

Real hypersurfaces in a complex projective space with constant principal curvatures

By Ryoichi TAKAGI^{*)}

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Introduction.

This paper is a continuation of the previous one [6], in which we classified those homogeneous real hypersurfaces in a complex projective space $P_n(\mathbb{C})$ of complex dimension $n (\geq 2)$ which are orbits under analytic subgroups of the projective unitary group $PU(n+1)$, and gave some characterization of those hypersurfaces. We shall call each of such hypersurfaces a model space for convenience' sake. The main purpose of this paper is to give another characterization of a geodesic hypersphere in $P_n(\mathbb{C})$ which is one of six kinds of the model space. It can be stated as follows.

THEOREM 1. *If M is a connected complete real hypersurface in $P_n(\mathbb{C})$ with two constant principal curvatures, then M is a geodesic hypersphere.*

In §1 we shall determine the principal curvatures of the model spaces. As a result of the determination we know that each model space has two or three or five constant principal curvatures. In §3 we shall prove Theorem 1 and in §4 give its application.

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§1. Principal curvatures of the model spaces.

In [6] we have seen that roughly speaking there is a one-to-one correspondence between the model spaces and the isotropy representations of various Hermitian symmetric spaces of rank two. The correspondence is given as follows. Let (\mathfrak{u}, θ) be a Hermitian effective orthogonal symmetric Lie algebra of compact type and of rank two. \mathfrak{u} is a compact semisimple Lie algebra and θ is an involutive automorphism of \mathfrak{u} (cf. [3]). Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of \mathfrak{u} into the eigenspaces of θ for the eigenvalues $+1$ and -1 , respectively. Then \mathfrak{k} and \mathfrak{p} satisfy $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. For the Killing form B of \mathfrak{u} we define a positive definite inner product \langle, \rangle

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on \mathfrak{p} by $\langle X, Y \rangle = -B(X, Y)$ for $X, Y \in \mathfrak{p}$. Let K be the analytic subgroup of the group of inner automorphisms of \mathfrak{u} with Lie algebra $\text{ad}(\mathfrak{k})$. Then K leaves the subspace \mathfrak{p} of \mathfrak{u} invariant and acts on \mathfrak{p} as an orthogonal transformation group with respect to \langle, \rangle . We define a representation ρ of K on \mathfrak{p} by $\rho(k) = k|_{\mathfrak{p}}$ for $k \in K$. The differential ρ_* of ρ is an isomorphism of \mathfrak{k} into the Lie algebra of the orthogonal group of \mathfrak{p} and satisfies $(\rho_*X)Y = [X, Y]$ for $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$. Let \mathfrak{a} be a maximal abelian subspace in \mathfrak{p} and S be a hypersphere in \mathfrak{p} centered at the origin. For simplicity we assume that the radius of S is equal to 1. Let a be a regular element of \mathfrak{p} in $S \cap \mathfrak{a}$. Since $\dim \mathfrak{a} = 2$, the orbit $N = \rho(K)a$ of a under $\rho(K)$ is a hypersurface in S . It is known ([3]) that there is an element Z_0 in the center of \mathfrak{k} such that

$$\begin{aligned} (\rho_*Z_0)^2 &= -1, \\ \langle (\rho_*Z_0)X, (\rho_*Z_0)Y \rangle &= \langle X, Y \rangle \quad \text{for } X, Y \in \mathfrak{p}. \end{aligned}$$

Thus we may regard \mathfrak{p} as a complex vector $(n+1)$ -space \mathbf{C}^{n+1} with complex structure $I = \rho_*Z_0$ and Hermitian inner product \langle, \rangle , where $2(n+1) = \dim \mathfrak{p}$. The image $M = \pi(N)$ of N by the canonical projection π of S onto $P_n(\mathbf{C})$ becomes a real hypersurface in $P_n(\mathbf{C})$. The Riemannian metric of S induced from \langle, \rangle will be denoted by g . Then g induces naturally what is called the Fubini-Study metric \tilde{g} on $P_n(\mathbf{C})$ through π (as stated later). Then with respect to \tilde{g} , $P_n(\mathbf{C})$ has constant holomorphic sectional curvature 4. Let C denote the circle group in K generated by Z_0 . Then the group $G = \rho(K)/\rho(C)$ is a compact analytic subgroup of $PU(n+1)$ which acts on M transitively as a transformation group of isometries of M , where as a Riemannian metric of M we take the one induced from \tilde{g} . Conversely every model space, that is, every real hypersurface in $P_n(\mathbf{C})$ being an orbit under an analytic subgroup of $PU(n+1)$ is congruent to a real hypersurface M obtained in this way with respect to the group of isometries of $P_n(\mathbf{C})$ (for the last several results, cf. [6]).

Since G is an analytic subgroup of $PU(n+1)$, all principal curvatures of M is constant ([7], §1) and so let us evaluate the principal curvatures of M at a special point, say $\pi(a)$. Since the tangent space of \mathbf{C}^{n+1} is identified with itself, a vector field on \mathbf{C}^{n+1} can be regarded as a mapping of \mathbf{C}^{n+1} into itself and the complex structure I of \mathbf{C}^{n+1} as a vector field on S . Under such an identification, \tilde{g} is expressed as

$$\tilde{g}(\tilde{U}, \tilde{V}) \circ \pi = g(U, V) - g(U, I)g(V, I)$$

for vector fields U, V on S and vector fields \tilde{U}, \tilde{V} on $P_n(\mathbf{C})$ such that $\pi_*U = \tilde{U}$ and $\pi_*V = \tilde{V}$. When we denote by ∇ (resp. $\tilde{\nabla}$) the Riemannian connection of S (resp. $P_n(\mathbf{C})$) associated to g (resp. \tilde{g}), the following relation between ∇

and $\tilde{\nabla}$ is fundamental :

$$(1.1) \quad \begin{aligned} \tilde{g}(\tilde{\nabla}_{\tilde{W}}\tilde{U}, \tilde{V}) \circ \pi = & g(\nabla_W U, V) - g(\nabla_W U, I)g(V, I) \\ & - g(U, I)g(I(W), V) - g(W, I)g(I(U), V) \end{aligned}$$

for vector fields U, V, W on S and vector fields $\tilde{U}, \tilde{V}, \tilde{W}$ on $P_n(\mathbf{C})$ such that $\pi_*U = \tilde{U}$, $\pi_*V = \tilde{V}$ and $\pi_*W = \tilde{W}$.

Making use of (1.1) we shall find a relation between the second fundamental form of M and N . We denote the tangent space of a manifold L at $x \in L$ by L_x . In general the second fundamental form of a submanifold M' in a Riemannian manifold for a normal vector ν at $x \in M'$ induces the symmetric linear transformation of M'_x , which is called the shape operator of M' for ν . Let b be a unit vector normal to N at a in S . Then $\{a, b\}$ is an orthonormal base of \mathfrak{a} and π_*b is a unit vector normal to M . Let N' denote the hyperplane in N_a orthogonal to the unit vector $I(a) \in N_a$. Let T (resp. \tilde{T}) denote the shape operator of N (resp. M) for b (resp. π_*b). For $X \in \rho_*(\mathfrak{k})$ we denote by X^* (resp. \tilde{X}) the Killing vector field on S (resp. $P_n(\mathbf{C})$) induced by X . In our notation a vector I_a^* is identified with a vector $I(a)$. It was proved in [7, pp. 471-473] that

$$\begin{aligned} g(T(X_a^*), Y_a^*) &= -g(\nabla_b X^*, Y_a^*), \\ \tilde{g}(\tilde{T}(\tilde{X}_{\pi(a)}), \tilde{Y}_{\pi(a)}) &= -\tilde{g}(\tilde{\nabla}_{\pi_*b} \tilde{X}, \tilde{Y}_{\pi(a)}), \end{aligned}$$

and

$$T(I(a)) = -I(b)$$

for $X, Y \in \rho_*(\mathfrak{k})$ such that $X_a^*, Y_a^* \in N'$. These equations, together with (1.1), imply

$$(1.2) \quad g(T(u), v) = \tilde{g}(\tilde{T}(\pi_*u), \pi_*v), \quad T(I(a)) = -I(b)$$

for $u, v \in N'$.

We want to express the eigenvalues of \tilde{T} (they are by definition the principal curvatures of M) in terms of a root system of \mathfrak{u} making use of (1.2). For a linear form α on \mathfrak{a} we put

$$\begin{aligned} \mathfrak{p}_\alpha &= \{X \in \mathfrak{p}; (\text{ad } H)^2 X = -\alpha(H)^2 X \quad \text{for all } H \in \mathfrak{a}\}, \\ \mathfrak{k}_\alpha &= \{X \in \mathfrak{k}; (\text{ad } H)^2 X = -\alpha(H)^2 X \quad \text{for all } H \in \mathfrak{a}\}. \end{aligned}$$

Then $\mathfrak{p}_{-\alpha} = \mathfrak{p}_\alpha$, $\mathfrak{k}_{-\alpha} = \mathfrak{k}_\alpha$, $\mathfrak{p}_0 = \mathfrak{a}$ and \mathfrak{k}_0 is the centralizer of \mathfrak{a} in \mathfrak{k} . Moreover $\text{ad } a$ maps \mathfrak{k}_α onto \mathfrak{p}_α , and \mathfrak{p}_α onto \mathfrak{k}_α isomorphically. A root of \mathfrak{u} with respect to \mathfrak{a} is by definition a linear form α on \mathfrak{a} such that $\mathfrak{k}_\alpha \neq \{0\}$. Select a suitable ordering in the dual space of \mathfrak{a} and denote by Δ the set of positive roots of \mathfrak{u} with respect to \mathfrak{a} . Then we have the orthogonal direct decompositions of \mathfrak{p} and \mathfrak{k} with respect to B :

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \mathcal{A}} \mathfrak{k}_\alpha, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \mathcal{A}} \mathfrak{p}_\alpha.$$

It is known that there exists the subset $\mathcal{A}' = \{\lambda, \mu\}$ of \mathcal{A} consisting of strongly orthogonal roots, that is, none of $\pm\lambda \pm \mu$ is contained in \mathcal{A} and $\dim \mathfrak{p}_\lambda = \dim \mathfrak{p}_\mu = 1$. By [4, Proposition 3.10] an element Z_0 of the center of \mathfrak{k} belongs to $\mathfrak{k}_0 + \mathfrak{k}_\lambda + \mathfrak{k}_\mu$. It follows that the vector space N' is spanned by \mathfrak{p}_α ($\alpha \in \mathcal{A} - \mathcal{A}'$) and $I(b)$. On the other hand, we proved in [7] that for each $\alpha \in \mathcal{A}$, $\kappa_\alpha = -\alpha(b)/\alpha(a)$ is an eigenvalue of T and \mathfrak{p}_α is contained in the eigenspace of T for κ_α . It follows from (1.2) that for each $\alpha \in \mathcal{A} - \mathcal{A}'$, κ_α is an eigenvalue of \hat{T} and $\pi_* \mathfrak{p}_\alpha$ is contained in the eigenspace of \hat{T} for κ_α . Hence $\pi_* I(b)$ is an eigenvector of \hat{T} for certain eigenvalue of \hat{T} , say κ . Then again from (1.2) we see that the set of all eigenvalues of T coincides with $\{\kappa_\alpha : \alpha \in \mathcal{A} - \mathcal{A}'\} \cup \{y; y^2 - \kappa y - 1 = 0\}$, from which we have $\kappa = \kappa_\lambda + \kappa_\mu$. Thus we proved

THEOREM 2. *Let the notation be in above. Then the principal curvatures of the model space M corresponding to (u, θ) are given by*

$$-\alpha(b)/\alpha(a) \quad (\alpha \in \mathcal{A} - \mathcal{A}')$$

and

$$-\lambda(b)/\lambda(a) - \mu(b)/\mu(a) \quad (\mathcal{A}' = \{\lambda, \mu\}).$$

By virtue of Theorem 2 we can read the distinct principal curvatures ξ_1, \dots, ξ_r of M and their multiplicities $m(\xi_1), \dots, m(\xi_r)$ from the table given by S. Araki [1]. It is easily checked that $\kappa_\alpha \neq \kappa_\beta$, $\kappa_\alpha \neq \kappa$ and $m(\kappa_\alpha) = \dim \mathfrak{p}_\alpha$ for all $\alpha, \beta \in \mathcal{A} - \mathcal{A}'$ with $\alpha \neq \beta$, and so $m(\kappa) = 1$. The values ξ_1, \dots, ξ_r depend on a position of a as well as (u, θ) and hence they include a parameter t . We have the following Table

Table

	u	\mathfrak{f}	$\dim M$	r	ξ_i	$m(\xi_i)$
A_1	$\mathfrak{su}(n+1) + \mathfrak{su}(2)$ ($n \geq 2$)	$\mathfrak{su}(n) + u(1)$ $+ \mathfrak{su}(1) + u(1)$	$2n-1$	2	$\xi_1 = \cot t$ $\xi_2 = 2 \cot 2t$	$m(\xi_1) = 2(n-1)$ $m(\xi_2) = 1$
A_2	$\mathfrak{su}(p+1) + \mathfrak{su}(q+1)$ ($p \geq q \geq 2$)	$\mathfrak{su}(p) + u(1)$ $+ \mathfrak{su}(q) + u(1)$	$2(p+q)-3$	3	$\xi_1 = \cot t$ $\xi_2 = -\tan t$ $\xi_3 = 2 \cot 2t$	$m(\xi_1) = 2(p-1)$ $m(\xi_2) = 2(q-1)$ $m(\xi_3) = 1$
B	$\mathfrak{o}(p+2)$ ($p \geq 3$)	$\mathfrak{o}(p) + \mathbf{R}$	$2p-3$	3	$\xi_1 = \cot(t-\pi/4)$ $\xi_2 = -\tan(t-\pi/4)$ $\xi_3 = 2 \cot 2t$	$m(\xi_1) = p-2$ $m(\xi_2) = p-2$ $m(\xi_3) = 1$
C	$\mathfrak{su}(p+2)$ ($p \geq 3$)	$\mathfrak{su}(p) + u(2)$	$4p-3$	5	$\xi_i = \cot(t-\pi i/4)$ ($i=1, 2, 3, 4$) $\xi_5 = 2 \cot 2t$	$m(\xi_i) = 2(p-2)$ ($i=0, 2$) $m(\xi_i) = 2$ ($i=1, 3$) $m(\xi_5) = 1$
D	$\mathfrak{o}(10)$	$\mathfrak{r}(5)$	17	5	$\xi_i = \cot(t-\pi i/4)$ ($i=1, 2, 3, 4$) $\xi_5 = 2 \cot 2t$	$m(\xi_i) = 4$ ($i=1, 2, 3, 4$) $m(\xi_5) = 1$
E	E_6	$\mathfrak{o}(10) + \mathbf{R}$	29	5	$\xi_i = \cot(t-\pi i/4)$ ($i=1, 2, 3, 4$) $\xi_5 = 2 \cot 2t$	$m(\xi_i) = 8$ ($i=0, 2$) $m(\xi_i) = 6$ ($i=1, 3$) $m(\xi_5) = 1$

As an example we work out the type C in detail. This corresponds to the type AIII in the table of [1] (Set p, l in AIII as $2, p+1$ respectively). Then we know that two simple roots α, β of \mathfrak{u} with respect to \mathfrak{a} have the diagram $\begin{array}{c} \circ \\ \longleftarrow \alpha \\ \longrightarrow \beta \\ \circ \end{array}$, and that $m(\alpha)=2, m(\beta)=2(p-2)$ and $m(2\alpha)=1$, where $m(\alpha) = \dim \mathfrak{p}_\alpha$. Since the Weyl group of (\mathfrak{u}, θ) is simply transitive on the set of Weyl chambers in \mathfrak{a} , we have $\Delta = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 2\alpha, 2\alpha + 2\beta\}$. Hence $m(2\alpha + \beta) = 2, m(\alpha + \beta) = 2(p-2)$, and $\Delta' = \{2\alpha, 2\alpha + 2\beta\}$. The values ξ_1, \dots, ξ_5 depend on the angle t between α and β .

The model space $M(A_1)$ of type A_1 is a geodesic hypersphere in $P_n(\mathbb{C})$. In fact, a regular element a of \mathfrak{p} is decomposed into $a = a' + a''$, where $a' \in \mathfrak{su}(n+1) \cap \mathfrak{p}$ and $a'' \in \mathfrak{su}(2) \cap \mathfrak{p}$. Then it is easily seen that the distance between each point of $M(A_1)$ and the point $\pi(a'')$ in $P_n(\mathbb{C})$ is equal to $\cot^{-1}(|a''|/|a'|)$. Thus we saw that every geodesic hypersurface in $P_n(\mathbb{C})$ with constant holomorphic sectional curvature 4 has two constant principal curvatures ξ, η such that the multiplicity of η is equal to one and $\xi^2 - \xi\eta - 1 = 0$.

REMARK 1.1. If we denote by \check{J} the complex structure of $P_n(\mathbb{C})$ induced from I and by ν a normal vector field on an arbitrary model space then $\check{J}(\nu)$ is a direction of principal curvature (that is an eigenvector of \check{T} for the eigenvalue κ) everywhere.

REMARK 1.2. Among model spaces of any type there is a minimal one because the mean curvature $(\xi_1 + \dots + \xi_r)/(2n-1)$ vanishes everywhere for some t .

REMARK 1.3. By a theorem of Tashiro-Tachibana [8] there is no totally umbilical real hypersurface in $P_n(\mathbb{C})$.

§2. Structure equations.

Hereafter let $P_n(\mathbb{C})$ ($n \geq 2$) be a complex projective space with the metric of constant holomorphic sectional curvature $4c$ and M be a connected Riemannian real hypersurface with the induced metric. We denote by $F(M)$ the bundle of orthonormal frames of M . Then $F(M)$ is a principal fibre bundle over M with structure group $O(2n-1)$. An element u of $F(M)$ can be expressed by $u = (p : e_1, \dots, e_{2n-1})$, where p is a point of M and e_1, \dots, e_{2n-1} is an ordered orthonormal base of M_p . We denote by θ_i, θ_{ij} and Θ_{ij} *) the canonical 1-forms, the connection forms and the curvature forms of $F(M)$ respectively. Then they satisfy

*) Hereafter the indices i, j, k, l run from 1 to $2n-1$ and the indices A, B, C, D run from 1 to $2n$.

$$(2.1) \quad d\theta_i + \sum_j \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0,$$

$$(2.2) \quad d\theta_{ij} + \sum_k \theta_{ik} \wedge \theta_{kj} = \Theta_{ij}.$$

We denote by $F(P)$ the bundle of orthonormal frames of $P_n(\mathbf{C})$, and by $\tilde{\theta}_A$, $\tilde{\theta}_{AB}$ and $\tilde{\Theta}_{AB}$ the canonical 1-forms, the connection forms and the curvature forms of $F(P)$ respectively. Then $\tilde{\theta}_A$ and $\tilde{\theta}_{AB}$ satisfy

$$(2.3) \quad d\tilde{\theta}_A + \sum_B \tilde{\theta}_{AB} \wedge \tilde{\theta}_B = 0, \quad \tilde{\theta}_{AB} + \tilde{\theta}_{BA} = 0,$$

and $\tilde{\Theta}_{AB}$ are given by

$$(2.4) \quad \begin{aligned} \tilde{\Theta}_{AB} &= d\tilde{\theta}_{AB} + \sum_C \tilde{\theta}_{AC} \wedge \tilde{\theta}_{CB} \\ &= c\tilde{\theta}_A \wedge \tilde{\theta}_B + c \sum_{C,D} (\tilde{J}_{AC}\tilde{J}_{BD} + \tilde{J}_{AB}\tilde{J}_{CD})\tilde{\theta}_C \wedge \tilde{\theta}_D, \end{aligned}$$

where $\tilde{J} = (\tilde{J}_{AB})$ denotes the complex structure of $P_n(\mathbf{C})$, that is, $\tilde{J}(\tilde{e}_A) = \sum_B \tilde{J}_{BA}\tilde{e}_B$ at $(\tilde{p} : \tilde{e}_1, \dots, \tilde{e}_{2n}) \in F(P)$. Moreover \tilde{J} satisfies

$$(2.5) \quad \sum_C \tilde{J}_{AC}\tilde{J}_{CB} = -\delta_{AB}, \quad \tilde{J}_{AB} + \tilde{J}_{BA} = 0,$$

$$(2.6) \quad d\tilde{J}_{AB} = \sum_C (\tilde{J}_{AC}\tilde{\theta}_{CB} - \tilde{J}_{BC}\tilde{\theta}_{CA}).$$

The equation (2.6) means that \tilde{J} is parallel.

The isometric inclusion mapping ι of M into $P_n(\mathbf{C})$ induces three tensor fields $H = (H_{ij})$, $J = (J_{ij})$ and $f = (f_i)$ on $F(M)$ as follows. For an element $u = (p : e_1, \dots, e_{2n-1}) \in F(M)$ there exists a unique tangent vector e_{2n} of $P_n(\mathbf{C})$ at $\iota(p)$ such that $\tilde{u} = (\iota(p) : \iota_*e_1, \dots, \iota_*e_{2n-1}, \tilde{e}_{2n})$ is an element of $F(P)$ compatible with the orientation of $P_n(\mathbf{C})$ determined by \tilde{J} . This mapping $u \rightarrow \tilde{u}$ of $F(M)$ into $F(P)$ is also denoted by the same letter ι . Then denoting by ι^* the dual mapping of ι_* we have $\theta_i = \iota^*\tilde{\theta}_i$ and $\iota^*\tilde{\theta}_{2n} = 0$, from which we know $\theta_{ij} = \iota^*\tilde{\theta}_{ij}$ and $0 = \iota^*d\tilde{\theta}_{2n} = -\sum_i \iota^*\tilde{\theta}_{2n,i} \wedge \theta_i = 0$. By Cartan's lemma we may write as

$$(2.7) \quad \phi_i \equiv \iota^*\tilde{\theta}_{2n,i} = \sum_j H_{ij}\theta_j, \quad H_{ij} = H_{ji}.$$

The quadratic form $\sum_i \phi_i \theta_i$ is called the *second fundamental form* of M for \tilde{e}_{2n} . Put $J_{ij} = \tilde{J}_{ij} \circ \iota$ and $f_i = \tilde{J}_{2n,i} \circ \iota$. The pair (J, f) is called the *almost contact structure* of M . From (2.2), (2.4) and (2.7) we have the equation of Gauss

$$(2.8) \quad \Theta_{ij} = \phi_i \wedge \phi_j + c\theta_i \wedge \theta_j + c \sum_{k,l} (J_{ik}J_{jl} + J_{ij}J_{kl})\theta_k \wedge \theta_l.$$

From (2.4) and (2.7) we have the equation of Codazzi

$$(2.9) \quad d\phi_i + \sum_j \phi_j \wedge \theta_{ji} = c \sum_{j,k} (f_j J_{ik} + f_i J_{jk})\theta_j \wedge \theta_k.$$

Moreover (J, f) satisfies by (2.5) and (2.6)

$$(2.10) \quad \sum_k J_{ik} J_{kj} = f_i f_j - \delta_{ij}, \quad \sum_j f_j J_{ji} = 0, \\ \sum_i f_i^2 = 1, \quad J_{ij} + J_{ji} = 0,$$

$$(2.11) \quad dJ_{ij} = \sum_k (J_{ik} \theta_{kj} - J_{kj} \theta_{ik}) - f_i \phi_j + f_j \phi_i, \\ df_i = \sum_j (f_j \theta_{ji} - J_{ji} \phi_j).$$

§ 3. Proof of Theorem 1.

Assume that M has two constant principal curvatures ξ and η ($\xi \neq \eta$). Let m be the multiplicity of η . Define the subbundle F' of $F(M)$ by

$$F' = \{u \in F(M); \phi_a = \xi \theta_a, \phi_r = \eta \theta_r \text{ at } u\}^{*})$$

and restrict all differential forms and tensor fields under consideration to F' . Hereafter we shall promise that " $f_a = 0$ " means " $f_a = 0$ for all a on a non-empty open set of F' ", and " $f_a \neq 0$ " means " $f_a \neq 0$ for some a on a nonempty open set of F' ", etc.

LEMMA 3.1.

$$(1) \quad f_a J_{bc} = 0 \quad \text{and} \quad f_r J_{st} = 0. \\ (2) \quad (\eta - \xi) \theta_{ar} = c \sum_b (f_b J_{ar} - f_r J_{ab} + 2f_a J_{br}) \theta_b \\ + c \sum_s (f_s J_{ra} - f_a J_{rs} + 2f_r J_{sa}) \theta_s.$$

PROOF. By (2.1) and (2.9) the exterior derivatives of $\phi_a = \xi \theta_a$ and $\phi_r = \eta \theta_r$ give

$$(3.1) \quad (\xi - \eta) \sum_r \theta_{ar} \wedge \theta_r = c \sum_{j,k} (f_j J_{ak} + f_a J_{jk}) \theta_j \wedge \theta_k,$$

$$(3.2) \quad (\eta - \xi) \sum_a \theta_{ar} \wedge \theta_a = c \sum_{j,k} (f_j J_{rk} + f_r J_{jk}) \theta_j \wedge \theta_k.$$

Taking account of the coefficients of $\theta_b \wedge \theta_c$ in (3.1) we have

$$(3.3) \quad f_b J_{ac} - f_c J_{ab} + 2f_a J_{bc} = 0.$$

Put $c = a$ in (3.3) to get $f_a J_{ba} = 0$. Multiplying (3.3) by f_a therefore we have $f_a J_{bc} = 0$. Similarly we have $f_r J_{st} = 0$ from (3.2). We can prove (2) easily by applying a method of indeterminate coefficients to (3.1) and (3.2). Q. E. D.

LEMMA 3.2. $f_a = 0$ or $f_r = 0$.

PROOF. From (2.10) and (1) of Lemma 3.1 we have

*) Hereafter the indices a, b, c run from 1 to $2n-1-m$ and the indices r, s, t run from $2n-m$ to $2n-1$.

$$0 = \sum_{a,b} f_a J_{ab} J_{br} = \sum_a f_a (-\sum_s J_{as} J_{sr} + f_a f_r) = f_r \sum_a f_a^2$$

since $\sum_a f_a J_{as} = -\sum_r f_r J_{rs} = 0$.

Q. E. D.

Without loss of generality we may assume $f_a = 0$ and so $f_r \neq 0$ by Lemma 3.2. Then (1) of Lemma 3.1 implies $J_{rs} = 0$. By (2.11) and (2) of Lemma 3.1 the derivative of $f_a = 0$ gives

$$(3.4) \quad (\xi^2 - \eta\xi - c)J_{ab} = 0.$$

Similarly the derivative of $J_{rs} = 0$ gives

$$(3.5) \quad (\eta^2 - \xi\eta + 2c)(f_r \delta_{st} - f_s \delta_{rt}) = 0.$$

LEMMA 3.3. $m = 1$.

PROOF. Suppose $m \geq 2$. Then (3.5) implies $\eta^2 - \xi\eta + 2c = 0$. Hence from (3.4) we have $J_{ab} = 0$ since $\xi^2 - \eta\xi - c \neq 0$. Take the exterior derivative of (2) of Lemma 3.1 making use of (2.1), (2.2), (2.8), (2.10) and (2) of Lemma 3.1 itself to obtain

$$\begin{aligned} & (\eta - \xi)(c + \xi\eta)\theta_a \wedge \theta_r + 3c\xi f_r \sum_s f_s \theta_a \wedge \theta_s \\ & + c \sum_{b,s} ((2\eta - 3\xi)J_{ar} J_{bs} + (\eta - 3\xi)J_{as} J_{br})\theta_b \wedge \theta_s \\ & + \frac{2c^2}{\eta - \xi} \sum_{b,s,t} (f_s^2 J_{ar} J_{bt} + f_r f_t J_{as} J_{bs})\theta_b \wedge \theta_t = 0. \end{aligned}$$

Summing up the coefficients of $\theta_a \wedge \theta_r$ in above equation on r and making use of $\sum_r J_{ar}^2 = \sum_r f_r^2 = 1$ and $2c = \eta(\xi - \eta)$ we have

$$\eta^2 + m\xi\eta + (m+3)c = 0,$$

which contradicts $\eta^2 - \xi\eta + 2c = 0$.

Q. E. D.

Let $S^{2n+1}(c)$ denote a $(2n+1)$ -sphere with constant sectional curvature c and π be the canonical projection of $S^{2n+1}(c)$ onto $P_n(\mathbf{C})$.

PROOF OF THEOREM 1. It follows from (1.2) and Lemma 3.3 that the principal curvatures of a hypersurface $N = \pi^{-1}(M)$ in $S^{2n+1}(c)$ are given by ξ and the roots of the equation $y^2 - \eta y - c = 0$. On the other hand, we have $J_{ab} \neq 0$ since $f_a = 0$ and $J_{ar} = 0$. Hence (3.4) implies $\xi^2 - \eta\xi - c = 0$. Thus N has two constant principal curvatures ξ with multiplicity $2n-1$ and $-c/\xi$ with multiplicity 1. By a theorem of E. Cartan [2, p. 180] we see that N is congruent to a product $S^{2n-1}(\xi^2 + c) \times S^1(c(\xi^2 + c)/\xi^2)$ of two spheres, which is exactly an orbit in $S^{2n+1}(c)$ of type A_1 in the Table in §1. By a comment below the Table M is a geodesic hypersphere in $P_n(\mathbf{C})$.

Q. E. D.

REMARK 2.1. The radius of above geodesic hypersphere M is equal to $(|\xi|/\sqrt{c}) \cot^{-1}(|\xi|/\sqrt{c})$.

§ 4. An application of Theorem 1.

Now modifying the condition of Theorem 1 we obtain

THEOREM 3. *If a connected complete real hypersurface M in $P_n(\mathbb{C})$ has two principal curvatures ξ with multiplicity $2n-2$ and η with multiplicity 1, then M is a geodesic hypersphere.*

PROOF. Owing to Theorem 1 it suffices to prove that both ξ and η are constant. We adopt the notation in § 3. Thus the index r stands for $2n-1$. By (2.1) and (2.4) the exterior derivative of $\phi_a = \xi\theta_a$ and $\phi_r = \eta\theta_r$ give

$$(4.1) \quad \sum_b \{ \delta_{ab} d\xi + c \sum_j (f_b J_{aj} + f_a J_{bj}) \theta_j \} \wedge \theta_b \\ + \{ (\eta - \xi) \theta_{ar} + c \sum_j (f_r J_{aj} + f_a J_{rj}) \theta_j \} \wedge \theta_r = 0,$$

$$(4.2) \quad \sum_a \{ (\eta - \xi) \theta_{ar} + c \sum_j (f_r J_{aj} + f_a J_{rj}) \theta_j \} \wedge \theta_a \\ + \{ d\eta + 2c f_r \sum_a J_{ra} \theta_a \} \wedge \theta_r = 0.$$

It follows from Cartan's lemma that $\{ \}$'s in (4.1) and (4.2) can be expressed as

$$(4.3) \quad \delta_{ab} d\xi + c \sum_j (f_b J_{aj} + f_a J_{rj}) \theta_j = \sum_c A_{abc} \theta_c + A_{ab} \theta_r,$$

$$(4.4) \quad (\eta - \xi) \theta_{ar} + c \sum_j (f_r J_{aj} + f_a J_{rj}) \theta_j = \sum_b A_{ab} \theta_b + A_a \theta_r,$$

$$(4.5) \quad d\eta + 2c f_r \sum_a J_{ra} \theta_a = \sum_a A_a \theta_a + A \theta_r,$$

where $A_{abc} = A_{acb} = A_{bac}$. From (4.3) we have $A_{abc} = c(f_b J_{ac} + f_a J_{bc})$ and $A_{ab} = c(f_b J_{ar} + f_a J_{br})$ for $a \neq b$. Hence $0 = A_{abc} - A_{acb} = c(f_b J_{ac} - f_c J_{ab} + 2f_a J_{bc})$ for $a \neq b$. From this we have $f_a J_{bc} = 0$ as in the proof of (1) of Lemma 3.1. Then putting $d\xi = \sum_a \xi_a \theta_a + \xi_r \theta_r$ and $d\eta = \sum_a \eta_a \theta_a + \eta_r \theta_r$ we have from (4.3) and (4.5)

$$A_{aa} = 2c f_a J_{ar} + \xi_r, \quad A_a = -\eta_a,$$

Thus (4.4) was reduced to

$$(4.6) \quad (\eta - \xi) \theta_{ar} = c \sum_b (f_b J_{ar} - f_r J_{ab} + 2f_a J_{br}) \theta_b + \xi_r \theta_a + \eta_a \theta_r.$$

Here we shall divide into two cases.

(1) The case where $f_a \neq 0$. Then we have $f_r = 0$ as in the proof of Lemma 3.2. By (2.11) and (4.6) the derivative of $f_r = 0$ gives $\eta_a = 0$ and

$$(4.7) \quad (\xi^2 - \eta\xi + 2c) J_{ar} + f_a \xi_r = 0.$$

Multiply (4.7) by f_a and sum up on a to obtain $\xi_r = 0$. Thus (4.7) again implies $\xi^2 - \eta\xi + 2c = 0$ since $J_{ar} \neq 0$. The derivative of $\xi^2 - \eta\xi + 2c = 0$ gives $\xi_a = 0$ and $\eta_r = 0$, which shows that both ξ and η are constant.

(2) The case $f_a = 0$. Then since $J_{ab} \neq 0$, the derivative of $f_a = 0$ gives

$\xi_r = 0$, $\eta_a = 0$ and $\xi^2 - \eta\xi - c = 0$ as in the case (1). The derivative of $\xi^2 - \eta\xi - c = 0$ gives $\eta_r = 0$, which implies that η is constant, and hence ξ is also constant. Q. E. D.

Following S. Tachibana and T. Kashiwada [5] we shall call a real hypersurface in $P_n(\mathbb{C})$ totally η -umbilic if $H_{ij} = \alpha\delta_{ij} + \beta f_i f_j$ holds good for some scalar functions α and β . For a matrix $Q = (x\delta_{\lambda\mu} + y_\lambda z_\mu)$ of degree D we see that $\det Q = x^D + (\sum_\lambda y_\lambda z_\lambda)x^{D-1}$ by induction on D and differentiation on x . Thus if M is η -umbilic hypersurface in $P_n(\mathbb{C})$ then each principal curvature κ of M satisfies

$$\begin{aligned} 0 &= \det(H - \kappa\delta_{ij}) = \det((\alpha - \kappa)\delta_{ij} + \beta f_i f_j) \\ &= (\alpha - \kappa)^{2n-2}(\alpha - \kappa + \beta) \end{aligned}$$

since $\sum_i f_i^2 = 1$. So $\kappa = \alpha$ or $\alpha + \beta$. But β does not vanish everywhere. In fact by [8, Theorem 3] the set $F' = \{u \in F(M); \beta = 0 \text{ at } u\}$ contains no non-empty open sets of $F(M)$. On the other hand, Theorem 3 shows that both α and β are constant on $F(M) - F'$ and hence $F' = \emptyset$ by continuity of β . Thus Theorem 3 again implies

COROLLARY 4. *If M is a connected complete totally η -umbilic hypersurface in $P_n(\mathbb{C})$ then M is a geodesic hypersphere.*

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Ryoichi TAKAGI
Department of Mathematics
Faculty of Science
Tokyo University of Education
Otsuka, Bunkyo-ku
Tokyo, Japan