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# A construction of $\beta$-normal sequences

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In this paper we define the normality of sequences in the scale of not necessarily integral $\beta$ and give a construction of $\beta$-normal sequences as a generalization of Champernowne's construction of normal sequences.

Let $\beta>1$ be a fixed real number. Define a transformation $T_\beta$ on the unit interval, which we call $\beta$-transformation, as follows: $T_\beta x=\beta x-[\beta x]$, $0\leqq x<1$, where $[z]$ is the integral part of $z$. Then $T_\beta$ has an invariant probability measure $\mu_\beta$, under which $T_\beta$ is ergodic, such that

$$1-\beta^{-1}<\frac{d\mu_\beta}{dx}=\frac{1}{E_\beta}\sum_{n=0}^{\infty}\frac{c_n(x)}{\beta^n}<(1-\beta^{-1})^{-1},$$

where

$$c_n(x)=\begin{cases}1 & \text{if} \quad x<T^n 1,\\ 0 & \text{if} \quad x\geqq T^n 1,\end{cases}$$

$$T^0 1=1, \qquad T^n 1=T_\beta^{n-1}(\beta-[\beta]),$$

and $E_\beta$ is the normalizing constant (see [2]). Recently the first named author and Y. Takahashi investigated in [1] the $\beta$-transformations as a class of symbolic dynamics and obtained various new results. Our theorem (in this paper) is a byproduct of these results.

Consider the $\beta$-adic expansion of a real number $x$, $0\leqq x<1$, i.e.

$$x=\sum_{n=0}^{\infty}\omega_n(x)\beta^{-n-1}$$

where $\omega_n(x)=[\beta T^n x]$, $n\geqq 0$. Then through the mapping $\pi_\beta(x)=\omega_0(x)\omega_1(x)\cdots$ $\beta$-transformation is isomorphic to a shift on the one-sided product space $A^N$ where $A$ is the state space $\{0,1,\cdots,\beta_0\}$ and $\beta_0$ is the greatest integer less than $\beta$. Of course the measure on $A^N$ is generated by $\pi_\beta\pi_\beta^{-1}$, which we again denote by $\mu_\beta$. Now we define the $\beta$-normality of a sequence in $A^N$.

A sequence $b=b_0b_1b_2\cdots$ in $A^N$ is said to be $\beta$-normal if for any positive integer $k$ and any word $u=u_1u_2\cdots u_k$ of length $k$ we have

$$\lim_{n\to\infty} n^{-1}F_n(u)=\mu_\beta(u)$$

where $F_n(u)=F_n(u, b)$ is the number of indices $i$, $0\leqq i\leqq n-1$, for which $b_i b_{i+1} \cdots b_{i+k-1}=u_1 u_2 \cdots u_k$. Then the following criterion for $\beta$-normality can be obtained easily as a special case of the theorem 6 in [3] (p. 46).

Criterion for $\beta$-normality. Let $b$ be a sequence in $A^N$. Suppose that there exists a constant $C$ depending at most on $\beta$ such that the relation

$$\limsup_{n\to\infty} n^{-1}F_n(u) < C\mu_\beta(u)$$

holds for any word $u$ of any length. Then $b$ is $\beta$-normal.

Construction. $A$ word $u=u_1 u_2 \cdots u_k$ of length $k$ is said to be $\beta$-admissible if there exists a number $x$, $0\leqq x<1$, and an integer $n\geqq 0$ such that $u_1 u_2 \cdots u_k =\omega_n(x)\omega_{n+1}(x) \cdots \omega_{n+k-1}(x)$ where $\omega_j(x)$, $j\geqq 0$ is the $j$-th coordinate of the $\beta$-expansion of $x$. The set of all $\beta$-admissible word of length $k$ will be denoted by $W_k$ and the cardinality of the set by $\mathrm{card}(W_k)$. Let

$$C_k=C_{k,1}C_{k,2} \cdots C_{k,\mathrm{card}(W_k)}$$

be the word of length $k\cdot\mathrm{card}(W_k)$ obtained by aligning all words in $W_k$ lexicographically. Consider the sequence defined by

$$b_\beta=C_1 C_2 \cdots C_k \cdots .$$

THEOREM. *The sequence $b_\beta$ is $\beta$-normal.*

REMARK 1. These arguments show that for $\beta$-normality of the sequence $b_\beta$, the ordering of $\beta$-admissible words of length $k$ in $C_k$ is not substantial and so we may obtain a set of $\beta$-normal sequence having the power of the continuum by making all possible permutation, for each $k\geqq 1$, on all $\beta$-admissible words in $W_k$. If $\beta$ is an integer greater than 1 then the sequence $b_\beta$ becomes the Champernowne sequence. In [4] A. G. Postnikov generalized the Champernowne's construction to the Markovian cases and to the case of continued fraction expansion.

PROOF OF THE THEOREM. For any word $u$ of length $k$ we denote by $\mathrm{card}(W_n(u))$ the number of words in $W_{n+k}$ whose first $k$ digits coincide with $u$. Then we know the following

LEMMA. *For any word $u$ of length $k$*

$$\lim_{n\to\infty} \beta^{-k-n}\,\mathrm{card}(W_n(u))=\frac{R_\beta(u)}{M_\beta(1-\beta^{-1})}$$

*and hence*

$$\lim_{n\to\infty} \beta^{-n}\,\mathrm{card}(W_n)=\frac{1}{M_\beta(1-\beta^{-1})}$$

*where $R_\beta(u)$ is the Lebesgue measure of the interval $\pi_\beta^{-1}u$ and $M_\beta$ is a constant which depends only on $\beta$.*

For the proof of this lemma see [2].

REMARK 2. From this lemma Sh. Ito and Y. Takahashi deduced in [1] several properties of the system $(T_\beta, \mu_\beta)$; for example, the absolute continuity of the invariant measure $\mu_\beta$ with respect to Lebesgue measure, the Bernoulli property and the fact that the metrical entropy of $(T_\beta, \mu_\beta)$ attains the topological entropy.

Let $F(u, c_n)$ be the number of $u$ appearing in $c_n$. Then we have

$$F(u, c_n) \leqq \sum_{j=0}^{n-k} \text{card}\,(W_j)\,\text{card}\,(W_{n-j-k}(u)) + (k-1)\,\text{card}\,(W_n)$$

and so

$$\frac{F(u, c_n)}{n\,\text{card}\,(W_n)}$$

$$\leqq \frac{1}{n-k+1} \sum_{j=0}^{n-k} \frac{\text{card}\,(W_j)}{\beta^j} \cdot \frac{\text{card}\,(W_{n-j-k}(u))}{\beta^{n-j}} \cdot \frac{\beta^n}{\text{card}\,(W_n)} + O\left(\frac{1}{n}\right).$$

From the above lemma we obtain

$$\limsup_{n\to\infty} \frac{F(u, c_n)}{n\,\text{card}\,(W_n)} \leqq \frac{R_\beta(u)}{M_\beta(1-\beta^{-1})}$$

$$\leqq \frac{\beta^{(u)}}{M_\beta(1-\beta^{-1})^2}$$

since $1-\beta^{-1} < d\mu_\beta/dx$.

Put $p_j = \sum_{i=1}^{j} i\cdot\text{card}\,(W_i)$ then

$$F_{p_j}(u) = F_{p_j}(u, b_\beta) = \sum_{i=1}^{j} F(u, c_i) + O(j)\,.$$

Hence we have

$$\limsup_{n\to\infty} \frac{F_{p_j}(u)}{p_j} \leqq \frac{\mu_\beta(u)}{M_\beta(1-\beta^{-1})}\,.$$

But for any $n \geqq 1$ we have

$$n^{-1}F_n(u) \leqq \frac{F_{p_{j+1}}(u)}{p_{j+1}} \cdot \frac{p_{j+1}}{p_j}\,,$$

where $k$ is the integer such that $p_j \leqq n < p_{j+1}$. Therefore we obtain

$$\limsup_{n\to\infty} n^{-1}F_n(u) \leqq \frac{\beta+1}{M_\beta(1+\beta^{-1})^2}\,\mu_\beta(u)\,.$$

The proof of our theorem is thus complete by the criterion.

REMARK 3. Let $A$ be a finite set with discrete topology and let $A^N = \prod_{k=1}^{\infty} A_k$, $A_k = A$ $(k=1, 2, \cdots)$. The shift transformation on the space $A^N$ is defined by the mapping

$$\sigma : (a_1a_2\cdots) \longrightarrow (a_2a_3\cdots)\,, \qquad (a_1a_2\cdots) \in A^N\,.$$

$A$ subshift is the pair $(X, \sigma)$ where $X$ is a closed, with respect to the product topology, $\sigma$-invariant subset of $A^N$. Let $W_k=W_k(X)$ be the set of all words of length $k$ appeared in $X$. Denote by

$$c_k=c_k(X)=c_{k,1}c_{k,2}\cdots c_{k,\mathrm{card}(W_k)}$$

the word of length $k\cdot\mathrm{card}(W_k)$ obtained by aligning all words in $W_k$ lexicographically and define the sequence

$$b(X)=c_1c_2\cdots c_k\cdots$$

as an analogue of the Champernowne sequence. If the orbit $\{\sigma^n b(X)\,;\,n=0,1,\cdots\}$ has such 'special uniformity' as is mentioned in Lemma, the sequence $b(X)$ is normal with respect to some $\sigma$-invariant measure. (The definition of the normality of a sequence with respect to an arbitrary measure on $X$ can be found in [3].) In general we may conjecture that the sequence $b(X)$ is normal with respect to the corresponding $\sigma$-invariant measure $\mu$ on $X$ (if it is unique) and moreover, the metrical entropy of the system $(X, \sigma, \mu)$ attains the topological entropy. This is the case for Markov subshifts (see [1]) and also for $\beta$-transformations as we have already shown though they are not necessarily Markov.

## References


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