we obtain

$$\begin{aligned}\operatorname{div}(X)(z_0, w_0) &= \operatorname{div}(X)(z_0 - iF(w_0, w_0), 0)\\ &= \operatorname{div}(X)(z_0 - iF(w_0, w_0) - \operatorname{Re} z_0, 0),\end{aligned}$$

if X satisfies the condition $[\mathfrak{g}^{-2} + \mathfrak{g}^{-1}, X - iJX] \subset \mathfrak{g}(D)$. q. e. d.

LEMMA 2.5.

$$(1) \qquad \operatorname{div}\left(\frac{\partial}{\partial z^k}\right) + \overline{\operatorname{div}\left(\frac{\partial}{\partial z^k}\right)} = 0 \quad \textit{for all } k.$$

$$(2) \qquad \operatorname{div}\left(\frac{\partial}{\partial w^\alpha}\right) = 0 \quad \textit{on } D' = \{(z, 0) \in D\} \quad \textit{for all } \alpha.$$

PROOF. Since $\partial/\partial z^k = s(e_k) \in \mathfrak{g}^{-2}$, the statement (1) is clear by Lemma 2.3. For every $c \in W$, $s(c) \in \mathfrak{g}^{-1}$. Hence by Lemma 2.3 $\operatorname{div}(s(c)) + \overline{\operatorname{div}(s(c))} = 0$. From (2.4) and (2.7)

$$\operatorname{div}(s(c)) = 2i \sum_k F^k(w, c) \operatorname{div}\left(\frac{\partial}{\partial z^k}\right) + \sum_\alpha c^\alpha \operatorname{div}\left(\frac{\partial}{\partial w^\alpha}\right).$$

Therefore we have for every $c \in W$

$$\sum_\alpha c^\alpha \operatorname{div}\left(\frac{\partial}{\partial w^\alpha}\right) + \sum_\alpha \bar{c}^{\alpha} \overline{\operatorname{div}\left(\frac{\partial}{\partial w^\alpha}\right)} = 0 \qquad \text{on } D'.$$

This implies the statement (2). q. e. d.

PROPOSITION 2.6 (cf. [8]).

(1) *Let X be a vector field on D of the form* (2.5). *Then $X \in \mathfrak{g}^1$ if and only if the condition $[\mathfrak{g}^{-1}, X] \subset \mathfrak{g}^0$ is satisfied.*

(2) *Let X be a vector field on D of the form* (2.6). *Then $X \in \mathfrak{g}^2$ if and only if the following conditions are satisfied:*

$$[\mathfrak{g}^{-2}, X] \subset \mathfrak{g}^0, \qquad [\mathfrak{g}^{-1}, X] \subset \mathfrak{g}^1 \qquad \textit{and}$$

$$\sum_\alpha B^\alpha_{j\alpha} = \sum_\alpha \bar{B}^\alpha_{j\alpha} \qquad \textit{for every } j \quad (1 \leqq j \leqq n),$$

where $B^\alpha_{j\beta}$ is the coefficient of P^α_{11}, i.e., $P^\alpha_{11} = \sum_j B^\alpha_{j\beta} z^j w^\beta$.

PROOF. If $X \in \mathfrak{g}^1$ then clearly $[\mathfrak{g}^{-1}, X] \subset \mathfrak{g}^0$. Conversely suppose that X is a vector field of the form (2.5) satisfying $[\mathfrak{g}^{-1}, X] \subset \mathfrak{g}^0$. By (2.7) and Lemma 2.5 we have

$$\operatorname{div}(X) = \sum_\alpha P^\alpha_{10} \operatorname{div}\left(\frac{\partial}{\partial w^\alpha}\right) = 0 \qquad \text{on } D'.$$

Hence by Lemma 2.4 we get $\operatorname{div}(X + \bar{X}) = 0$. Then the statement (1) follows immediately from Lemma 2.3.

Let $X \in \mathfrak{g}^2$. Then it is clear that $[\mathfrak{g}^{-2}, X] \subset \mathfrak{g}^0$ and $[\mathfrak{g}^{-1}, X] \subset \mathfrak{g}^1$. Now let X be a vector field of the form (2.6) satisfying $[\mathfrak{g}^{-2}, X] \subset \mathfrak{g}^0$ and $[\mathfrak{g}^{-1}, X] \subset \mathfrak{g}^1$. We write

$$X = \sum_{hjk} A^k_{hj} z^h z^j \frac{\partial}{\partial z^k} + \sum_{j\alpha\beta} B^\alpha_{j\beta} z^j w^\beta \frac{\partial}{\partial w^\alpha} \quad (A^k_{hj} = A^k_{jh}).$$

Then from the formula (2.4)

$$\begin{aligned}[s(e_j), X] &= \left[\frac{\partial}{\partial z^j}, X\right] \\ &= 2\sum_{hk} A^k_{hj} z^h \frac{\partial}{\partial z^k} + \sum_{\alpha\beta} B^\alpha_{j\beta} w^\beta \frac{\partial}{\partial w^\alpha}.\end{aligned}$$

Therefore from Proposition 2.1, the formula (2.4) and Theorem 2.2, each A^k_{hj} is real. Hence by using (2.7) and Lemma 2.5 we have

$$\begin{aligned}\operatorname{div}(X+\bar{X}) &= 2\sum_{jk} A^k_{jk}(z^j+\bar{z}^j) + \sum_{hjk} A^k_{hj}(z^h z^j - \bar{z}^h \bar{z}^j) \operatorname{div}\left(\frac{\partial}{\partial z^k}\right) \\ &\quad + \sum_{j\alpha}(B^\alpha_{j\alpha} z^j + \bar{B}^\alpha_{j\alpha} \bar{z}^j) \\ &\quad + \sum_{j\alpha\beta} B^\alpha_{j\beta} z^j w^\beta \operatorname{div}\left(\frac{\partial}{\partial w^\alpha}\right) + \sum_{j\alpha\beta} \bar{B}^\alpha_{j\beta} \bar{z}^j \bar{w}^\beta \, \overline{\operatorname{div}\left(\frac{\partial}{\partial w^\alpha}\right)} \\ &= \sum_j \left(\sum_\alpha B^\alpha_{j\alpha} - \sum_\alpha \bar{B}^\alpha_{j\alpha}\right) i \operatorname{Im} z^j,\end{aligned}$$

at all points of $(iV, 0) \subset D$. In fact, where $\operatorname{Re} z = 0$ we have $z^h z^j - \bar{z}^h \bar{z}^j = (z^h + \bar{z}^h) z^j - \bar{z}^h (z^j + \bar{z}^j) = 0$. It follows immediately by Lemma 2.3 and Lemma 2.4 that $X \in \mathfrak{g}^2$ if and only if $\sum_\alpha B^\alpha_{j\alpha} = \sum_\alpha \bar{B}^\alpha_{j\alpha}$ for all j. q. e. d.

§ 3. Infinitesimal automorphisms of a Siegel domain, II.

3.1. Let $\mathfrak{g}(D)$ be the Lie algebra of all infinitesimal automorphisms of D. By Theorem 2.2, $\mathfrak{g}(D) = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$ as a graded Lie algebra where $\mathfrak{g}^\lambda = \{X \in \mathfrak{g}(D); [E, X] = \lambda X\}$.

LEMMA 3.1.

(1) "$X \in \mathfrak{g}^0$, $[X, \mathfrak{g}^{-2} + \mathfrak{g}^{-1}] = 0$" *implies* $X = 0$.

(2) "$X \in \mathfrak{g}^1$, $[X, \mathfrak{g}^{-2}] = 0$" *implies* $X = 0$.

(3) "$X \in \mathfrak{g}^2$, $[X, \mathfrak{g}^{-2}] = 0$" *implies* $X = 0$.

PROOF. (1) It is obvious from (2.2).

(2) Let $X \in \mathfrak{g}^1$, then X is of the form (2.5), i.e.,

$$X = \sum_k P^k_{11} \frac{\partial}{\partial z^k} + \sum_\alpha P^\alpha_{10} \frac{\partial}{\partial w^\alpha} + \sum_\alpha P^\alpha_{20} \frac{\partial}{\partial w^\alpha}.$$

It is easy to see that "$[s(e_j), X] = 0$ for all j" implies $P^k_{11} = P^\alpha_{10} = 0$ for all k and for all α. Then we have $[I, X] = iX$. Therefore both X and iX belong to $\mathfrak{g}(D)$ and hence by a theorem of H. Cartan $X=0$.

(3) Let $X\in\mathfrak{g}^2$. Then X is of the form (2.6), i. e.,

$$X=\sum_k P_{20}^k\frac{\partial}{\partial z^k}+\sum_\alpha P_{11}^\alpha\frac{\partial}{\partial w^\alpha}.$$

Then "$[s(e_j), X]=0$ for all j" implies $P_{20}^k=P_{11}^\alpha=0$ for all k and for all α.

q. e. d.

By Lemma 3.1 we may regard $\mathfrak{g}^0$ as a subalgebra of the Lie algebra of all derivations of $\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$ (as the graded Lie algebra). Let $\hat{\mathfrak{g}}=\sum_{\lambda\geqq-2}\hat{\mathfrak{g}}^\lambda$ be the algebraic prolongation of $(\mathfrak{g}^{-2}+\mathfrak{g}^{-1}, \mathfrak{g}^0)$ (cf. [**10**]). Then $\hat{\mathfrak{g}}=\sum_{\lambda\geqq-2}\hat{\mathfrak{g}}^\lambda$ is the graded Lie algebra determined uniquely by the following properties:

1) $\hat{\mathfrak{g}}^{-2}+\hat{\mathfrak{g}}^{-1}+\hat{\mathfrak{g}}^0=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0$ as a graded algebra.

2) For each $\lambda>0$ the condition "$X\in\hat{\mathfrak{g}}^\lambda$, $[X, \mathfrak{g}^{-2}+\mathfrak{g}^{-1}]=0$" implies $X=0$.

3) $\hat{\mathfrak{g}}$ is maximum among the graded algebra satisfying the conditions 1) and 2). More precisely, let $\mathfrak{h}=\sum_{\lambda\geqq-2}\mathfrak{h}^\lambda$ be any graded Lie algebra satisfying the conditions 1) and 2). Then $\mathfrak{h}$ is imbedded in $\hat{\mathfrak{g}}$ as a graded subalgebra of $\hat{\mathfrak{g}}$.

By Lemma 3.1 and the property 3) of $\hat{\mathfrak{g}}$ we may regard $\mathfrak{g}(D)$ as a graded subalgebra of $\hat{\mathfrak{g}}$.

3.2. In this paragraph we shall show $\hat{\mathfrak{g}}^1=\mathfrak{g}^1$. Let X be an element of $\mathfrak{g}^1$. Then X has the form (2.5). We write

$$X=\sum_{kj\alpha}H_{j\alpha}^k z^j w^\alpha\frac{\partial}{\partial z^k}+\sum_{k\alpha}a_k^\alpha z^k\frac{\partial}{\partial w^\alpha}+\sum_{\alpha\beta\gamma}b_{\beta\gamma}^\alpha w^\beta w^\gamma\frac{\partial}{\partial w^\alpha}$$
$$(b_{\beta\gamma}^\alpha=b_{\gamma\beta}^\alpha).$$

Then

$$[s(e_j), X]=\sum_{k\alpha}H_{j\alpha}^k w^\alpha\frac{\partial}{\partial z^k}+\sum_\alpha a_j^\alpha\frac{\partial}{\partial w^\alpha}.$$

Therefore by Theorem 2.2 and the formula (2.4) we have

$$X=\sum_{jk}2iF^k(w, a_j)z^j\frac{\partial}{\partial z^k}+\sum_{k\alpha}a_k^\alpha z^k\frac{\partial}{\partial w^\alpha}+\sum_{\alpha\beta\gamma}b_{\beta\gamma}^\alpha w^\beta w^\gamma\frac{\partial}{\partial w^\alpha} \tag{3.1}$$

where $a_j\in W$ such that $w^\alpha(a_j)=a_j^\alpha$. For every $c\in W$, we have

$$\begin{aligned}[s(c), X]&=[s(c), \sum_j z^j s(a_j)]+\Big[s(c), \sum_{\alpha\beta\gamma}b_{\beta\gamma}^\alpha w^\beta w^\gamma\frac{\partial}{\partial w^\alpha}\Big]\\ &=4\sum_{jk}z^k\,\mathrm{Im}\,F^j(a_k, c)\frac{\partial}{\partial z^j}\\ &\quad-2\sum_j(2\sum_{k\alpha\beta\gamma\delta}F_{\gamma\delta}^k F_{\beta\alpha}^j\bar{c}^\delta\bar{a}_k^\alpha w^\beta w^\gamma+i\sum_{\alpha\beta\gamma\delta}F_{\alpha\delta}^j b_{\beta\gamma}^\alpha\bar{c}^\delta w^\beta w^\gamma)\frac{\partial}{\partial z^j}\\ &\quad+2\sum_\alpha(i\sum F^k(w, c)a_k^\alpha+\sum_{\beta\gamma}b_{\beta\gamma}^\alpha c^\beta w^\gamma)\frac{\partial}{\partial w^\alpha},\end{aligned}$$

where $F^k_{\alpha\beta}$ is the component of F^k, i. e., $F^k(w, w')=\sum_{\alpha\beta} F^k_{\alpha\beta} w^\alpha \overline{w}'^\beta$. Since $[s(c), X] \in \mathfrak{g}^0$, the second part of the above sum is necessarily 0 for all $c \in W$. This is equivalent to the following equality:

$$\sum_\alpha b^\alpha_{\beta\gamma} F^j_{\alpha\delta} = i\sum_k (F^k_{\beta\delta}\bar{a}^\alpha_k F^j_{\gamma\alpha} + F^k_{\gamma\delta}\bar{a}^\alpha_k F^j_{\beta\alpha}) \qquad \text{for all } j, \beta, \gamma, \delta . \tag{3.2}$$

We put

$$A(c)^j_k = 4\,\mathrm{Im}\, F^j(a_k, c) \tag{3.3}$$

$$B(c)^\alpha_\gamma = \sum_{k\beta} 2iF^k_{\gamma\beta}\bar{c}^\beta a^\alpha_k + \sum_\beta 2b^\alpha_{\beta\gamma} c^\beta . \tag{3.4}$$

Then we have

$$[s(c), X] = \sum_{jk} A(c)^j_k z^k \frac{\partial}{\partial z^j} + \sum_{\alpha\beta} B(c)^\alpha_\beta w^\beta \frac{\partial}{\partial w^\alpha}.$$

And the matrices $A(c)=(A(c)^j_k)$ and $B(c)=(B(c)^\alpha_\beta)$ satisfy the following conditions:

$$\exp tA(c)V = V \qquad t \in \boldsymbol{R} . \tag{3.5}$$

$$A(c)F(w, w') = F(B(c)w, w') + F(w, B(c)w') \qquad \text{for} \quad w, w' \in W . \tag{3.6}$$

By a direct computation, we can show the formula (3.2) and (3.6) are equivalent to each other. Thus we obtain the following lemma by using Proposition 2.6.

LEMMA 3.2. *A vector field of the form* (3.1) *belongs to* $\mathfrak{g}^1$ *if and only if the following conditions are satisfied:*

(1) *Let* $A(c)$ *be the linear endomorphism of* R *given by* (3.3). *Then* $A(c)$ *satisfies* (3.5).

(2) *The equality* (3.2) *holds, that is,* $A(c)$ *and* $B(c)$ *given by* (3.3) *and* (3.4) *satisfy* (3.6) *for all* $c \in W$.

Let $f \in \hat{\mathfrak{g}}^1$. For every k $(k=1, \cdots, n)$ there exists a unique vector $a_k \in W$ such that

$$s(a_k) = [s(e_k), f] . \tag{3.7}$$

And for every $c \in W$, there exist $A(c) \in \mathfrak{gl}(R)$ and $B(c) \in \mathfrak{gl}(W)$ which are uniquely determined by the equality

$$\rho(A(c), B(c)) = [s(c), f] . \tag{3.8}$$

We put

$$b^\gamma_{\alpha\beta} = \frac{1}{2} B^\gamma_{\beta\alpha} - i\sum_j F^j_{\beta\alpha} a^\gamma_j , \tag{3.9}$$

where $B^\gamma_{\alpha\beta} = B(v_\alpha)^\gamma_\beta$ given by (3.8).

LEMMA 3.3. *Let* $f \in \hat{\mathfrak{g}}^1$. *And we give* a_k, $A(c)$, $B(c)$ *and* $b^\gamma_{\alpha\beta}$ *by* (3.7), (3.8) *and* (3.9). *Then*

(1) *$A(c)$ and a_k satisfy* (3.3).

(2) $b^{\gamma}_{\alpha\beta}=b^{\gamma}_{\beta\alpha}$.

(3) *$B(c)$, a_k and $b^{\gamma}_{\alpha\beta}$ satisfy* (3.4).

PROOF. (1) By using (2.2) we have

$$\begin{aligned}[[s(c), f], s(e_k)] &= [\rho(A(c), B(c)), s(e_k)] \\ &= -s(A(c)e_k) = -\sum_j A(c)^j_k s(e_j)\,.\end{aligned}$$

On the other hand, by using (2.1)

$$\begin{aligned}[[s(c), f], s(e_k)] &= [s(c), [f, s(e_k)]] \\ &= [s(a_k), s(c)] = -4\sum_j \operatorname{Im} F^j(a_k, c)s(e_j)\,.\end{aligned}$$

Hence we obtain (3.3).

(2) By using (2.1)

$$\begin{aligned}[[s(v_\alpha), s(v_\beta)], f] &= -4[s(\operatorname{Im} F(v_\alpha, v_\beta)), f] \\ &= -4\sum_j \operatorname{Im} F^j_{\alpha\beta} s(a_j)\,.\end{aligned}$$

On the other hand by using (2.2)

$$\begin{aligned}[[s(v_\alpha), s(v_\beta)], f] &= [\rho(A(v_\alpha), B(v_\alpha)), s(v_\beta)] + [s(v_\alpha), \rho(A(v_\beta), B(v_\beta))] \\ &= -s(B(v_\alpha)v_\beta) + s(B(v_\beta)v_\alpha)\,.\end{aligned}$$

Hence we get

$$B(v_\alpha)v_\beta - B(v_\beta)v_\alpha = -2i\sum_j (F^j_{\alpha\beta} - F^j_{\beta\alpha})a_j\,.$$

Consequently

$$B^{\gamma}_{\alpha\beta} - B^{\gamma}_{\beta\alpha} = -2i\sum_j F^j_{\alpha\beta}a^{\gamma}_j + 2i\sum_j F^j_{\beta\alpha}a^{\gamma}_j\,.$$

Therefore we get $b^{\gamma}_{\alpha\beta}=b^{\gamma}_{\beta\alpha}$.

(3) By using (2.2)

$$[[f, s(iv_\alpha)], s(v_\beta)] = s(B(iv_\alpha)v_\beta)\,.$$

On the other hand by using (2.1) and (2.2)

$$\begin{aligned}[[f, s(iv_\alpha)], s(v_\beta)] &= s(iB(v_\beta)v_\alpha) - 4[f, \sum_j \operatorname{Re} F^j_{\alpha\beta} s(e_j)] \\ &= s(iB(v_\beta)v_\alpha) + 4\sum_j \operatorname{Re} F^j_{\alpha\beta} s(a_j)\,.\end{aligned}$$

Hence we have

$$\begin{aligned}B(iv_\alpha)^{\gamma}_{\beta} &= iB^{\gamma}_{\beta\alpha} + 2\sum_j (F^j_{\alpha\beta} + F^j_{\beta\alpha})a^{\gamma}_j \\ &= 2\sum_j F^j_{\beta\alpha}a^{\gamma}_j + 2ib^{\gamma}_{\alpha\beta}\,.\end{aligned}$$

Therefore we have (3.4) for $c=v_1, \cdots, v_m, iv_1, \cdots, iv_m$. By the definition (3.8), $B(c)$ is real linear in c. And both sides of the equation (3.4) are real linear in c. q. e. d.

PROPOSITION 3.4. $\hat{\mathfrak{g}}^1=\mathfrak{g}^1$.

PROOF. Let $f\in\hat{\mathfrak{g}}^1$ and let a_k, $A(c)$, $B(c)$ and $b^{\gamma}_{\alpha\beta}$ be the same as in Lemma 3.3. Then $b^{\gamma}_{\alpha\beta}=b^{\gamma}_{\beta\alpha}$. Hence we can define the vector field X_f of the form (2.5) by (3.1). Then by Lemma 3.2 and Lemma 3.3 $X_f\in\mathfrak{g}^1$, since $A(c)$ and $B(c)$ satisfy (3.5) and (3.6). Therefore we get the linear mapping of $\hat{\mathfrak{g}}^1$ into $\mathfrak{g}^1$ which maps $f\in\hat{\mathfrak{g}}^1$ to $X_f\in\mathfrak{g}^1$. It is clear that if $X_f=0$ then $f=0$ and that this mapping coincides with the identity on $\mathfrak{g}^1\subset\hat{\mathfrak{g}}^1$. q. e. d.

3.3. Next we investigate $\hat{\mathfrak{g}}^2$. We first show the following lemma.

LEMMA 3.5.

(1) "$f\in\hat{\mathfrak{g}}^2$, $[f,\mathfrak{g}^{-2}]=0$" *implies* $f=0$.

(2) $[I,\hat{\mathfrak{g}}^2]=0$.

PROOF. For any $X\in\mathfrak{g}^{-2}$ and for any $Y\in\mathfrak{g}^{-1}$, we have

$$[X,[Y,f]]=[Y,[X,f]]=0\,.$$

Since $[Y,f]\in\hat{\mathfrak{g}}^1$, and $\hat{\mathfrak{g}}^1=\mathfrak{g}^1$ by Proposition 3.4, the above equation shows $[Y,f]=0$ for any $Y\in\mathfrak{g}^{-1}$ by Lemma 3.1 and hence $f=0$.

(2) Since I is in the center of $\mathfrak{g}^0$ and $[I,\mathfrak{g}^{-2}]=0$ by (2.3) and Theorem 2.2, we have $[\mathfrak{g}^{-2},[I,\hat{\mathfrak{g}}^2]]=0$. On the other hand $[I,\hat{\mathfrak{g}}^2]\subset\hat{\mathfrak{g}}^2$, and hence $[I,\hat{\mathfrak{g}}^2]=0$ by (1). q. e. d.

We denote by $\tilde{\mathfrak{g}}^2$ the set of all vector field X of the form (2.6) satisfying $[\mathfrak{g}^{-2},X]\subset\mathfrak{g}^0$ and $[\mathfrak{g}^{-1},X]\subset\mathfrak{g}^1$. Let $X\in\tilde{\mathfrak{g}}^2$. Then we can write

$$X=\sum_{hjk}A^k_{hj}z^hz^j\frac{\partial}{\partial z^k}+\sum_{k\alpha\beta}B^{\alpha}_{k\beta}z^kw^{\beta}\frac{\partial}{\partial w^{\alpha}}\qquad(A^k_{hj}=A^k_{jh})\,. \tag{3.10}$$

Clearly "$[\mathfrak{g}^{-2},X]=0$" implies $X=0$. Since $\hat{\mathfrak{g}}^2$ consists of all element f of $\mathrm{Hom}(\mathfrak{g}^{-2},\mathfrak{g}^0)+\mathrm{Hom}(\mathfrak{g}^{-1},\hat{\mathfrak{g}}^1)$ satisfying the following condition (cf. [**10**]):

$$f([Y,Z])=[f(Y),Z]+[Y,f(Z)]\qquad Y,Z\in\mathfrak{g}^{-2}+\mathfrak{g}^{-1}\,,$$

we may regard X as an element of $\hat{\mathfrak{g}}^2$, where we consider X as an element of $\mathrm{Hom}(\mathfrak{g}^{-2},\mathfrak{g}^0)+\mathrm{Hom}(\mathfrak{g}^{-1},\hat{\mathfrak{g}}^1)$ such that

$$X(Y)=[X,Y]\qquad Y\in\mathfrak{g}^{-2}+\mathfrak{g}^{-1}\,.$$

Conversely let f be an element of $\hat{\mathfrak{g}}^2$. Then there exist $A_j\in\mathfrak{gl}(R)$ and $B_j\in\mathfrak{gl}(W)$ for every j ($j=1,\cdots,n$) which are uniquely determined by

$$\rho(2A_j,B_j)=[s(e_j),f]\,. \tag{3.11}$$

LEMMA 3.6. *Let* $f\in\hat{\mathfrak{g}}^2$ *and let* A_j *be as in* (3.11). *Then* $A^h_{jk}=A^h_{kj}$, *where* $A_j=(A^h_{jk})$.

PROOF.

$$\begin{aligned}0=[f,[s(e_j),s(e_k)]]&=-[\rho(2A_j,B_j),s(e_k)]-[s(e_j),\rho(2A_k,B_k)]\\&=2\sum_h A^h_{jk}s(e_h)-2\sum_h A^h_{kj}s(e_h)\,.\end{aligned}$$

q. e. d.

For every $f\in\hat{\mathfrak{g}}^{2}$ we can define the vector field X_{f} by (3.10) with $A_{j}=(A_{jh}^{k})$ and $B_{j}=(B_{j\beta}^{\alpha})$ given by (3.11). Then the correspondence: $f\rightarrow X_{f}$ gives a linear mapping of $\hat{\mathfrak{g}}^{2}$ into the vector space of all vector fields of the form (2.6). This mapping is injective by Lemma 3.5 and clearly equal to the identity on $\tilde{\mathfrak{g}}^{2}\subset\hat{\mathfrak{g}}^{2}$. It is also clear that for every $a\in R$ the following equality holds:

$$[s(a), f]=[s(a), X_{f}]. \tag{3.12}$$

For every $c\in W$, $[s(c), f]$ belongs to $\mathfrak{g}^{0}$ and hence is a vector field on D. We put $X(c)=[s(c), f]-[s(c), X_{f}]$.

LEMMA 3.7.

$$X(c)=0 \qquad \textit{for every} \quad c\in W.$$

PROOF. First we prove the following equality holds for all c, $c'\in W$:

$$[s(c), X(c')]=[s(c'), X(c)]. \tag{3.13}$$

In fact

$$[[s(c), s(c')], X_{f}]=[[s(c), X_{f}], s(c')]+[s(c), [s(c'), X_{f}]].$$

On the other hand since $[s(c), s(c')]\in\mathfrak{g}^{-2}$ we get by (3.12)

$$\begin{aligned}[][[s(c), s(c')], X_{f}]&=[[s(c), s(c')], f]\\ &=[[s(c), f], s(c')]+[s(c), [s(c'), f]].\end{aligned}$$

Hence we obtain (3.13). By a direct calculation

$$\begin{aligned}[][s(c), X_{f}]=&\sum_{hjk}4iF^{k}(w, c)A_{jk}^{h}z^{j}\frac{\partial}{\partial z^{h}}+\sum_{k\alpha\beta}2iF^{k}(w, c)B_{k\beta}^{\alpha}w^{\beta}\frac{\partial}{\partial w^{\alpha}}\\ &-\sum_{jk}2iF^{j}(B_{k}w, c)z^{k}\frac{\partial}{\partial z^{j}}+\sum_{j\alpha\beta}B_{j\beta}^{\alpha}c^{\beta}z^{j}\frac{\partial}{\partial w^{\alpha}}.\end{aligned}$$

It follows that $X(c)$ has the form

$$\begin{aligned}X(c)=&\sum_{jk\alpha}\Gamma(c)_{j\alpha}^{k}z^{j}w^{\alpha}\frac{\partial}{\partial z^{k}}+\sum_{j\alpha}\Gamma(c)_{j}^{\alpha}z^{j}\frac{\partial}{\partial w^{\alpha}}\\ &+\sum_{\alpha\beta\gamma}\Gamma(c)_{\beta\gamma}^{\alpha}w^{\beta}w^{\gamma}\frac{\partial}{\partial w^{\alpha}},\end{aligned}$$

and all coefficients are real linear in c. By using (3.12) we get for every k $(k=1, \cdots, n)$

$$[s(e_{k}), X(c)]=[s(c), [s(e_{k}), f]]=[s(c), [s(e_{k}), X_{f}]]=0.$$

This implies $\Gamma(c)_{j\alpha}^{k}=\Gamma(c)_{j}^{\alpha}=0$, and hence

$$X(c)=\sum_{\alpha\beta\gamma}\Gamma(c)_{\beta\gamma}^{\alpha}w^{\beta}w^{\gamma}\frac{\partial}{\partial w^{\alpha}}.$$

Now

$$X(ic)=[s(ic), f]-[s(ic), X_f]$$
$$=-[[I, s(c)], f]+[[I, s(c)], X_f]$$
$$=-[I, [s(c), f]]-[s(c), [I, f]]$$
$$+[[I, X_f], s(c)]+[I, [s(c), X_f]].$$

By Lemma 3.5, $[I, f]=0$. And

$$[I, X_f]=\Big[\sum_\alpha iw^\alpha\frac{\partial}{\partial w^\alpha}, \sum_{hjk} A^k_{hj}z^h z^j\frac{\partial}{\partial z^k}+\sum_{j\alpha\beta} B^\alpha_{j\beta}z^j w^\beta\frac{\partial}{\partial w^\alpha}\Big]=0.$$

It follows

$$X(ic)=-[I, X(c)]$$
$$=-\Big[\sum_\alpha iw^\alpha\frac{\partial}{\partial w^\alpha}, \sum_{\alpha\beta\gamma}\Gamma(c)^\alpha_{\beta\gamma}w^\beta w^\gamma\frac{\partial}{\partial w^\alpha}\Big]$$
$$=-iX(c).$$

This implies that $X(c)$ has the form

$$X(c)=\sum_{\alpha\beta\gamma\delta}\Gamma^\alpha_{\beta\gamma\delta}\bar{c}^\delta w^\beta w^\gamma\frac{\partial}{\partial w^\alpha}\qquad(\Gamma^\alpha_{\beta\gamma\delta}=\Gamma^\alpha_{\gamma\beta\delta}).$$

Then

$$[s(c'), X(c)]=\Big[\sum_k 2iF^k(w, c')\frac{\partial}{\partial z^k}+\sum_\alpha c'^\alpha\frac{\partial}{\partial w^\alpha}, X(c)\Big]$$
$$=2\sum_{\alpha\beta\gamma\delta}\Gamma^\beta_{\alpha\gamma\delta}\bar{c}^\delta c'^\alpha w^\gamma\frac{\partial}{\partial w^\beta}\quad\Big(\mathrm{mod}\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\Big).$$

It follows from (3.13)

$$\sum_{\alpha\delta}\Gamma^\beta_{\alpha\gamma\delta}c'^\alpha\bar{c}^\delta=\sum_{\alpha\delta}\Gamma^\beta_{\alpha\gamma\delta}c^\alpha\bar{c}'^\delta\qquad\text{for all }\beta, \gamma.$$

This implies $\Gamma^\beta_{\alpha\gamma\delta}=0$ for all α, β, γ, δ. q.e.d.

By Lemma 3.7, $X_f\in\tilde{\mathfrak{g}}^2$. Thus we obtain the following proposition.

PROPOSITION 3.8. $\hat{\mathfrak{g}}^2=\tilde{\mathfrak{g}}^2$.

Since $(ad\, I)^2=-id$ on $\mathfrak{g}^{-1}$, we may regard $\mathfrak{g}^{-1}$ as a complex vector space. Let X be an element of $\mathfrak{g}^0$. Then $ad\, X|_{\mathfrak{g}^{-1}}$ is a complex linear endomorphism because I is in the center of $\mathfrak{g}^0$.

THEOREM 3.9. *Let D be a Siegel domain and let $\mathfrak{g}(D)=\sum_{\lambda=-2}^{2}\mathfrak{g}^\lambda$ be the graded Lie algebra of all infinitesimal automorphisms of D. Then $\mathfrak{g}(D)$ is a graded subalgebra of the prolongation $\hat{\mathfrak{g}}=\sum_{\lambda\geqq-2}\hat{\mathfrak{g}}^\lambda$ of $(\mathfrak{g}^{-2}+\mathfrak{g}^{-1}, \mathfrak{g}^0)$, and $\mathfrak{g}^1$ and $\mathfrak{g}^2$ are determined as follows:*

(1) $\mathfrak{g}^1=\hat{\mathfrak{g}}^1$.

(2) $\mathfrak{g}^2$ *consists of all* $X\in\hat{\mathfrak{g}}^2$ *such that* $\mathrm{Im}\, Tr\,(ad\,[X, Y]|_{\mathfrak{g}^{-1}})=0$ *for all* $X\in\mathfrak{g}^{-2}$.

PROOF. (1) is already proved (Proposition 3.4). Let X be an element of $\tilde{\mathfrak{g}}^2$. We write

$$X=\sum_{hjk} A_{hj}^{k} z^{h} z^{j} \frac{\partial}{\partial z^{k}}+\sum_{k\alpha\beta} B_{k\beta}^{\alpha} z^{k} w^{\beta} \frac{\partial}{\partial w^{\alpha}} .$$

Then

$$[s(e_{j}), X]=2\sum_{kh} A_{hj}^{k} z^{h} \frac{\partial}{\partial z^{k}}+\sum_{\alpha\beta} B_{j\beta}^{\alpha} w^{\beta} \frac{\partial}{\partial w^{\alpha}} .$$

It follows

$$Tr(ad[s(e_{j}), X]|_{\mathfrak{g}^{-1}})=\sum_{\alpha} B_{j\alpha}^{\alpha} .$$

Then the assertion (2) follows immediately from Proposition 2.6 and Proposition 3.8. q. e. d.

COROLLARY 3.10. *Let D be a Siegel domain of the first kind. Then* $\mathfrak{g}(D)=\mathfrak{g}^{-2}+\mathfrak{g}^{0}+\mathfrak{g}^{2}$ *and* $\mathfrak{g}^{2}$ *is the first prolongation of the linear Lie algebra* $\mathfrak{g}^{0}\subset\mathfrak{gl}(\mathfrak{g}^{-2})$, *i.e.,* $\mathfrak{g}^{2}$ *consists of all element f of* Hom $(\mathfrak{g}^{-2}, \mathfrak{g}^{0})$ *satisfying the following condition:*

$$f(X)Y=f(Y)X \qquad \textit{for all} \quad X, Y\in\mathfrak{g}^{-2} .$$

COROLLARY 3.11 (cf. [4]). *Let $D=D(V, F)$ be a Siegel domain and let* $\mathfrak{g}(D)$ *and* $\hat{\mathfrak{g}}$ *be as in Theorem* 3.9. *Then* $\mathfrak{g}(D)$ *is maximum among the subalgebras* $\mathfrak{h}$ *of* $\hat{\mathfrak{g}}$ *satisfying the following conditions:*

1) $\mathfrak{h}\subset\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^{0}+\hat{\mathfrak{g}}^{1}+\hat{\mathfrak{g}}^{2}$,

2) $\mathfrak{h}\supset\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^{0}$.

PROOF. Since $\mathfrak{h}$ contains the element E, $\mathfrak{h}$ is a graded subalgebra of $\hat{\mathfrak{g}}$. Therefore $\mathfrak{h}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^{0}+\mathfrak{h}^{1}+\mathfrak{h}^{2}$ ($\mathfrak{h}^{1}=\mathfrak{h}\cap\hat{\mathfrak{g}}^{1}$ and $\mathfrak{h}^{2}=\mathfrak{h}\cap\hat{\mathfrak{g}}^{2}$). Let $\mathfrak{r}$ be the radical of $\mathfrak{h}$. Being invariant by $ad\,E$, $\mathfrak{r}$ is a graded subalgebra of $\hat{\mathfrak{g}}$, i.e., $\mathfrak{r}=\mathfrak{r}^{-2}+\mathfrak{r}^{-1}+\mathfrak{r}^{0}+\mathfrak{r}^{1}+\mathfrak{r}^{2}$ ($\mathfrak{r}^{\lambda}=\mathfrak{r}\cap\hat{\mathfrak{g}}^{\lambda}$). We put $\mathfrak{r}'=\mathfrak{r}^{-2}+\mathfrak{r}^{-1}+\mathfrak{r}^{0}+\mathfrak{r}^{1}+[\mathfrak{r}^{1}, \mathfrak{r}^{1}]$. Since $\hat{\mathfrak{g}}^{1}=\mathfrak{g}^{1}$ by Theorem 3.9, $\mathfrak{r}'$ is a solvable subalgebra of $\mathfrak{g}(D)$. Let $X\in\mathfrak{r}^{1}$ and $v\in V$. Then $X+ad\,I\circ ad\,s(v)X$ is contained in the isotropy subalgebra of $\mathfrak{g}(D)$ at $(iv, 0)\in D$ (cf. [5]). On the other hand $X+ad\,I\circ ad\,s(v), X\in\mathfrak{r}'$. Therefore by the same argument as in the proof of Lemma 4.2 in [5], we have $X=0$ and hence $\mathfrak{r}^{1}=0$. It follows

$$[\mathfrak{g}^{-1}, [\mathfrak{g}^{-2}, \mathfrak{r}^{2}]]=[\mathfrak{g}^{-2}, [\mathfrak{g}^{-1}, \mathfrak{r}^{2}]]\subset[\mathfrak{g}^{-2}, \mathfrak{r}^{1}]=0 .$$

Then by Theorem 3.9 we get $\mathfrak{r}^{2}\subset\mathfrak{g}^{2}$, and hence $\mathfrak{r}\subset\mathfrak{g}(D)$. Let $X\in\mathfrak{r}^{2}$. Then $X+\frac{1}{2}(ad\,s(v))^{2}X$ is contained in the isotropy subalgebra of $\mathfrak{g}(D)$ at $(iv, 0)$ (cf. [5]). Clearly $X+\frac{1}{2}(ad\,s(v))^{2}X\in\mathfrak{r}$. Therefore we have $X=0$ analogously, and hence $\mathfrak{r}^{2}=0$. We put $\mathfrak{h}'=\mathfrak{h}/\mathfrak{r}$. Then $\mathfrak{h}'=\mathfrak{h}'^{-2}+\mathfrak{h}'^{-1}+\mathfrak{h}'^{0}+\mathfrak{h}^{1}+\mathfrak{h}^{2}$ with $\mathfrak{h}'^{\lambda}=\mathfrak{g}^{\lambda}/\mathfrak{r}^{\lambda}$ for $\lambda\leqq 0$. Denote by E' the image in $\mathfrak{h}'^{0}$ of E. Let $\mathfrak{h}'=\sum_{j}\mathfrak{h}_{j}$ be the decomposition of $\mathfrak{h}'$ into simple ideals. Being invariant by $ad\,E'$, each $\mathfrak{h}_{j}$ is a graded ideal of $\mathfrak{h}'$. Thus we can write $\mathfrak{h}_{j}=\mathfrak{h}_{j}^{-2}+\mathfrak{h}_{j}^{-1}+\mathfrak{h}_{j}^{0}+\mathfrak{h}_{j}^{1}+\mathfrak{h}_{j}^{2}$ ($\mathfrak{h}_{j}^{\lambda}=\mathfrak{h}_{j}\cap\mathfrak{h}'^{\lambda}$). Suppose that $\mathfrak{h}_{j}^{1}=0$. Then $[\mathfrak{g}^{-1}, \mathfrak{h}_{j}^{2}]=0$. It follows $[\mathfrak{g}^{-1}, [\mathfrak{g}^{-2}, \mathfrak{h}_{j}^{2}]]=0$ and hence from Theorem 3.9 $\mathfrak{h}_j^2 \subset \mathfrak{g}^2$. Next we investigate the case where $\mathfrak{h}_j^1 \neq 0$. Then $[\mathfrak{h}_j^{-1}, \mathfrak{h}_j^{-1}]+\mathfrak{h}_j^{-1}+[\mathfrak{h}_j^{-1}, \mathfrak{h}_j^1]+\mathfrak{h}_j^1+[\mathfrak{h}_j^1, \mathfrak{h}_j^1]$ is an ideal of $\mathfrak{h}_j$. Therefore $\mathfrak{h}_j^2=[\mathfrak{h}_j^1, \mathfrak{h}_j^1]$. Since $\mathfrak{h}_j^1 \subset \mathfrak{g}^1$ by Theorem 3.9, we have $\mathfrak{h}_j^2 \subset \mathfrak{g}^2$. q. e. d.

REMARK 3. Theorem 3.9 is proved in [10] with the assumption of homogeneity of D, in connection with the study of "real submanifolds".

§ 4. Irreducibility of Siegel domains.

4.1. Let $D=D(V, F)$ be a Siegel domain and let $\mathfrak{g}(D)=\sum_{\lambda=-2}^{2} \mathfrak{g}^\lambda$ be the graded Lie algebra of all infinitesimal automorphisms of D. It is easy to see that any ideal $\mathfrak{a}$ of $\mathfrak{g}(D)$ is decomposed into the form:

$$\mathfrak{a}=\mathfrak{a}^{-2}+\mathfrak{a}^{-1}+\mathfrak{a}^0+\mathfrak{a}^1+\mathfrak{a}^2, \qquad \mathfrak{a}^\lambda=\mathfrak{a} \cap \mathfrak{g}^\lambda. \tag{4.1}$$

Assume that $\mathfrak{g}(D)=\mathfrak{g}_1+\mathfrak{g}_2$ (direct sum), where both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are non-trivial ideals of $\mathfrak{g}(D)$. Then by (4.1) we have

$$\mathfrak{g}_\mu=\mathfrak{g}_\mu^{-2}+\mathfrak{g}_\mu^{-1}+\mathfrak{g}_\mu^0+\mathfrak{g}_\mu^1+\mathfrak{g}_\mu^2$$
$$\mathfrak{g}^\lambda=\mathfrak{g}_1^\lambda+\mathfrak{g}_2^\lambda$$
$$\mathfrak{g}_\mu^\lambda=\mathfrak{g}^\lambda \cap \mathfrak{g}_\mu \qquad (\mu=1, 2,\ \lambda=-2, -1, 0, 1, 2).$$

We assert $\mathfrak{g}_\mu^{-2} \neq (0)$. In fact, suppose $\mathfrak{g}_\mu^{-2}=(0)$. Then for each element $s(c)$ of $\mathfrak{g}_\mu^{-1}$ $(c \in W)$, we have

$$[[I, s(c)], s(c)] \in \mathfrak{g}^{-2} \cap \mathfrak{g}_\mu=\mathfrak{g}_\mu^{-2}=(0).$$

On the other hand by using (2.1) and (2.3)

$$[[I, s(c)], s(c)]=s(4F(c, c)).$$

It follows that $\mathfrak{g}_\mu^{-1}=(0)$. As a result $\mathfrak{g}_\mu^0=\mathfrak{g}_\mu^1=\mathfrak{g}_\mu^2=(0)$ by Lemma 3.1. This contradicts the assumption that $\mathfrak{g}_\mu$ is not trivial, proving our assertion. We set for $\mu=1, 2$

$$R_\mu=\{a \in R\,;\ s(a) \in \mathfrak{g}_\mu^{-2}\}$$
$$W_\mu=\{c \in W\,;\ s(c) \in \mathfrak{g}_\mu^{-1}\}.$$

Then we have $R=R_1+R_2$ (direct sum) and $W=W_1+W_2$ (direct sum). Since $[I, s(c)]=-s(ic)$ for $c \in W$, W_μ is a complex subspace of W. We denote by F_μ the R_μ^c-component of F corresponding to the direct sum $R^c=R_1^c+R_2^c$.

LEMMA 4.1.

(1) *$F(w, w')=F_1(w_1, w_1')+F_2(w_2, w_2')$, where $w=w_1+w_2$ and $w'=w_1'+w_2'$ (w_μ, $w_\mu' \in W_\mu$, $\mu=1, 2$).*

(2) *"$w_\mu \in W_\mu$, $F_\mu(w_\mu, w_\mu)=0$" implies $w_\mu=0$.*

PROOF. By using (2.1) we have

$$4s(\mathrm{Im}\, F(w, w')) = -[s(w), s(w')]$$
$$= -[s(w_1), s(w_1')] - [s(w_2), s(w_2')] .$$

It follows

$$4s(\mathrm{Im}\, F_\mu(w, w')) = -[s(w_\mu), s(w'_\mu)]$$
$$= 4s(\mathrm{Im}\, F(w_\mu, w'_\mu)) = 4s(\mathrm{Im}\, F_\mu(w_\mu, w'_\mu)) .$$

Hence we get $\mathrm{Im}\, F(w, w') = \mathrm{Im}\, F_1(w_1, w_1') + \mathrm{Im}\, F_2(w_2, w_2')$. And

$$\mathrm{Re}\, F(w, w') = \mathrm{Im}\, F(iw, w')$$
$$= \mathrm{Im}\, F_1(iw_1, w_1') + \mathrm{Im}\, F_2(iw_2, w_2')$$
$$= \mathrm{Re}\, F_1(w_1, w_1') + \mathrm{Re}\, F_2(w_2, w_2') .$$

Therefore we have proved the assertion (1). The assertion (2) follows immediately from (1). q. e. d.

We can write uniquely $E = E_1 + E_2$ ($E_\mu \in g^0_\mu$, $\mu = 1, 2$). Then we have

LEMMA 4.2.

$$\mathrm{Exp}\, tE_1(z_1, z_2, w_1, w_2) = (e^{2t} z_1, z_2, e^t w_1, w_2) .$$
$$\mathrm{Exp}\, tE_2(z_1, z_2, w_1, w_2) = (z_1, e^{2t} z_2, w_1, e^t w_2) .$$

PROOF. E_μ is the unique element of $\mathfrak{g}^0$ such that

$$[E_\mu, X_1 + X_2] = -2X_\mu , \qquad X_\mu \in \mathfrak{g}_\mu^{-2}$$
$$[E_\mu, Y_1 + Y_2] = -Y_\mu , \qquad Y_\mu \in \mathfrak{g}_\mu^{-1},\ \mu = 1, 2 .$$

Then Lemma 4.2 follows immediately from (2.2). q. e. d.

Let D_μ (resp. V_μ) be the image of D (resp. of V) by the natural projection: $R_1^c \times R_2^c \times W_1 \times W_2 \rightarrow R_\mu^c \times W_\mu$ (resp. $R \rightarrow R_\mu$). Then D_μ and V_μ are open sets of $R_\mu^c \times W_\mu$ and R_μ respectively.

LEMMA 4.3. *D_μ is the Siegel domain of the second kind associated with a convex cone V_μ and a V_μ-hermitian form F_μ ($\mu = 1, 2$).*

PROOF. We prove this for $\mu = 1$. Let $y_1 \in V_1$. Then there exists $y_2 \in V_2$ such that $(y_1, y_2) \in V$. By Lemma 4.2, $(y_1, ty_2) \in V$ for all $t > 0$. Taking the limit as $t \rightarrow 0$, we have $(y_1, 0) \in \bar{V}$. As a result V_1 contains no entire straight lines. Also by Lemma 4.2 $(ty_1, y_2) \in V$ for all $t > 0$ and hence V_1 is a cone. Clearly V_1 is convex. Therefore V_1 is a convex cone. From Lemma 4.1 it is clear that F_1 is a V_1-hermitian form on W_1. Let $(z_1, w_1) \in D_1$. Then there exists $(z_2, w_2) \in D_2$ such that $(z_1, z_2, w_1, w_2) \in D$. By using Lemma 4.1 we have $\mathrm{Im}\, z_1 - F_1(w_1, w_1) \in V_1$. Conversely suppose that the point (z_1, w_1) satisfies $\mathrm{Im}\, z_1 - F_1(w_1, w_1) \in V_1$. Then there exists $y_2 \in V_2$ such that $(\mathrm{Im}\, z_1 - F_1(w_1, w_1), y_2) \in V$. Therefore $(z_1, iy_2, w_1, 0) \in D$. This implies $(z_1, w_1) \in D_1$. q. e. d.

Let $y_1 \in V_1$ and $y_2 \in V_2$. Then $(y_1, 0), (0, y_2) \in \bar{V}$ as in the proof of Lemma

4.3, and hence $(y_1, y_2)\in \bar{V}$. It follows that $V\subset V_1\times V_2\subset \bar{V}$. Since $V_1\times V_2$ is open we conclude $V=V_1\times V_2$. Let $(z_1, w_1)\in D_1$ and $(z_2, w_2)\in D_2$. Then $\operatorname{Im} z_1-F_1(w_1, w_1)\in V_1$ and $\operatorname{Im} z_2-F_2(w_2, w_2)\in V_2$, by Lemma 4.3. Since $V=V_1\times V_2$, we have $(\operatorname{Im} z_1-F_1(w_1, w_1), \operatorname{Im} z_2-F_2(w_2, w_2))\in V$, and hence $(z_1, z_2, w_1, w_2)\in D$. As a consequence $D=D_1\times D_2$. Finally it is not difficult to see that $\mathfrak{g}_\mu$ is identified with the Lie algebra of all infinitesimal automorphisms of D_μ[1]. Thus we have proved the following proposition.

PROPOSITION 4.4. *Let $D=D(V, F)$ be a Siegel domain in $R^c\times W$, and let $\mathfrak{g}(D)$ be the Lie algebra of all infinitesimal automorphisms of D. Suppose that $\mathfrak{g}(D)=\mathfrak{g}_1+\mathfrak{g}_2$ (direct sum), where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are non-trivial ideals of $\mathfrak{g}(D)$. Then we have $D=D_1\times D_2$ where D_μ is the Siegel domain $D(V_\mu, F_\mu)$ in $R^c_\mu\times W_\mu$ ($\mu=1, 2$) such that*

(1) *$R=R_1+R_2$, $W=W_1+W_2$ and $V=V_1\times V_2$.*

(2) *$F(w, w')=F_1(w_1, w_1')+F_2(w_2, w_2')$, $w=w_1+w_2$, $w'=w_1'+w_2'$, where $w_\mu, w_\mu'\in W_\mu$.*

(3) *$\mathfrak{g}_\mu$ is identified with the Lie algebra of all infinitesimal automorphisms of D_μ.*

REMARK 4. In Proposition 4.4 we may make the assumption that $\mathfrak{g}^a=\mathfrak{a}_1+\mathfrak{a}_2$ where $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are non-trivial ideals of $\mathfrak{g}^a$ instead of the assumption $\mathfrak{g}(D)=\mathfrak{g}_1+\mathfrak{g}_2$.

4.2. Let D be a Siegel domain. Since D is complete simply connected[2] Kähler manifold by Theorem 1.3, D is uniquely decomposed into the direct product of complete simply connected irreducible Kähler manifolds D_i's ([**1**], [**7**]),

$$D=D_1\times \cdots \times D_s\,.$$

This decomposition is called the de Rham decomposition of D. We denote by A and A_i the identity components of the isometry groups of D and D_i respectively. Then it is well known ([**1**], [**7**]) that

$$A=A_1\times \cdots \times A_s\,. \tag{4.2}$$

A holomorphic transformation f_i of D_i extended to a holomorphic transformation of D in a trivial manner is an isometry of D. We denote by ds^2 and

---

1) Let $\mathfrak{g}(D_\mu)=\mathfrak{u}_\mu^{-2}+\mathfrak{u}_\mu^{-1}+\mathfrak{u}_\mu^0+\mathfrak{u}_\mu^1+\mathfrak{u}_\mu^2$ be the Lie algebra of $\operatorname{Aut}(D_\mu)$. Then $\mathfrak{g}(D_\mu)\subset\mathfrak{g}(D)$. Clearly $\mathfrak{u}_\mu^{-2}=\mathfrak{g}_\mu^{-2}$, $\mathfrak{u}_\mu^{-1}=\mathfrak{g}_\mu^{-1}$ and $\mathfrak{u}_\mu^0\supset\mathfrak{g}_\mu^0$. Moreover by using Theorem 3.9 we know $\mathfrak{g}_\mu^1\subset\mathfrak{u}_\mu^1$ and $\mathfrak{g}_\mu^2\subset\mathfrak{u}_\mu^2$. Therefore $\mathfrak{g}_\mu\subset\mathfrak{g}(D_\mu)$. The inverse inclusion follows immediately from the equation

$$[\mathfrak{g}_\nu^{-2}+\mathfrak{g}_\nu^{-1}, \mathfrak{u}_\mu^0+\mathfrak{u}_\mu^1+\mathfrak{u}_\mu^2]=0 \qquad (\nu\neq\mu)\,.$$

2) Let η be a diffeomorphism of $R^c\times W$ to $R\times R\times W$ defined by $\eta(z, w)=(\operatorname{Im} z-F(w, w), \operatorname{Re} z, w)$. Then $\eta(D)=V\times R\times W$. Since the convex cone V is simply connected, so is D.

ds_i^2 the metric tensor fields of D and D_i respectively, and denote by ι_i the imbedding of D_i into D. Then

$$f_i^* ds_i^2 = f_i^*(\iota_i^* ds^2) = \iota_i^*(f_i^* ds^2) = \iota_i^* ds^2 = ds_i^2 .$$

This implies that f_i is an isometry of D_i and hence the automorphism group of D_i is a closed subgroup of the isometry group of D_i. Denote by G and G_i the identity components of $\mathrm{Aut}(D)$ and $\mathrm{Aut}(D_i)$ respectively. Then we have

LEMMA 4.5. $G = G_1 \times \cdots \times G_s$.

PROOF. Since $G \subset A$, every element $g \in G$ may be written as $g = (g_1, \cdots, g_s)$ by (4.2), where $g_i \in A_i$. We denote by J and by J_i the complex structures of D and D_i respectively. Then $J = J_1 \times \cdots \times J_s$. Since $g_* \circ J = J \circ g_*$, we have $g_{i*} \circ J_i = J_i \circ g_{i*}$. As a result we have $g_i \in G_i$. Thus we have proved $G \subset G_1 \times \cdots \times G_s$. The inverse inclusion is clear. q. e. d.

LEMMA 4.6. *Let M be a complex submanifold of D which is invariant by G. Then* $\dim M = \dim D$.

PROOF. From the condition of M every element of $\mathfrak{g}^{-2} + \mathfrak{g}^{-1}$ is tangent to M. Since M is a complex submanifold of D, every element of $J(\mathfrak{g}^{-2})$ is also tangent to M. Then from the expression (2.4), we easily see $\dim_{\boldsymbol{R}}(J(\mathfrak{g}^{-2}) + \mathfrak{g}^{-2} + \mathfrak{g}^{-1}) = \dim_{\boldsymbol{R}} D$. q. e. d.

We say that a Siegel domain is *irreducible* if it is irreducible as a Kähler manifold.

THEOREM 4.7. *Let D be a Siegel domain and let $D = D_1 \times \cdots \times D_s$ be its de Rham decomposition. Then each D_i is holomorphically equivalent to an irreducible Siegel domain.*

PROOF. We denote by $\mathfrak{g}_i$ the Lie algebra of all infinitesimal automorphisms of D_i. Then by Lemma 4.5 each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}(D)$ and $\mathfrak{g}(D) = \sum_{i=1}^{s} \mathfrak{g}_i$ (direct sum). By Lemma 4.6, each $\mathfrak{g}_i$ is not trivial and hence by Proposition 4.4 there exist Siegel domain D_j''s such that $D = D_1' \times \cdots \times D_s'$. It is well known ([**6**]) that $ds^2 = ds_1^2 + \cdots + ds_s^2$, where ds^2 and ds_i^2 are the Bergmann metrics on D and D_i' respectively. This implies that each D_i' is irreducible and holomorphically equivalent to some D_j, since the de Rham decomposition is unique up to the order. q. e. d.

We call a convex cone V (in R) *reducible* if there exist convex cones V_1 (in R_1) and V_2 (in R_2) such that $V = V_1 \times V_2$ $(R = R_1 + R_2)$, and otherwise *irreducible*. From Proposition 4.4 and Theorem 4.7, we have the following corollaries.

COROLLARY 4.8. *Let $D(V, F)$ be a Siegel domain. Suppose that V is irreducible. Then $D(V, F)$ is irreducible.*

COROLLARY 4.9. *A Siegel domain D is irreducible if and only if the Lie* *algebra of* Aut(D) *is not split into the direct sum of ideals.*

REMARK 5. When $D(V, F)$ is homogeneous, Theorem 4.7 and Corollary 4.8 are already proved in [**2**] and the converse of Corollary 4.8 is also true ([**2**]). But there exists an inhomogeneous Siegel domain for which the converse of Corollary 4.8 does not hold (Example 1). We state another property of homogeneous Siegel domains which does not hold for inhomogeneous domains. The Siegel domain $D=D(V, F)$ is called *non-degenerate*, if $\{F(w, w)\,;\ w\in W\}$ generates the vector space R, or equivalently if $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$ (cf. [**3**]). Let S be the submanifold of $R^c\times W$ defined by

$$S=\{(z, w)\in R^c\times W\,;\ \mathrm{Im}\, z-F(w, w)=0\}\,.$$

S is called the Silov boundary of D. We denote by $GL(S)$ the closed subgroup of $GL(R^c\times W)$ leaving S invariant. Then we have $GL(D)\subset GL(S)$ ([**9**]). Assume that D is homogeneous. Then $GL(D)=GL(S)$ if and only if D is non-degenerate ([**3**]). But in general cases the equality $GL(D)=GL(S)$ does not hold even if D is non-degenerate (Example 2).

EXAMPLE 1. Let $V=\boldsymbol{R}^{2+}=\{(y^1, y^2)\in \boldsymbol{R}^2\,;\ y^1>0, y^2>0\}$, and let F be the V-hermitian form on $\boldsymbol{C}^1$ defined by $F(w, w')=(w\bar{w}', w\bar{w}')$, $w, w'\in \boldsymbol{C}^1$. Clearly V is reducible. Assume that $D=D(V, F)$ is reducible. Then $D(V, F)=D(V_1, F_1)\times D(V_2, F_2)$ and $V=V_1\times V_2$. Since $F(w, w')$ is 1-dimensional, we may assume $F=F_1$ and $F_2=0$. Then $V_1=\{F(w, w)\,;\ w\neq 0\}$. Clearly there is no vector v_2 in $\boldsymbol{R}^2$ such that $V=V_1\times\{tv_2\,;\ t>0\}$. As a result D is irreducible. We can see that $\mathfrak{g}^0$ is generated by E and I and that $\mathfrak{g}(D)=\mathfrak{g}^a$.

EXAMPLE 2. Let $V=\boldsymbol{R}^{2+}$ and let F be the V-hermitian form on $\boldsymbol{C}^2$ defined by

$$F(w, w')=(w^1\bar{w}'^1+w^2\bar{w}'^2,\ w^2\bar{w}'^2)$$
$$w=(w^1, w^2),\ w'=(w'^1, w'^2)\,.$$

Clearly $D=D(V, F)$ is non-degenerate. Every element of the Lie algebra of $GL(S)$ has the following form:

$$(z, w)\longrightarrow (Az, Bw),\ \text{where } A=\begin{pmatrix} a_{11} & a_{12}\\ 0 & a_{22}\end{pmatrix} \text{ and}$$
$$B=\begin{pmatrix} b_{11} & 0\\ 0 & b_{22}\end{pmatrix} \text{ such that } a_{11}=2\,\mathrm{Re}\, b_{11},\ a_{22}=2\,\mathrm{Re}\, b_{22}$$
$$\text{and } a_{12}=a_{22}-a_{11}\,.$$

Therefore $\dim GL(S)=4$. And every element of $\mathfrak{g}^0$ has the form:

$$(z, w)\longrightarrow (Az, Bw),\ \text{where } A=\begin{pmatrix} a_{11} & 0\\ 0 & a_{22}\end{pmatrix} \text{ and}$$
$$B=\begin{pmatrix} b_{11} & 0\\ 0 & b_{22}\end{pmatrix} \text{ such that } a_{11}=a_{22}=2\,\mathrm{Re}\, b_{11}=2\,\mathrm{Re}\, b_{22}\,.$$

Therefore $\dim GL(D)=3$, and hence $GL(D) \neq GL(S)$. It is not difficult to see that $\mathfrak{g}(D)=\mathfrak{g}^a$.

Bibliography

[1] J. Hano and Y. Matsushima, Some studies on Kählerian homogeneous spaces, Nagoya Math. J., 11 (1957), 77-92.

[2] S. Kaneyuki, On the automorphism groups of homogeneous bounded domains, J. Fac. Sci. Univ. Tokyo, 14 (1967), 89-130.

[3] S. Kaneyuki and M. Sudo, On Silov boundaries of Siegel domains, J. Fac. Sci. Univ. Tokyo, 15 (1968), 131-146.

[4] W. Kaup, Einige Bemerkungen über polynomiale Vektorfelder, Jordanalgebren und die Automorphismen von Siegelschen Gebieten, Math. Ann., 204 (1973), 131-144.

[5] W. Kaup, Y. Matsushima and T. Ochiai, On the automorphisms and equivalences of generalized Siegel domains, Amer. J. Math., 92 (1970), 475-497.

[6] S. Kobayashi, Geometry of bounded domains, Trans. Amer. Math. Soc., 92 (1959), 267-290.

[7] S. Kobayashi and K. Nomizu, Foundation of differential geometry, in Interscience Tracts, No. 15, **I** (1963), **II** (1968). Wiley, New York.

[8] S. Murakami, On automorphisms of Siegel domains, Lecture Notes in Math., 286, Springer, 1972.

[9] I. I. Pyatetski-Shapiro, Geometry of classical domains and theory of automorphic functions, Fizmatgiz, Moscow, 1961. French translation, Paris, 1966.

[10] N. Tanaka, On infinitesimal automorphisms of Siegel domains, J. Math. Soc. Japan, 22 (1970), 180-212.

Kazufumi NAKAJIMA
Department of Mathematics
Faculty of Science
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto, Japan