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Curvature and critical Riemannian metric

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Let M be a compact orientable C^∞ manifold and g a C^∞ Riemannian metric on M satisfying

$$\int_M dV=1 \tag{0.1}$$

where dV is the volume element of M measured by g. We denote the set of all such metrics by $\mathcal{M}(M)$ or $\mathcal{M}$. When g is fixed we have a Riemannian manifold (M, g).

Let us take a covering $\{U\}$ of M by coordinate neighborhoods and denote the local coordinates in U by x^h. In each neighborhood U we use the natural frame. Then the components of the curvature tensor of (M, g) in U is given by

$$K_{kji}{}^h=\partial_k\left\{ {h \atop ji} \right\}-\partial_j\left\{ {h \atop ki} \right\}+\left\{ {h \atop kp} \right\}\left\{ {p \atop ji} \right\}-\left\{ {h \atop jp} \right\}\left\{ {p \atop ki} \right\}$$

where $\left\{ {h \atop ji} \right\}$ are the Christoffel symbols derived from the components g_{ji} of g, Latin indices run the range $\{1, \cdots, n\}$, and the summation convention is adopted. The Ricci tensor and the scalar curvature are given respectively by

$$K_{ji}=K_{pji}{}^p, \qquad K=g^{ji}K_{ji}$$

where g^{ji} are defined by $g_{ik}g^{kh}=\delta_i^h$. Similarly all tensors will be expressed in terms of their components.

In a Riemannian manifold (M, g) indices can be raised and lowered by g^{ji} and g_{ji} so that for example $K^{kjih}=K_{dcb}{}^h g^{dk}g^{cj}g^{bi}$ are the contravariant components of the curvature tensor. Thus $K_{kjih}K^{kjih}$ is a scalar. Considering this at each point of M we get a scalar field.

Now let us consider the integral

$$J[g]=\int_M K_g dV_g ,$$

where we write K_g and dV_g for K and dV respectively in order to emphasize that these depend on the metric g. $J[g]$ is the image of g by a map $J: \mathcal{M}(M)\rightarrow \boldsymbol{R}$. Critical points of this map are known as Einstein metrics.

M. Berger studied the second derivative of $J[g(t)]$ for curves $g(t)$ of $\mathcal{M}(M)$ and showed that it is not true that the index of $J[g]$ and also the index of $-J[g]$ are finite for every critical point [**1**]. Recently the present author proved that the index of $J[g]$ and the index of $-J[g]$ are both positive at each critical point [**6**]. This result diminishes our interest in J to a certain extent.

Let us consider the integral

$$I[g]=\int_M K_{kjih}K^{kjih}dV_g\,.$$

Then I is also a mapping $I:\mathcal{M}(M)\to\boldsymbol{R}$. This integral has a remarkable property that $I[g]$ is non-negative and moreover that, if M does not admit a locally flat metric, then $I[g]$ is strictly positive.

If η is a diffeomorphism of M and η^* its pull back, we have $I[g]=I[\eta^*(g)]$. Hence we can deduce a mapping $\tilde{I}:\mathcal{M}/\mathcal{D}\to\boldsymbol{R}$ from the mapping $I:\mathcal{M}\to\boldsymbol{R}$ where $\mathcal{D}$ is the diffeomorphism group of M and $\mathcal{M}/\mathcal{D}$ is the space of orbits generated by $\mathcal{D}$ of Riemannian metrics [**2**].

If $\bar{g}$ is a critical point of I, then the orbit of $\bar{g}$ by $\mathcal{D}$ is a critical point of $\tilde{I}$ and vice versa. In this case let us say that $\bar{g}$ is a critical point of $\tilde{I}$ for convenience sake. This convention is useful since there can exist no local minimum, in the strict sense, of the mapping I but $\tilde{I}$ may possibly have a local minimum and the present paper concerns this.

Some years ago M. Berger obtained differential equations of g for the critical point, namely the critical Riemannian metric, of the mapping I [**1**]. In the present paper it is shown that a metric of constant curvature is a critical Riemannian metric. Moreover, the second derivative of $I[g(t)]$ at a critical point is calculated and the following Main Theorem is obtained.

THEOREM. *If M is diffeomorphic to S^n and $\bar{g}$ is a metric of positive constant curvature on M, then the index of I and also of $\tilde{I}$ at $\bar{g}$ is zero and $\tilde{I}$ has a local minimum at $\bar{g}$.*

§ 1. Critical Riemannian metric.

Here and in the sequel we write

$$I[g]=\int K_{kjih}K^{kjih}dV\,. \tag{1.1}$$

We have dropped M and g in this formula. A C^∞ curve in $\mathcal{M}$ will be represented locally by $g_{ji}(x^1,\cdots,x^n;\,t)$ and we define a tensor field D_{ji} on $(M,g(t))$ by

$$D_{ji}(x,t)=\frac{\partial g_{ji}(x,t)}{\partial t}\,. \tag{1.2}$$

This symmetric (0, 2)-tensor satisfies

$$\int D_p{}^p dV = 0 \tag{1.3}$$

because of (0.1).

The curvature tensor $K_{kji}{}^h$ changes with g and we get [**6**]

$$\frac{\partial}{\partial t} K_{kji}{}^h = \nabla_k D_{ji}{}^h - \nabla_j D_{ki}{}^h \tag{1.4}$$

where the tensor $D_{ji}{}^h$ is defined by

$$D_{ji}{}^h = \frac{\partial}{\partial t}\left\{ {h \atop ji} \right\}$$

and satisfies

$$D_{ji}{}^h = \frac{1}{2}(\nabla_j D_i{}^h + \nabla_i D_j{}^h - \nabla^h D_{ji}) \tag{1.5}$$

and ∇ means covariant differentiation with respect to the metric tensor $g(t)$.

As we have

$$\frac{\partial}{\partial t} g^{ih} = -g^{ki} g^{jh} \frac{\partial}{\partial t} g_{kj} = -D^{ih},$$

we get

$$\begin{aligned}\frac{\partial}{\partial t}(K_{kjih}K^{kjih}) &= \frac{\partial}{\partial t}(K_{kji}{}^h K_{dcb}{}^a g^{kd} g^{jc} g^{ib} g_{ha}) \\ &= 2(\nabla_k D_{ji}{}^h - \nabla_j D_{ki}{}^h) K^{kji}{}_h \\ &\quad + K_{kji}{}^h K_{dcb}{}^a(-D^{kd} g^{jc} g^{ib} g_{ha} - g^{kd} D^{jc} g^{ib} g_{ha} \\ &\quad - g^{kd} g^{jc} D^{ib} g_{ha} + g^{kd} g^{jc} g^{ib} D_{ha}).\end{aligned}$$

Substituting (1.5) into this formula and taking some property of the curvature tensor into account, we can deduce

$$\frac{\partial}{\partial t}(K_{kjih}K^{kjih}) = 4K^{kji}{}_h \nabla_k \nabla_i D_j{}^h - 2K_{kji}{}^b K^{kjia} D_{ba}. \tag{1.6}$$

Now, from (1.1) we get

$$\frac{d}{dt} I[g(t)] = \int \left[\frac{\partial}{\partial t}(K_{kjih}K^{kjih}) + \frac{1}{2} K_{kjih}K^{kjih} g^{qp} D_{qp} \right] dV.$$

Substituting (1.6) into this and applying Green's theorem, we get

$$\begin{aligned}\frac{d}{dt} I[g(t)] = \int \Big[& 4(\nabla_i \nabla_k K^{kji}{}_h) D_j{}^h \\ & - 2K_{kji}{}^q K^{kjip} D_{qp} + \frac{1}{2} K_{kjih}K^{kjih} D_p{}^p \Big] dV.\end{aligned}$$

Then, applying Ricci's identity and Bianchi's identity, we get

(1.7) $$\frac{d}{dt}I[g(t)]=\int\Big[2\nabla^j\nabla^i K-4\nabla_p\nabla^p K^{ji}+4K^j{}_pK^{pi}$$
$$-4K^j{}_{qp}{}^iK^{qp}-2K^{srqj}K_{srq}{}^i+\frac{1}{2}K_{dcba}K^{dcba}g^{ji}\Big]D_{ji}dV\,.$$

A point $\bar{g}$ is a critical point of I if and only if (1.7) vanishes at $t=0$ for all curves $g(t)$ such that $g(0)=\bar{g}$. Let us consider the integral which is obtained from the right-hand member of (1.7) by replacing $g(t)$ with $\bar{g}$. Then we can say that $\bar{g}$ is a critical point of I if and only if this integral vanishes for all tensor fields D_{ji} satisfying (1.3) in which g is replaced with $\bar{g}$. Thus we see that $\bar{g}$ is a critical point of I if and only if

(1.8) $$2\nabla^j\nabla^i K-4\nabla_p\nabla^p K^{ji}+4K^j{}_pK^{pi}-4K^j{}_{qp}{}^iK^{qp}$$
$$-2K^{srqj}K_{srq}{}^i+\frac{1}{2}K_{dcba}K^{dcba}g^{ji}=cg^{ji}$$

is satisfied by $g=\bar{g}$ and some constant c. Precisely, c is obtained by transvecting with g_{ji}, hence

(1.9) $$c=-\frac{2}{n}\nabla_p\nabla^p K+\Big(\frac{1}{2}-\frac{2}{n}\Big)K_{dcba}K^{dcba}\,.$$

Thus we get the following theorem [1].

THEOREM 1.1. *Let M be a compact orientable C^∞ manifold and $\mathcal{M}$ be the space of Riemannian metrics on M satisfying* (0.1). *Then a necessary and sufficient condition for a Riemannian metric g to be a critical point of the functional* (1.1) *is that* (1.8) *is satisfied by this metric g where c is a constant in M. At that time c is given by* (1.9).

EXAMPLE. If M is diffeomorphic to S^n, M admits a Riemannian metric of positive constant curvature. Then

(1.10) $$K_{kjih}=\frac{K}{n(n-1)}(g_{kh}g_{ji}-g_{jh}g_{ki})$$

satisfies (1.8) with

$$c=\Big(\frac{1}{2}-\frac{2}{n}\Big)\frac{2K^2}{n(n-1)}\,.$$

Hence this metric is a critical point of the functional (1.1) with the critical value

$$\frac{2K^2}{n(n-1)}\,.$$

Now, the purpose of the present paper is to prove the following theorem which is equivalent with the Main Theorem stated in the introduction.

THEOREM 1.2. *Let M be a C^∞ manifold diffeomorphic to S^n. Then the mapping $\tilde{I}:\mathcal{M}/\mathcal{D}\to\boldsymbol{R}$ given by the functional* (1.1) *has a local minimum at the Riemannian metric $\bar{g}$ of positive constant curvature* (1.10).

§ 2. The second derivative of $I[g(t)]$.

In order to prove the main theorem let us first calculate the second derivative $I''[g(t)]$ for an arbitrary curve $g(t)$ of $\mathcal{M}$.

If we define W^{ji} by

$$W^{ji}=2\nabla^j\nabla^i K-4\nabla_p\nabla^p K^{ji}+4K^j{}_pK^{pi}-4K^j{}_{qp}{}^iK^{qp} -2K^{srqj}K_{srq}{}^i+\frac{1}{2}K_{dcba}K^{dcba}g^{ji}, \tag{2.1}$$

we get

$$I''[g(t)]=\int\Big[\frac{\partial W^{ji}}{\partial t}\frac{\partial g_{ji}}{\partial t}+W^{ji}\Big(\frac{\partial^2 g_{ji}}{\partial t^2}+\frac{1}{2}\frac{\partial g_{ji}}{\partial t}g^{qp}\frac{\partial g_{qp}}{\partial t}\Big)\Big]dV \tag{2.2}$$

where

$$\frac{\partial W^{ji}}{\partial t}=2\frac{\partial}{\partial t}\nabla^j\nabla^i K-4\frac{\partial}{\partial t}\nabla_p\nabla^p K^{ji} +4\frac{\partial}{\partial t}(K^j{}_pK^{pi})-4\frac{\partial}{\partial t}(K^j{}_{qp}{}^iK^{qp})-2\frac{\partial}{\partial t}(K^{srqj}K_{srq}{}^i) +\frac{1}{2}\frac{\partial}{\partial t}(K_{dcba}K^{dcba}g^{ji}). \tag{2.3}$$

From the definition of $D_{ji}{}^h$ we have the following identity which is valid for any C^∞ tensor field depending on t differentiably, for example $T_i{}^h$,

$$\frac{\partial}{\partial t}\nabla_j T_i{}^h=\nabla_j\frac{\partial}{\partial t}T_i{}^h-D_{ji}{}^pT_p{}^h+D_{jp}{}^hT_i{}^p. \tag{2.4}$$

Applying this identity to the first and the second terms in the right-hand member of (2.3), we get

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^j\nabla^i K&=\frac{\partial}{\partial t}(g^{jq}g^{ip}\nabla_q\nabla_p K)\\ &=-D^{jp}\nabla_p\nabla^i K-D^{ip}\nabla^j\nabla_p K\\ &\quad+\nabla^j\nabla^i\frac{\partial}{\partial t}K-D^{jip}\nabla_p K,\end{aligned} \tag{2.5}$$

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_p\nabla^p K^{ji}&=\frac{\partial}{\partial t}(g^{qp}\nabla_q\nabla_p K^{ji})\\ &=-D^{qp}\nabla_q\nabla_p K^{ji}\\ &\quad+g^{qp}(-D_{qp}{}^r\nabla_r K^{ji}+D_{qr}{}^j\nabla_p K^{ri}+D_{qr}{}^i\nabla_p K^{jr})\\ &\quad+g^{qp}\nabla_q\Big(\nabla_p\frac{\partial}{\partial t}K^{ji}+D_{pr}{}^jK^{ri}+D_{pr}{}^iK^{jr}\Big).\end{aligned} \tag{2.6}$$

From (1.4) and (1.5) we get

(2.7) $$\frac{\partial}{\partial t}K^{ji}=-D^{jp}K_p{}^i-D^{ip}K_p{}^j+\frac{1}{2}(\nabla_p\nabla^j D^{ip}+\nabla_p\nabla^i D^{jp}-\nabla_p\nabla^p D^{ji}-\nabla^j\nabla^i D_p{}^p),$$

(2.8) $$\frac{\partial}{\partial t}K=-D^{qp}K_{qp}+\nabla_q\nabla_p D^{qp}-\nabla_q\nabla^q D_p{}^p,$$

(2.9) $$\frac{\partial}{\partial t}(K^j{}_pK^{pi})=-D^{jq}K_{qp}K^{pi}-D^{iq}K_{qp}K^{pj}-K^{jq}D_{qp}K^{pi}+\frac{1}{2}K^{qi}(\nabla_p\nabla^j D_q{}^p+\nabla_p\nabla_q D^{jp}-\nabla_p\nabla^p D_q{}^j-\nabla_q\nabla^j D_p{}^p)+\frac{1}{2}K^{qj}(\nabla_p\nabla^i D_q{}^p+\nabla_p\nabla_q D^{ip}-\nabla_p\nabla^p D_q{}^i-\nabla_q\nabla^i D_p{}^p),$$

(2.10) $$\frac{\partial}{\partial t}(K^j{}_{qp}{}^iK^{qp})=-K_{rqp}{}^iK^{qp}D^{rj}-K^j{}_{qp}{}^i(D^{qr}K_r{}^p+D^{pr}K_r{}^q)+\frac{1}{2}K^{qp}(\nabla^j\nabla_q D_p{}^i+\nabla^j\nabla_p D_q{}^i-\nabla^j\nabla^i D_{qp}-\nabla_q\nabla^j D_p{}^i-\nabla_q\nabla_p D^{ji}+\nabla_q\nabla^i D_p{}^j)+\frac{1}{2}K^j{}_{qp}{}^i(\nabla_r\nabla^q D^{pr}+\nabla_r\nabla^p D^{qr}-\nabla_r\nabla^r D^{qp}-\nabla^q\nabla^p D_r{}^r),$$

(2.11) $$\frac{\partial}{\partial t}(K^{srqj}K_{srq}{}^i)=(\nabla_c\nabla_b D_a{}^j+\nabla_c\nabla_a D_b{}^j-\nabla_c\nabla^j D_{ba})K^{cbai}+(\nabla_c\nabla_b D_a{}^i+\nabla_c\nabla_a D_b{}^i-\nabla_c\nabla^i D_{ba})K^{cbaj}-K^{bsrj}K^a{}_{sr}{}^iD_{ba}-K^{sbrj}K_s{}^a{}_r{}^iD_{ba}-K^{srbj}K_{sr}{}^{ai}D_{ba},$$

(2.12) $$\frac{\partial}{\partial t}(K_{dcba}K^{dcba}g^{ji})=(4K^{dcba}\nabla_d\nabla_b D_{ca}-2K_{dcb}{}^qK^{dcbp}D_{qp})g^{ji}-K_{dcba}K^{dcba}D^{ji}.$$

Applying (1.5), (2.7) and (2.8) we get from (2.5) and (2.6)[1]

(2.13) $$\frac{\partial}{\partial t}\nabla^j\nabla^i K=-D^{jq}\nabla_q\nabla^i K-D^{iq}\nabla_q\nabla^j K+\nabla^j\nabla^i(-D^{qp}K_{qp}+\nabla_q\nabla_p D^{qp}-\nabla_q\nabla^q D_p{}^p)-\frac{1}{2}(\nabla^j D^{iq}+\nabla^i D^{jq}-\nabla^q D^{ji})\nabla_q K,$$

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1) In this paper ∇AB always means $(\nabla A)B$. Thus we have $\nabla(AB)=\nabla AB+A\nabla B$.

(2.14)
$$\frac{\partial}{\partial t}\nabla_q\nabla^q K^{ji}$$
$$=-D^{qp}\nabla_q\nabla_p K^{ji}-\nabla_q D^{qr}\nabla_r K^{ji}+\frac{1}{2}\nabla^r D_q{}^q\nabla_r K^{ji}$$
$$+(\nabla_q D_r{}^j+\nabla_r D_q{}^j-\nabla^j D_{qr})\nabla^q K^{ri}$$
$$+(\nabla_q D_r{}^i+\nabla_r D_q{}^i-\nabla^i D_{qr})\nabla^q K^{rj}$$
$$+\frac{1}{2}\nabla^p(\nabla_p D_r{}^j+\nabla_r D_p{}^j-\nabla^j D_{pr})K^{ri}$$
$$+\frac{1}{2}\nabla^p(\nabla_p D_r{}^i+\nabla_r D_p{}^i-\nabla^i D_{pr})K^{rj}$$
$$-\nabla_q\nabla^q\Big(D^{js}K_s{}^i+D^{is}K_s{}^j-\frac{1}{2}\nabla_s\nabla^j D^{is}-\frac{1}{2}\nabla_s\nabla^i D^{js}$$
$$+\frac{1}{2}\nabla_s\nabla^s D^{ji}+\frac{1}{2}\nabla^j\nabla^i D_s{}^s\Big).$$

Let us define F by

(2.15)
$$F=\frac{\partial W^{ji}}{\partial t}D_{ji}$$

and the notation $\cong$ by $A\cong A+$divergence. Then we get

(2.16)
$$F\cong D_{ji}\Big[-4D^{jq}\nabla_q\nabla^i K$$
$$+2\nabla^j\nabla^i(-D^{qp}K_{qp}+\nabla_q\nabla_p D^{qp}-\nabla_q\nabla^q D_p{}^p)$$
$$-2\nabla^j D^{iq}\nabla_q K+\nabla^q D^{ji}\nabla_q K+4D^{qp}\nabla_q\nabla_p K^{ji}$$
$$+4\nabla_q D^{qp}\nabla_p K^{ji}-2\nabla^q D_p{}^p\nabla_q K^{ji}$$
$$-8(\nabla_q D_p{}^j+\nabla_p D_q{}^j-\nabla^j D_{qp})\nabla^q K^{pi}$$
$$-4\nabla^q(\nabla_q D_p{}^j+\nabla_p D_q{}^j-\nabla^j D_{qp})K^{pi}$$
$$+2\nabla_q\nabla^q(4D^{jp}K_p{}^i-2\nabla_p\nabla^j D^{ip}+\nabla_p\nabla^p D^{ji}+\nabla^j\nabla^i D_p{}^p)$$
$$-8D^{jq}K_{qp}K^{pi}-4K^{jq}D_{qp}K^{pi}$$
$$+4K^{jq}(\nabla_p\nabla^i D_q{}^p+\nabla_p\nabla_q D^{ip}-\nabla_p\nabla^p D_q{}^i-\nabla_q\nabla^i D_p{}^p)$$
$$+4K_{rqp}{}^i K^{qp}D^{rj}+4K^j{}_{qp}{}^i(D^{qr}K_r{}^p+D^{pr}K_r{}^q)$$
$$-2K^{qp}(\nabla^j\nabla_q D_p{}^i+\nabla^j\nabla_p D_q{}^i-\nabla^j\nabla^i D_{qp}$$
$$-\nabla_q\nabla^j D_p{}^i-\nabla_q\nabla_p D^{ji}+\nabla_q\nabla^i D_p{}^j)$$
$$-2K^j{}_{qp}{}^i(\nabla_r\nabla^q D^{pr}+\nabla_r\nabla^p D^{qr}-\nabla_r\nabla^r D^{qp}-\nabla^q\nabla^p D_r{}^r)$$
$$-4K^{cbaj}(\nabla_c\nabla_b D_a{}^i+\nabla_c\nabla_a D_b{}^i-\nabla_c\nabla^i D_{ba})$$
$$+4K^{bqpj}K^a{}_{qp}{}^i D_{ba}+2K^{qpbj}K_{qp}{}^{ai}D_{ba}$$
$$+(2K^{dcba}\nabla_d\nabla_b D_{ca}-K_{dcb}{}^q K^{dcbp}D_{qp})g^{ji}$$
$$-\frac{1}{2}K_{dcba}K^{dcba}D^{ji}\Big].$$

This formula is valid for any value of t.

We now want to evaluate the integral (2.2) at $t=0$, namely, at a critical point.

As we have $W^{ji}=cg^{ji}$ at $t=0$, we get

$$I''[g(0)]=\int\Big[F+cg^{ji}\Big(\frac{\partial^2 g_{ji}}{\partial t^2}+\frac{1}{2}\frac{\partial g_{ji}}{\partial t}g^{qp}\frac{\partial g_{qp}}{\partial t}\Big)\Big]_0 dV$$

where $[\ ]_0$ means that we have put $t=0$.

From (0.1) we get

$$\int\Big[g^{ji}\frac{\partial^2 g_{ji}}{\partial t^2}-D^{ji}D_{ji}+\frac{1}{2}(D_p{}^p)^2\Big]dV=0\,.$$

Hence we can deduce

$$I''[g(0)]=\int G\,dV \tag{2.17}$$

where G is given by

$$G\cong\Big[F+\Big\{-\frac{2}{n}\nabla_q\nabla^q K+\Big(\frac{1}{2}-\frac{2}{n}\Big)K_{dcba}K^{dcba}\Big\}D_{ji}D^{ji}\Big]_0\,, \tag{2.18}$$

for we have (1.9).

§ 3. The integrand of $I''[g(0)]$.

According to M. Berger and D. Ebin [1], [2] we can deduce properties of the Hessian of $I[g]$ at a critical point $\bar{g}$ if we study $I''[g(0)]$ for all curves $g(t)$ such that $g(0)=\bar{g}$ and such that the tensor field $D_{ji}=[\partial g_{ji}/\partial t]_0$ satisfies $\nabla^j D_{ji}=0$.

If D_{ji} satisfies

$$\nabla^j D_{ji}=0\,, \tag{3.1}$$

we get

$$\nabla_q\nabla^j D^{qi}=K^j{}_q D^{qi}+K_q{}^j{}_p{}^i D^{qp} \tag{3.2}$$

and $D_{ji}\nabla^j T^i\cong 0$ for any vector field T^h on M. Applying such results we get after some straightforward calculation

$$\begin{aligned}G\cong&-4D_{ji}D^{jq}\nabla_q\nabla^i K-2D_{ji}\nabla^j D^{iq}\nabla_q K\\&+D_{ji}\nabla^q D^{ji}\nabla_q K+4D_{ji}D^{qp}\nabla_q\nabla_p K^{ji}\\&-2\nabla^q D_p{}^p D_{ji}\nabla_q K^{ji}\\&-8D_{ji}(\nabla_q D_p{}^j+\nabla_p D_q{}^j-\nabla^j D_{qp})\nabla^q K^{pi}\\&-4D_{ji}(\nabla^q\nabla_q D_p{}^j+K_p{}^q D_q{}^j-K^{jq}D_{qp})K^{pi}\end{aligned} \tag{3.3}$$

$$+2\nabla_q\nabla^q D_{ji}(4D^{jp}K_p{}^i-2K_r{}^j{}_p{}^i D^{rp}-2K_p{}^j D^{ip}+\nabla^j\nabla^i D_p{}^p+\nabla_p\nabla^p D^{ji})$$
$$-8D_{ji}D^{jq}K_{qp}K^{pi}$$
$$+4D_{ji}K^{jq}(K_{qp}D^{ip}-\nabla_p\nabla^p D_q{}^i-\nabla_q\nabla^i D_p{}^p)$$
$$+4K_{rqp}{}^i K^{qp}D^{rj}D_{ji}$$
$$-2K^{qp}(2D_{ji}\nabla^j\nabla_q D_p{}^i-D_{ji}\nabla^j\nabla^i D_{qp}-D_{ji}\nabla_q\nabla_p D^{ji})$$
$$-2D_{ji}K^j{}_{qp}{}^i\{2(K_r{}^q{}_a{}^p D^{ar}+K_r{}^q D^{pr})-\nabla_r\nabla^r D^{qp}-\nabla^q\nabla^p D_r{}^r\}$$
$$+2D_{ji}K^{cbaj}K_{cba}{}^p D_p{}^i$$
$$-4D_{ji}K^{cbaj}(\nabla_c\nabla_a D_b{}^i-\nabla_c\nabla^i D_{ba})$$
$$+4K^{bqpj}K^a{}_{qp}{}^i D_{ba}D_{ji}$$
$$+2K^{dcba}\nabla_d\nabla_b D_{ca}D_p{}^p$$
$$-K_{dcb}{}^q K^{dcbp}D_{qp}D_r{}^r$$
$$+\Big(-\frac{2}{n}\nabla_p\nabla^p K-\frac{2}{n}K_{dcba}K^{dcba}\Big)D_{ji}D^{ji}$$

at $t=0$, that is, at $\bar{g}=g(0)$.

§ 4. $I''[g(0)]$ when $g(0)$ is the metric of constant curvature.

If $g(0)$ is a metric of constant curvature, we have

$$K_{kjih}=\frac{K}{n(n-1)}(g_{kh}g_{ji}-g_{jh}g_{ki}),$$
$$K^{dcbj}K_{dcb}{}^i=\frac{2K^2}{n^2(n-1)}g^{ji},$$
$$K^{kqpj}K^i{}_{qp}{}^h=\frac{K^2}{n^2(n-1)^2}\{(n-2)g^{kj}g^{ih}+g^{ki}g^{jh}\}$$

and $\nabla_k K^{ji}=0$, $\nabla_k K=0$.

Hence we get from (3.3)

$$G\cong 2\nabla_q\nabla^q D_{ji}\Big\{\nabla^j\nabla^i D_p{}^p+\nabla_p\nabla^p D^{ji}+\frac{2K}{n(n-1)}(g^{ji}D_p{}^p-D^{ji})\Big\}$$
$$-\frac{2K}{n}D_{ji}\nabla_q\nabla^q D^{ji}$$
$$-\frac{2K}{n(n-1)}(D_q{}^q g_{ji}-D_{ji})\Big\{\frac{2K}{n}D^{ji}-\frac{2K}{n(n-1)}(D_p{}^p g^{ji}-D^{ji})$$
$$-\nabla_p\nabla^p D^{ji}-\nabla^j\nabla^i D_p{}^p\Big\}$$
$$+\frac{4K^2}{n^2(n-1)}D_{ji}D^{ji}$$

$$-\frac{4K}{n(n-1)}D_{ji}(\nabla_q\nabla^j D^{iq}-\nabla_q\nabla^q D^{ji})$$
$$+\frac{4K^2}{n^2(n-1)^2}\{(n-2)D_{ji}D^{ji}+(D_p{}^p)^2\}$$
$$-\frac{2K}{n(n-1)}D_q{}^q\nabla_r\nabla^r D_p{}^p-\frac{2K^2}{n^2(n-1)}(D_p{}^p)^2$$
$$-\frac{4K^2}{n^2(n-1)}D_{ji}D^{ji}$$

since such terms as $D_{ji}\nabla^j\nabla^i D_p{}^p$ are divergences because of (3.1).

We use Ricci's identities and identities such as (3.2) and get

$$2\nabla_q\nabla^q D_{ji}\nabla^j\nabla^i D_p{}^p \cong -2\nabla^j\nabla_q\nabla^q D_{ji}\nabla^i D_p{}^p$$
$$\cong -\frac{4K}{n(n-1)}D_p{}^p\nabla_q\nabla^q D_r{}^r$$

by virtue of (3.1). We also get

$$D_{ji}\nabla_p\nabla^j D^{ip}\cong\frac{K}{n(n-1)}\{D_{ji}D^{ji}-(D_p{}^p)^2\}+\frac{K}{n}D_{ji}D^{ji}$$

and finally

$$\begin{aligned}G\cong\ &2\nabla_q\nabla^q D_{ji}\nabla_p\nabla^p D^{ji}\\&-\frac{2K}{n-1}\left(D_{ji}\nabla_p\nabla^p D^{ji}-\frac{1}{n}D_p{}^p\nabla_q\nabla^q D_r{}^r\right)\\&+\frac{4(n-2)}{n^2(n-1)^2}K^2D_{ji}D^{ji}\\&-\frac{2(n-3)}{n^2(n-1)^2}K^2(D_p{}^p)^2.\end{aligned}\tag{4.1}$$

§5. Proof of the main theorem.

If we put

$$D_{ji}=H_{ji}+\frac{H}{n}g_{ji},\qquad H=D_p{}^p,\tag{5.1}$$

we get $g^{ji}H_{ji}=0$. Let us define G_1, G_2 by

$$G_1=2\left[\nabla_q\nabla^q H_{ji}\nabla_p\nabla^p H^{ji}-\frac{K}{n-1}H_{ji}\nabla_p\nabla^p H^{ji}+\frac{2(n-2)}{n^2(n-1)^2}K^2H_{ji}H^{ji}\right],\tag{5.2}$$

$$G_2=\frac{2}{n}\left[\nabla_q\nabla^q H\nabla_p\nabla^p H-\frac{n-4}{n^2(n-1)}K^2H^2\right].\tag{5.3}$$

Then we have $G\cong G_1+G_2$ and

$$I''[g(0)]=\int G_1\,dV+\int G_2\,dV.\tag{5.4}$$

Writing $\Delta f=-\nabla_p\nabla^p f$ we get

$$\int(\Delta f)^2 dV=\int f\nabla_q\nabla^q\nabla^p\nabla_p f\, dV$$

$$=\int f\nabla_q(\nabla^p\nabla^q\nabla_p f-K_p{}^q\nabla^p f)\,dV$$

$$=\int(\nabla^q\nabla^p f)(\nabla_q\nabla_p f)\,dV+\frac{K}{n}\int f\Delta f\, dV.$$

Since we have

$$\Big(\nabla^q\nabla^p f+\frac{1}{n}g^{qp}\Delta f\Big)\Big(\nabla_q\nabla_p f+\frac{1}{n}g_{qp}\Delta f\Big)=(\nabla^q\nabla^p f)(\nabla_q\nabla_p f)-\frac{1}{n}(\Delta f)^2,$$

we get

$$\int(\Delta f)^2 dV=\int\Big(\nabla^q\nabla^p f+\frac{1}{n}g^{qp}\Delta f\Big)\Big(\nabla_q\nabla_p f+\frac{1}{n}g_{qp}\Delta f\Big)dV$$

$$+\frac{1}{n}\int(\Delta f)^2 dV+\frac{K}{n}\int f\Delta f\, dV,$$

hence

$$\int(\Delta f)^2 dV\geqq\frac{K}{n-1}\int\nabla_p f\nabla^p f\, dV.$$

It is well-known [3] that, if λ_1 is the first positive eigenvalue of the equation $\Delta\rho=\lambda\rho$, and if f satisfies $\int f dV=0$, then

$$\int\nabla_p f\nabla^p f dV\geqq\lambda_1\int f^2 dV.$$

Moreover,

$$\lambda_1=\frac{1}{n-1}K$$

for a space of positive constant curvature [5], [7]. Hence we have

$$\int(\Delta f)^2 dV\geqq\frac{1}{(n-1)^2}K^2\int f^2 dV$$

in our case.

If H does not vanish identically, we have

$$\int(\Delta H)^2 dV-\frac{n-4}{n^2(n-1)}K^2\int H^2 dV\geqq\Big[\frac{1}{(n-1)^2}-\frac{n-4}{n^2(n-1)}\Big]K^2\int H^2 dV>0,$$

hence $\int G_2 dV>0$. Since we have $\int G_1 dV\geqq 0$ always, we have proved $\int(G_1+G_2)dV>0$ in this case.

Let us now assume $H\equiv 0$. If moreover H_{ji} also vanishes everywhere on M, we have $D_{ji}\equiv 0$. We need not consider this case. If H_{ji} does not identically vanish, we have $\int G_1 dV > 0$, hence $\int (G_1+G_2) dV > 0$.

Thus we have proved the main theorem.

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