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Wave and scattering operators for an evolving system $\frac{d}{dt}-iA(t)$

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§ 1. Introduction.

In the preface of their book 'Scattering Theory' [8], Lax and Phillips wrote that 'Scattering theory compares the asymptotic behavior of an evolving system as t tends to $-\infty$ with its asymptotic behavior as t tends to $+\infty$. It is especially fruitful for studying systems constructed from a simpler system by a perturbation provided that any motion of the perturbed system for large $|t|$ is indistinguishable from a motion of the unperturbed system. Thus, if $U(t;0)$ and $U_0(t;0)$ denote the operators relating the states of the perturbed and unperturbed systems at time zero to their respective states at time t, then to each state u of the perturbed system, there correspond two states u_- and u_+ of the unperturbed system such that $U(t;0)u$ behaves like $U_0(t;0)u_-$ as $t\to-\infty$ and like $U_0(t;0)u_+$ as $t\to+\infty$. The scattering operator is defined as the mapping:

$$S: u_- \longrightarrow u_+ .\text{'}$$

There exist many excellent studies of the scattering theory when the operators $U(t;s)$ and $U_0(t;s)$ depend only on the difference $t-s$, i.e. $U(t;s)=U(t-s;0)$ and $U_0(t;s)=U_0(t-s;0)$. e.g. T. Kato [7], P.D. Lax-R.S. Phillips [8].

Even in the case the operator $U(t;s)$ is non-linear, there are works of I.E. Segal [11], W.A. Strauss [12].

But there are not so many papers in which the time-dependent scattering theory where $U(t;s)$ depends on t and s is studied explicitly.

In 1966, J.D. Dollard [1] studied the following problem.

If the generator A_0 of the unperturbed system is given by $-\Delta$ (Δ=Laplacian) and if the generator $A_\varepsilon(t)$ of the perturbed system is given by $A_0+e^{-\varepsilon|t|}B$ where B is the multiplication operator by a 'nice' function $B(x)$, then does there exist a scattering operator between A_0 and $A_\varepsilon(t)$? (This is a part of [1].)

His answer is affirmative when the operator B is relatively small compared with A_0, that is, when $D(A_0)$ (=the domain of the operator A_0) is included in $D(B)$ and we have

$$\|Bu\| \leqq \alpha \|A_0 u\| + \beta \|u\| \qquad \text{for} \quad u \in D(A_0)\,, \tag{1.1}$$

where $0 \leqq \alpha < 1$, $\beta \geqq 0$ and $\|\cdot\|$ is the norm of the considered space.

On the other hand, in [4], we gave an example of the system $\frac{d}{dt} - iA(t)$ where we can construct the wave and scattering operators when $A(t)$ converges to $A^{\pm}$ $(t \to \pm\infty)$ in a suitable sense. (This example is an answer to the problem (II) posed in [3] without non-linear term u^3.)

In order to explain our ideas briefly, we consider a system with $A(t)$ complex valued.

Let $a(t)$ be a complex valued smooth function in t defined on $\boldsymbol{R}$. Suppose that the real part of $a(t)$, $\mathrm{Re}\, a(t)$, approaches to α^- and α^+ respectively when t tends to $-\infty$ and $+\infty$ in L^1 sense. In other words, the following inequalities hold.

$$\int_{-\infty}^{0} |\mathrm{Re}\, a(\tau) - \alpha^-|\, d\tau < \infty \qquad \text{and} \qquad \int_{0}^{\infty} |\mathrm{Re}\, a(\tau) - \alpha^-|\, d\tau < \infty\,. \tag{1.2}$$

And we suppose that the imaginary part of $a(t)$, $\mathrm{Im}\, a(t)$, is integrable on $\boldsymbol{R}$. But we remark here that these assumptions are too restrictive to contain the time-independent perturbation.

Then, we have

THEOREM 1.1. *Let $z(t)$ be a solution of*

$$\frac{d}{dt} z(t) = i a(t) z(t)\,. \tag{1.3}$$

If $z^-(t)$ is a solution of

$$\frac{d}{dt} z^-(t) = i\alpha^- z^-(t)\,, \tag*{(1.4)$^-$}$$

such that $\lim_{t \to -\infty} |z(t) - z^-(t)| = 0$, *then there exists a solution $z^+(t)$ of*

$$\frac{d}{dt} z^+(t) = i\alpha^+ z^+(t)\,, \tag*{(1.4)$^+$}$$

such that $\lim_{t \to +\infty} |z(t) - z^+(t)| = 0$.

PROOF. Define the operators as follows.

$$\left\{ \begin{aligned} & U(t\,;\,s) = e^{i\int_s^t a(\tau)d\tau}\,, \\ & {}^*W_+(t) = e^{-i\alpha^+ t} U(t\,;\,0) = e^{i\int_0^t t(a(\tau)-\alpha^+)d\tau}\,, \\ & W_-(t) = U(0\,;\,t) e^{i\alpha^- t} = e^{i\int_t^0 (a(\tau)-\alpha^-)d\tau}\,. \end{aligned} \right. \tag{1.5}$$

Then by the hypothesis (1.2) and $\int_{-\infty}^{\infty}|\mathrm{Im}\, a(\tau)|\, d\tau<\infty$, there exist the operators $^*W_+$ and W_- such that

$$\lim_{t\to+\infty}|^*W_+(t)-{}^*W_+|=0 \quad \text{and} \quad \lim_{t\to-\infty}|W_-(t)-W_-|=0\,. \tag{1.6}$$

More precisely, we have

$$^*W_+=e^{i\int_0^{\infty}(a(\tau)-\alpha^+)d\tau} \quad \text{and} \quad W_-=e^{i\int_{-\infty}^{0}(a(\tau)-\alpha^-)d\tau}\,. \tag{1.7}$$

If we define $z^+(t)=e^{it\alpha^+}\cdot{}^*W_+\cdot W_-z^-(0)$, we have the result desired, because the hypothesis $\lim_{t\to-\infty}|z(t)-z^-(t)|=0$ implies the relation $z(0)=W_-z^-(0)$. Q. E. D.

We shall denote the operators $^*W_+$ and W_- defined by (1.7) as $^*W_+^{(0)}$ and $W_-^{(0)}$.

Then, we have easily the following proposition.

PROPOSITION 1.2. *For every s, we have*

$$^*W_+^{(0)}U(0\,;s)=U_0^+(0\,;s)^*W_+^{(s)}\,, \tag{1.8}$$

$$U(s\,;0)W_-^{(0)}=W_-^{(s)}U_0^-(s\,;0)\,, \tag{1.9}$$

where

$$U_0^{\pm}(t\,;s)=e^{i\alpha^{\pm}(t-s)} \quad \textit{and}$$

$$^*W_+^{(s)}=e^{i\int_s^{\infty}(a(\tau)-\alpha^+)d\tau}\,, \qquad W_-^{(s)}=e^{i\int_{-\infty}^{s}(a(\tau)-\alpha^-)d\tau}\,.$$

REMARK 1.3. It is well-known that if there exist the wave operators between two groups $\{U_0(t)\}$ and $\{U(t)\}$ having the generators independent of t, then the wave operators are intertwining transformations for the groups $\{U_0(t)\}$ and $\{U(t)\}$. The above proposition is an analogue of this property.

We define other wave operators $W_+^{(s)}$ and $^*W_-^{(s)}$ as follows.

$$W_+^{(s)}=\lim_{t\to+\infty}U(s\,;t)U_0^+(t\,;s)=e^{-i\int_s^{\infty}(a(\tau)-\alpha^+)d\tau}\,. \tag{1.10}$$

$$^*W_-^{(s)}=\lim_{t\to-\infty}U_0^-(s\,;t)U(t\,;s)=e^{-i\int_{-\infty}^{s}(a(\tau)-\alpha^-)d\tau}\,. \tag{1.11}$$

And we define also the scattering operators $S_{\pm}^{(s)}$ as

$$S_+^{(s)}={}^*W_+^{(s)}W_-^{(s)} \quad \text{and} \quad S_-^{(s)}={}^*W_-^{(s)}W_+^{(s)}\,. \tag{1.12}$$

Then, we have the following proposition by easy calculation.

PROPOSITION 1.4. *For every s, t, we have*

$$U_0^+(t\,;0)S_+^{(0)}=U_0^+(t\,;s)S_+^{(s)}U_0^-(s\,;0)\,, \tag{1.13}$$

$$U_0^-(t\,;s)S_-^{(0)}=U_0^-(t\,;s)S_-^{(s)}U_0^+(s\,;0)\,. \tag{1.14}$$

We enumerate other properties of the W's and S's.

PROPOSITION 1.5. *A necessary and sufficient condition for* $W_+^{(s)*}$ (*=the complex conjugate of* $W_+^{(s)}$) $= {}^*W_+^{(s)}$ *is*

$$\int_s^\infty \operatorname{Im} a(\tau)d\tau = 0 . \tag{1.15}$$

PROOF. As $\alpha^\pm$ are real, we have $(W_+^{(s)})^* = e^{-i\int_s^\infty (a^*(\tau)-\alpha^+)d\tau}$. Combining this with the definition of ${}^*W_+^{(s)}$, we have the result.

COROLLARY 1.6. *Under the condition* (1.15), *we have*

$$|W_+^{(s)}| = |{}^*W_+^{(s)}| = 1 . \tag{1.16}$$

Analogously, we have

PROPOSITION 1.7. *A necessary and sufficient condition for* ${}^*W_-^{(s)} = W_-^{(s)*}$ *is*

$$\int_{-\infty}^s \operatorname{Im} a(\tau)d\tau = 0 . \tag{1.17}$$

It is clear that we have

PROPOSITION 1.8. $S_+^{(s)} = S_-^{(s)*}$ *is equivalent to*

$$\int_{-\infty}^\infty \operatorname{Im} a(\tau)d\tau = 0 . \tag{1.18}$$

We now consider an 'inverse problem'. First of all, we give the following.

DEFINITION 1.9. We say that a function $a(t)$ defined on $\boldsymbol{R}$ belongs to the class $(S\text{-}L^1; \{\alpha^\pm\})$ if (i) it is smooth, (ii) it satisfies the condition (1.2) and (iii) $\operatorname{Im} a(t)$ is integrable on $\boldsymbol{R}$.

Let us denote the operators $S_\pm^{(0)}$ defined by (1.12) as $S_\pm(a(\cdot))$.

PROPOSITION 1.10. *Let two functions* $a(t)$ *and* $b(t)$ *belong to the class* $(S\text{-}L^1 : \{\alpha^\pm\})$. *Then* $S_+(a(\cdot)) = S_+(b(\cdot))$ *holds if and only if there exists an integer* n *such that*

$$\int_{-\infty}^\infty (a(\tau)-b(\tau))d\tau = 2n\pi . \tag{1.19}$$

The proof is omitted.

In § 2, we shall extend Theorem 1.1, Proposition 1.2 and 1.4, to the theorems where $A(t)$ is a closed operator in a Hilbert space. These are also abstract versions of § 3, [4]. But, for the time being, it is difficult to formulate analogous properties of Propositions 1.5~1.10.

We shall apply the results of § 2, to a Schrödinger equation with variable coefficients in t in § 3, and to a hyperbolic system of first order with variable coefficient in t in § 4.

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§ 2. Abstract theorems.

Let H be a Hilbert space with a norm denoted by $|\cdot|_H$ and a scalar product denoted by $(\cdot, \cdot)_H$.

Let $A(t)$ be a closed linear operator in H with the domain $D(A(t))$ independent of t. We denote $D(A(t))$ by V and we suppose that V has its own norm $\|\cdot\|_V$ such that V is continuously imbedded in H. (From this assumption, $A(t)$ is a bounded operator from V to H by Banach's closed graph theorem.)

We pose the following hypotheses on $\{iA(t)\}$:

(I) The operator $iA(t)$ generates an evolution operator $U(t; s)$.

(II) The operator $iA(t)$ converges to $iA^{\pm}$ in a suitable sense when t tends to $\pm\infty$.

In order to guarantee the hypothesis (I), we suppose that there exist Hilbert structures $H(t)$ in H such that the norm $|\cdot|_{H(t)}$ is equivalent to $|\cdot|_H$ for all t, i. e. there exists a constant $c_0>0$ such that we have

$$c_0^{-1}|u|_{H(t)} \leqq |u|_H \leqq c_0|u|_{H(t)} \quad \text{for any} \quad t. \tag{2.1}$$

From now on, we shall use c_j's as various constants independent of t.

With the aid of the space $H(t)$, a sufficient conditions for (I) is given as follows (due to T. Kato [6], S. Mizohata [9], M. Ikawa [2] and K. Yosida [13]).

(I.1) V is dense in H and there exists a constant $\delta \geqq 0$ such that

$$|(\lambda I - iA(t))^{-1}|_{H(t)} \leqq (|\lambda| - \delta)^{-1} \quad \text{whenever} \quad |\lambda| > \delta \ (\lambda \text{ real}). \tag{2.2}$$

(I.2) For some λ_0, $B(t; s) = (\lambda_0 I - iA(t))(\lambda_0 I - iA(s))^{-1} \in \boldsymbol{B}(H)$ (=the space of bounded linear operators in H) is, at least for some s, weakly differentiable in t and $B_t(t; s)$ is strongly continuous in t.

(I.3) For any $u \in H$, $(u, u)_{H(t)}$ is continuously differentiable and there exists a constant $c_1>0$ such that we have

$$\left|\frac{d}{dt}(u, u)_{H(t)}\right| \leqq c_1(u, u)_H. \tag{2.3}$$

DEFINITION 2.1. A function $u(t)$, defined on $\boldsymbol{R}$ with values in H, is said to be a solution of $\frac{d}{dt}u(t) = iA(t)u(t)$ if it satisfies that (i) $u(t) \in \mathcal{E}_t^1(H)$ (=the space of H-valued continuously differentiable functions in $t \in \boldsymbol{R}$), (ii) for any t, $u(t) \in V$ and (iii) $\frac{d}{dt}u(t) = iA(t)u(t)$.

We denote by $\mathcal{E}_t^k(X)$, the set of functions in $t \in \boldsymbol{R}$ with values in X and k-times strongly continuously differentiable where X is a Banach space.

Then, we have the following theorem.

THEOREM 2.2 ([2], [6], [9], 13]). *We suppose that* (I.1), (I.2) *and* (I.3) *hold.*

Then, there exists a continuous operator $U(t;s)$ (=the evolution operator for $iA(t)$) such that

(i) *$U(t;s)$ is strongly continuous in s and t,*

(ii) *$U(t;s)U(s;r)=U(t;r)$, $U(t;t)=I$,*

(iii) *$U(t;s)V\subset V$,*

(iv) *$u(t)=U(t;s)u_0$ with $u_0\in V$, gives a solution of the problem*

$$\begin{cases} \dfrac{d}{dt}u(t)=iA(t)u(t)\,, \\ u(s)=u_0\,, \end{cases}$$

(v) *for $u_0\in V$, $U(t;s)u_0$ is strongly continuously differentiable in s and satisfies*

$$\frac{d}{ds}U(t;s)u_0=-iU(t;s)A(s)u_0\,.$$

We suppose that there also exist Hilbert structures $H^{\pm}$ in H such that the norms $|\cdot|_{H^{\pm}}$ are equivalent to the norm $|\cdot|_H$. Moreover, we suppose that there exist closed operators $A^{\pm}$ in H with domain $D(A^{\pm})=V$ and self-adjoint in $H^{\pm}$ respectively.

DEFINITION 2.3. We say that the operator $iA(t)$ converges to the operators $iA^{\pm}$ in the sense of $(0\text{-}L^1)$ when t tends to $\pm\infty$ if the following conditions are satisfied.

(II. a) There exists a function $\phi(t)\in L^1(\boldsymbol{R})\cap C^0(\boldsymbol{R})$ such that

(a. 1) $|(\mathrm{Re}\,(iA(t)-iA^+)u,\,u)_{H^+}|\leqq c_2\cdot\phi(t)\cdot|u|^2_{H^+}$ for $t\gg 1$, $u\in V$,

(a. 2) $|(iA(t)-iA^+)u|_{H^+}\leqq c_2\cdot\phi(t)\|u\|_V$ for $t\gg 1$, $u\in V$.

(II. b) We have also

(b. 1) $|(\mathrm{Re}\,(iA(t)-iA^-)u,\,u)_{H^-}|\leqq c_2\cdot\phi(t)\cdot|u|^2_{H^-}$ for $-t\gg 1$, $u\in V$,

(b. 2) $|(iA(t)-iA^-)u|_{H^-}\leqq c_2\cdot\phi(t)\cdot\|u\|_V$ for $-t\gg 1$, $u\in V$.

(II. c) For any $u\in V$, there exists the limit

$$A'(t)u=\lim_{h\to 0}h^{-1}(A(t+h)-A(t))u \quad \text{in } H$$

such that

$$|iA'(t)u|_{H^{\pm}}\leqq c_3\cdot\phi(t)\|u\|_V \quad \text{for } \pm t\gg 1\,.$$

(II. d) There exists a number $\lambda_1\in\boldsymbol{C}$ such that

$$\|u\|_V\leqq c_4|(\lambda_1 I-iA(t))u|_{H^{\pm}} \quad \text{for any } t$$

(coercive inequality).

The sense of 'suitable' in (II) means the convergence of $iA(t)$ to $iA^{\pm}$ in the sense of $(0\text{-}L^1)$.

THEOREM 2.4. *We suppose that* (I.1)~(I.3) *and* (II.a)~(II.d) *hold. Then, there exist the operators* $W_{\pm}$, $^*W_{\pm}$ *defined as follows:*

$$(2.4)\qquad {}^*W_+=\lim_{t\to+\infty} {}^*W_+(t)\,,\qquad {}^*W_+(t)=U_0^+(0\,;\,t)U(t\,;\,0)\,,$$

$$(2.5)\qquad W_-=\lim_{t\to-\infty} W_-(t)\,,\qquad W_-(t)=U(0\,;\,t)U_0^-(t\,;\,0)\,,$$

$$(2.6)\qquad W_+=\lim_{t\to+\infty} W_+(t)\,,\qquad W_+(t)=U(0\,;\,t)U_0^+(t\,;\,0)\,,$$

$$(2.7)\qquad {}^*W_-=\lim_{t\to-\infty} {}^*W_-(t)\,,\qquad {}^*W_-(t)=U_0^-(0\,;\,t)U(t\,;\,0)\,,$$

where lim *denotes the strong limit of operators in* H *and* $U_0^{\pm}(t;s)=\exp(i(t-s)A^{\pm})$.

We state an elementary lemma without proof.

LEMMA (Gronwall's inequality). *Let* $\alpha(t)$, $\beta(t)$ *and* $\gamma(t)$ *be continuous functions on an interval* $[a, b]$ *and suppose* $\beta(t)\geqq 0$ *there. If*

$$(2.8)\qquad \gamma(t)\leqq\alpha(t)+\int_a^t\beta(s)\gamma(s)ds\qquad for\quad t\in[a, b]\,,$$

then

$$(2.9)\qquad \gamma(t)\leqq\alpha(t)+\int_a^t\alpha(s)\beta(s)\Big(\exp\Big(\int_s^t\beta(r)dr\Big)\Big)ds$$

on the interval.

In order to prove the theorem, we begin with an a priori estimate.

LEMMA 2.5. *Let* $u(t)$ *be a solution of the following problem.*

$$(2.10)\qquad \begin{cases} \dfrac{d}{dt}u(t)=iA(t)u(t)\,,\\ u(0)=u_0\in V\,.\end{cases}$$

Then, the following inequalities hold.

$$(2.11)\qquad |u(t)|_H\leqq c_5|u_0|_H\,,$$

$$(2.12)\qquad \|u(t)\|_V+|u'(t)|_H\leqq c_5\|u_0\|_V\qquad \Big(u'(t)=\frac{d}{dt}u(t)\Big)\,.$$

PROOF. It is sufficient to prove the above inequalities when $t\geqq 0$ and H is replaced by H^+. Because other cases are proved analogously.

$$\begin{aligned}\left|\frac{1}{2}\frac{d}{dt}|u(t)|_{H^+}^2\right|&=\left|\mathrm{Re}\Big(\frac{d}{dt}u(t),\,u(t)\Big)_{H^+}\right|\\ &=|\mathrm{Re}\,((iA(t)-iA^+)u(t),\,u(t))_{H^+}+\mathrm{Re}\,(iA^+u(t),\,u(t))_{H^+}|\\ &\leqq c_2\cdot\phi(t)|u(t)|_{H^+}^2\end{aligned}$$

where we use (II.a) and $\mathrm{Im}\,(A^+u,\,u)_{H^+}=0$. Dividing the both sides of the above inequality by $|u(t)|_{H^+}$ and applying Gronwall's inequality when $\beta(t)$ is constant, we prove (2.11) immediately.

Put $\partial_h u(t)=h^{-1}(u(t+h)-u(t))$. Then $\partial_h u(t)\in V$, satisfies

$$\frac{d}{dt}\partial_h u(t)=iA(t+h)\cdot\partial_h u(t)+i\cdot\partial_h A(t)\cdot u(t)\,, \tag{2.13}$$

where

$$\partial_h A(t)\cdot u=h^{-1}(A(t+h)u-A(t)u)\,.$$

As above, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\partial_h u(t)|_{H^+}^2 \\ \leqq c_2\cdot\phi(t+h)|\partial_h u(t)|_{H^+}^2+|i\cdot\partial_h A(t)\cdot u(t)|_{H^+}\cdot|\partial_h u(t)|_{H^+}\,.\end{aligned} \tag{2.14}$$

Dividing both sides by $|\partial_h u(t)|_{H^+}$, and integrating from 0 to t and making $h\to 0$, we have

$$|u'(t)|_{H^+}\leqq|u'(0)|_{H^+}+\int_0^t|iA'(s)u(s)|_{H^+}ds+c_2\int_0^t\phi(s)|u'(s)|_{H^+}ds\,. \tag{2.15}$$

By the condition (II.c), (II.d) and (2.15), we have

$$\begin{aligned}\|u(t)\|_V+|u'(t)|_{H^+}&\leqq c_4|(\lambda_1 I-iA(t))u(t)|_{H^+}+|u'(t)|_{H^+}\\ &\leqq c'\Big(|u_0|_{H^+}+|u'(0)|_{H^+}+\int_0^t\phi(s)(\|u(s)\|_V+|u'(s)|_{H^+})ds\Big)\,.\end{aligned}$$

Using the fact that $|u'(0)|_{H^+}=|iA(0)u(0)|_{H^+}\leqq c''\|u(0)\|_V$, and applying Gronwall's inequality we prove (2.12). Q. E. D.

PROOF OF THEOREM 2.4. Differentiating formally $^*W_+(t)u_0$ for $u_0\in V$, we have

$$\begin{aligned}{}^*W_+(t)u_0-{}^*W_+(0)u_0&=\int_0^t\frac{d}{ds}{}^*W_+(s)u_0ds\\ &=\int_0^t U_0^+(0\,;s)(iA(s)-iA^+)U(s\,;0)u_0ds\,.\end{aligned} \tag{2.16}$$

If $u_0\in V$, we have that (i) $U(t\,;0)u_0\in V$, for any $t\in\boldsymbol{R}$, (ii) $(iA(t)-iA^+)\cdot U(t\,;0)u_0$ is continuous in t and (iii) $|(iA(t)-iA^+)U(t\,;0)u_0|_{H^+}$ is integrable on $\boldsymbol{R}^+$ by (II.a) and (2.12). So making $t\to+\infty$ in (2.16) and applying Theorem X.3.7. of T. Kato [7], we prove that there exists $^*W_+={}^*W_+(\infty)$.

Differentiating $W_-(t)u_0$ formally in t for $u_0\in V$, we have

$$\begin{aligned}W_-(t)u_0-W_-(0)u_0&=\int_0^t\frac{d}{ds}W_-(s)u_0ds\\ &=\int_t^0 U(0\,;s)(iA(s)-iA^-)U_0^-(s\,;0)u_0ds\,.\end{aligned} \tag{2.17}$$

Combining (II.d) with (II.a), (II.b) and $|A^\pm\cdot u|_{H^\pm}\leqq c^\pm\|u\|_V$, we have the coercive inequalities for $iA^\pm$, i. e. we have

(2.18) $$\|u\|_V \leqq c_6 |(\lambda_1 I - iA^{\pm})u|_{H^{\pm}} \qquad \text{for} \quad u \in V.$$

Using this inequality, we have, calculating analogously as in Lemma 2.5,

(2.19) $$\|U_0^{\pm}(t\,;0)u_0\|_V \leqq c_7 \|u_0\|_V \qquad \text{for} \quad u_0 \in V.$$

By (2.11), we may examine the above properties (ii) and (iii) for the function $U(0\,;s)(iA(s)-iA^-)U_0^-(s\,;0)u_0$. So we may prove the existence of W_-.

The existence of other operators W_+, $*W_-$ are proved analogously.

Q. E. D.

REMARK 2.6. If $u(t)$ is a solution of (2.10), then $u(t)$ belongs to $\mathcal{E}_t^0(V)$.

PROOF. For each t, we have

$$\lambda_1 u(t) - u'(t) = \lambda_1 u(t) - iA(t)u(t).$$

So inserting s instead of t in the above equality and subtracting one from the other, we have

$$\lambda_1(u(t)-u(s)) - iA(t)(u(t)-u(s))$$
$$= \lambda_1(u(t)-u(s)) - (u'(t)-u'(s)) + (iA(t)-iA(s))u(s).$$

By (II.d), we have

$$\|u(t)-u(s)\|_V \leqq c_4 |\lambda_1(u(t)-u(s)) - (u'(t)-u'(s))|_H$$
$$+ c_4 |(iA(t)-iA(s))u(s)|_H.$$

As $u(t) \in \mathcal{E}_t^1(H)$, the first term in the right hand side tends to zero when t tends to s. The second term in the right hand side tends also to zero when t tends to s because there exists $A'(t)u$ in H. So, we have the desired result.

The above defined operators $*W_+$, W_+ are denoted, from now on, by $*W_+^{(0)}$, $W_+^{(0)}$. (This means that we begin to observe the phenomenon governed by (2.10) at time zero.)

Put $*W_{\pm}^{(s)} = \lim_{t \to \pm\infty} U_0^+(s\,;t)U(t\,;s)$, $W_{\pm}^{(s)} = \lim_{t \to \pm\infty} U(s\,;t)U_0^+(t\,;s)$. Then, we have the following proposition.

PROPOSITION 2.7. *For any s, we have*

(2.20) $$*W_+^{(0)}U(0\,;s) = U_0^+(0\,;s)*W_+^{(s)},$$

(2.21) $$U(s\,;0)W_-^{(0)} = W_-^{(s)}U_0^-(s\,;0),$$

(2.22) $$U(s\,;0)W_+^{(0)} = W^{(s)}U_0^+(s\,;0),$$

(2.23) $$*W_-^{(0)}U(0\,;s) = U_0^-(0\,;s)*W_-^{(s)}.$$

PROOF. We prove only (2.20) because we may prove the others in the same way.

$$\begin{aligned} {}^*W_+^{(0)}U(0;s) &= \lim_{t\to+\infty} U_0^+(0;t)U(t;0)U(0;s) \\ &= \lim_{t\to+\infty} U_0^+(0;s)U_0^+(s;t)U(t;s) \\ &= U_0^+(0;s){}^*W_+^{(s)}. \end{aligned}$$

Q. E. D.

We define the scattering operators $S_+^{(s)}$ and $S_-^{(s)}$ as $S_+^{(s)}={}^*W_+^{(s)}W_-^{(s)}$ and $S_-^{(s)}={}^*W_-^{(s)}W_+^{(s)}$.

PROPOSITION 2.8. *For any s, we have*

$$U_0^+(t;0)S_+^{(0)}U_0^-(0;0)=U_0^+(t;s)S_+^{(s)}U_0^-(s;0)\,, \tag{2.24}$$

$$U_0^-(t;0)S_-^{(0)}U_0^+(0;0)=U_0^-(t;s)S_-^{(s)}U_0^+(s;0)\,. \tag{2.25}$$

PROOF.

$$\begin{aligned} U_0^+(t;0)S_+^{(0)} &= \lim_{r\to+\infty} U_0^+(t;0)U_0^+(0;r)U(r;0)U(0;-r)U_0^-(-r;0) \\ &= \lim_{r\to+\infty} U_0^+(t;s)U_0^+(s;r)U(r;s)U(s;-r)U_0^-(-r;s)U_0^-(s;0) \\ &= U_0^+(t;s)S_+^{(s)}U_0^-(s;0)\,. \end{aligned}$$

This proves (2.24). (2.25) may be proved analogously.

Before stating an analogue of Theorem 1.1, we introduce another notion of 'solution'.

DEFINITION 2.9. A function $u^\pm(t)\in\mathcal{E}_t^0(H)$ is said to be a weak solution of $\frac{d}{dt}u^\pm(t)=iA^\pm u^\pm(t)$ if there exists an element $u^\pm\in H$ such that $u^\pm(t)$ is represented by $u^\pm(t)=U_0^\pm(t;0)u^\pm$.

THEOREM 2.10. *Assume that the operator $iA(t)$ satisfy the conditions* (I.1)$\sim$(I.3) *and* (II.a)$\sim$(II.d), *and let $u(t)\in\mathcal{E}_t^0(V)\cap\mathcal{E}_t^1(H)$ be a solution of*

$$\frac{d}{dt}u(t)=iA(t)u(t)\,. \tag{2.26}$$

If there exists a weak solution $u^-(t)\in\mathcal{E}_t^0(H)$ of

$$\frac{d}{dt}u^-(t)=iA^-u^-(t) \tag{2.27}$$

satisfying

$$\lim_{t\to-\infty}|u(t)-u^-(t)|_H=0\,, \tag{2.28}$$

then there exists a unique weak solution $u^+(t)\in\mathcal{E}_t^0(H)$ of

$$\frac{d}{dt}u^+(t)=iA^+u^+(t) \tag{2.29}$$

satisfying

$$\lim_{t\to+\infty}|u(t)-u^+(t)|_H=0\,. \tag{2.30}$$

PROOF. By (2.28), we have $u(0)=W_{-}^{(0)}u^{-}(0)$. In fact,

$$|u(t)-u^{-}(t)|_{H}=|U(t;0)u(0)-U_{0}^{-}(t;0)u^{-}(0)|_{H}$$
$$=|U(t;0)(u(0)-U(0;t)U_{0}^{-}(t;0)u^{-}(0))|_{H}$$
$$\geqq c_{5}^{-1}|u(0)-W_{-}^{(0)}(t)u^{-}(0)|_{H}.$$

Putting $u^{+}(t)=U_{0}^{+}(t;0)S_{+}^{(0)}u^{-}(0)$, we may easily prove that $u^{+}(t)$ satisfies (2.29) weakly and (2.30). The uniqueness can be proved easily.

§ 3. Schrödinger equation.

Let Ω be a domain in $\boldsymbol{R}^{n}$ with a smooth boundary. The points in $\boldsymbol{R}^{n}$ have for coordinates $x=(x_{1}, x_{2}, \cdots, x_{n})$. Put $D_{j}=\frac{\partial}{\partial x_{j}}$, $D^{\alpha}=D_{1}^{\alpha_{1}}D_{2}^{\alpha_{2}}\cdots D_{n}^{\alpha_{n}}$ for $\alpha=(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n})$ and $D_{t}=\frac{\partial}{\partial t}$. We often use $u_{x_{j}}$, u_{t} respectively in place of $D_{j}u$, $D_{t}u$.

Let us introduce function spaces on Ω which will be used below.

$H^{l}(\Omega)$ stands for the Sobolev space of order l on Ω with the norm given by

$$\|v\|_{l}^{2}=\sum_{|\alpha|\leqq l}\|D^{\alpha}v\|^{2} \quad \text{where} \quad |\alpha|=\sum_{i=1}^{n}\alpha_{i} \quad \text{and} \quad \|v\|^{2}=\int_{\Omega}|v(x)|^{2}dx.$$

We denote $H^{0}(\Omega)$ by $L^{2}(\Omega)$.

$H_{0}^{l}(\Omega)$ is the closure of $C_{0}^{\infty}(\Omega)$ (=the set of infinitely differentiable functions with compact support on Ω) in $H^{l}(\Omega)$.

$\mathscr{B}^{m}(\Omega)$ denotes the space of functions having bounded smooth derivatives of order not exceeding m with the norm

$$|v|_{\mathscr{B}^{m}}=\sum_{|\alpha|\leqq m}\sup_{x\in\Omega}|D^{\alpha}v(x)|.$$

Let us consider the following equation.

$$\text{(3.1)} \qquad \begin{cases} \frac{1}{i}\frac{\partial}{\partial t}u(x, t)=L(x, t; D)u(x, t), \\ u(x, t)|_{\partial\Omega}=0 \qquad (\partial\Omega=\text{the boundary of } \Omega), \end{cases}$$

where

$$L(x, t; D)=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_{i}}\Big(a_{ij}(x, t)\frac{\partial}{\partial x_{j}}\Big)-q(x, t).$$

We assume the following:

(S.1) The coefficients $a_{ij}(x, t)$ and $q(x, t)$ belong to $\mathscr{B}^{\infty}(\Omega\times(-\infty, \infty))$ and are real-valued.

(S.2) $a_{ij}(x, t)=a_{ji}(x, t)$ and there exists a constant $\delta>0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \geqq \delta \sum_{j=1}^{n} \xi_j^2 \quad \text{for} \quad \xi=(\xi_1, \xi_2, \cdots, \xi_n) \in \boldsymbol{R}^n . \tag{3.2}$$

(S.3) There exist functions $a_{ij}^{\pm}(x)$, $q^{\pm}(x) \in \mathscr{B}^{\infty}(\Omega)$ and $0 \leqq \phi(t) \in L^1(\boldsymbol{R}^n)$ such that

$$\begin{cases} |a_{ij}(x, t)-a_{ij}^{\pm}(x)|_{\mathscr{B}^1(\Omega)} \leqq \phi(t) , & \left|\frac{\partial}{\partial t} a_{ij}(x, t)\right|_{\mathscr{B}^1(\Omega)} \leqq \phi(t) , \\ |q(x, t)-q^{\pm}(x)|_{\mathscr{B}^0(\Omega)} \leqq \phi(t) . \end{cases} \tag{3.3}$$

Define the operators $L^{\pm}(x ; D)$ as

$$L^{\pm}(x ; D)=\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}^{\pm}(x) \frac{\partial}{\partial x_j}\Big)-q^{\pm}(x) .$$

DEFINITION 3.1. A function $u^{\pm}(x, t) \in \mathscr{E}_t^0(L^2(\Omega))$ is said to be a weak solution of the problem

$$\begin{cases} \frac{1}{i} \frac{\partial}{\partial t} u^{\pm}(x, t)=L^{\pm}(x ; D) u^{\pm}(x, t) , \\ u^{\pm}(x, t)|_{\partial\Omega}=0 , \end{cases} \tag{3.4$^{\pm}$}$$

if it satisfies the following equality

$$\frac{1}{i} \int_{\Omega} u^{\pm}(x, t)\overline{\Phi^{\pm}(x, t)} dx-\frac{1}{i} \int_{\Omega} u^{\pm}(x, s)\overline{\Phi^{\pm}(x, s)} dx = \int_s^t dr \int_{\Omega} u^{\pm}(x, r)\overline{\Big(-\frac{1}{i} \frac{\partial}{\partial r} \Phi^{\pm}(x, r)+L^{\pm}(x ; D) \Phi^{\pm}(x, r)\Big)} dx \tag{3.5$^{\pm}$}$$

for any $s, t \in \boldsymbol{R}$ and any $\Phi^{\pm}(x, t) \in \mathscr{E}_t^1(L^2(\Omega)) \cap \mathscr{E}_t^0(H_0^1(\Omega) \cap H^2(\Omega))$.

Then, we have

THEOREM 3.2. *We assume* (S.1)~(S.3). *Let* $u(x, t) \in \mathscr{E}_t^1(L^2(\Omega)) \cap \mathscr{E}_t^0(H_0^1(\Omega) \cap H^2(\Omega))$ *be a solution of* (3.1).

If there exists a weak solution $u^-(x, t)$ *of* (3.4)$^-$ *such that*

$$\lim_{t \to -\infty} \|u(x, t)-u^-(x, t)\|=0 , \tag{3.6}$$

then there exists uniquely a weak solution $u^+(x, t)$ *of* (3.4)$^+$ *satisfying*

$$\lim_{t \to -\infty} \|u(x, t)-u^+(x, t)\|=0 . \tag{3.7}$$

This theorem is proved in the following way.

We define $H=H^{\pm}=L^2(\Omega)$, $V=H_0^1(\Omega) \cap H^2(\Omega)$ and $A(t)$, $A^{\pm}$ are the realizations of $L(x, t ; D)$, $L^{\pm}(x ; D)$ respectively in H with domain $D(A(t))=D(A^{\pm})=V$. Then, it is well-known that the operators $A(t)$, $A^{\pm}$ are self-adjoint in H (e. g. [**10**]). So, we may prove easily that the definition of the weak solution in Definition 3.1 is equivalent to that in Definition 2.9.

By the smoothness of the coefficients and by the self-adjointness of the operator $A(t)$, we readily prove the conditions (I.1)$\sim$(I.3).

LEMMA 3.3. *The operator* $iA(t)$ *converges to* $iA^{\pm}$ *in the sense of* (0-L^1) *when* t *tends to* $\pm\infty$.

PROOF. For any $u\in V$, we have

$$\|(iA(t)-iA^{+})u\|\leqq|a_{ij}(x,t)-a_{ij}^{+}(x)|_{\mathscr{B}^1(\Omega)}\|u\|_2+|q(x,t)-q^{+}(x)|_{\mathscr{B}^0(\Omega)}\|u\|$$
$$\leqq 2\phi(t)\|u\|_2\,,$$

which shows (II.a.2). The conditions (II.a.1), (II.b), and (II.c) are proved readily. (II.d) is the well-known coercive inequality for $L(x,t;D)$ (e.g. [10]).

By these facts, we may apply Theorem 2.10 to the present case, and obtain Theorem 3.2.

REMARK 3.4. Applying our theorems to $A_\varepsilon(t)=-\Delta+e^{-\varepsilon|t|}B$, we have a part of Dollard's result [1].

§4. Symmetric hyperbolic systems.

In this section, we study the equation in $\boldsymbol{R}^n$

$$E(x,t)\frac{\partial}{\partial t}u(x,t)=\sum_{j=1}^{n}A_j(x,t)\frac{\partial}{\partial x_j}u(x,t)+B(x,t)u(x,t)\,, \tag{4.1}$$

and compare it with

$$E^{\pm}(x)\frac{\partial}{\partial t}u^{\pm}(x,t)=\sum_{j=1}^{n}A_j^{\pm}(x)\frac{\partial}{\partial x_j}u^{\pm}(x,t)+B^{\pm}(x)u^{\pm}(x,t)\,. \tag{4.2}$$

Here, $u(t)$ $(=u(x,t))$, $u^{\pm}(t)$ are $m\times 1$-matrices, $E(x,t)$, $E^{\pm}(x)$, $A_j(x,t)$, $A_j^{\pm}(x)$, $B(x,t)$ and $B^{\pm}(x)$ are $m\times m$-matrices with the following properties:

(H.1) All matrices which appear in (4.1) and (4.2), belong to $\mathscr{B}^{\infty}(\boldsymbol{R}^n\times\boldsymbol{R})$ and $\mathscr{B}^{\infty}(\boldsymbol{R}^n)$.

(H.2) $E(x,t)$ and $E^{\pm}(x)$ are real, symmetric and positive definite matrices and there exists a constant $c>0$, independent of (x,t) such that

$$\begin{cases} c^{-1}|\zeta|^2\leqq {}^t\zeta\cdot E(x,t)\bar{\zeta}\leqq c|\zeta|^2 \\ c^{-1}|\zeta|^2\leqq {}^t\zeta\cdot E^{\pm}(x)\bar{\zeta}\leqq c|\zeta|^2 \end{cases}\quad \text{for any}\quad \zeta=(\zeta_1,\zeta_2,\cdots,\zeta_m)\in\boldsymbol{C}^m\,. \tag{4.3}$$

(H.3) $A_j(x,t)$, $A_j^{\pm}(x)$ are Hermitian and uniformly elliptic, i.e. there exists a constant $\delta_0>0$ independent of (x,t) such that

$$\begin{cases} \delta_0|\xi|\leqq\left|\sum_{j=1}^{n}A_j(x,t)\cdot\xi_j\right| \\ \delta_0|\xi|\leqq\left|\sum_{j=1}^{n}A_j^{\pm}(x)\cdot\xi_j\right| \end{cases}\quad \text{for any}\quad \xi=(\xi_1,\xi_2,\cdots,\xi_n)\in\boldsymbol{R}^n\,. \tag{4.4}$$

(H.4) $\sum_{j=1}^{n} \frac{\partial A_j^{\pm}(x)}{\partial x_j} = B^{\pm}(x) + B^{\pm}(x)^*$.

(H.5) There exists a function $\phi(t) \in L^1(\boldsymbol{R}) \cap C^0(\boldsymbol{R})$ such that

$$(4.5) \qquad \begin{cases} |A_j(x, t) - A_j^{\pm}(x)|_{\mathscr{B}^1(R^n)} \leqq \phi(t)\,, \\ |B_j(x, t) - B_j^{\pm}(x)|_{\mathscr{B}^0(R^n)} \leqq \phi(t)\,, \\ |E(x, t) - E^{\pm}(x)|_{\mathscr{B}^0(R^n)} \leqq \phi(t)\,. \end{cases}$$

(From here on, we use the usual notations for matrices, i. e. ${}^tA$ = transposed matrix of A, $A^* = ({}^t\bar{A})$ = adjoint matrix of A. Inequalities in (4.5) mean that each component of matrices satisfies (4.5).)

Let $\mathscr{H}$ be $[L^2(\boldsymbol{R}^n)]^m$ with the norm $\|\cdot\|_{\mathscr{H}}$ defined by

$$\|u\|_{\mathscr{H}}^2 = \int_{R^n} {}^tu(x) \cdot \overline{u(x)} dx\,.$$

In the space $\mathscr{H}$, we define also Hilbert structures $\mathscr{H}(t)$ and $\mathscr{H}^{\pm}$ as follows:

$$(4.6) \qquad \begin{cases} \|u\|_{\mathscr{H}(t)}^2 = \int_{R^n} {}^tu(x) \cdot E(x, t)\overline{u(x)} dx\,, \\ \|u\|_{\mathscr{H}^{\pm}}^2 = \int_{R^n} {}^tu(x) \cdot E^{\pm}(x)\overline{u(x)} dx\,. \end{cases}$$

It is obvious by (4.3) that these structures in $\mathscr{H}$ are equivalent for all t.

Now, we put $\mathcal{V} = [H^1(\boldsymbol{R}^n)]^m$ and define the norm $\|\cdot\|_{\mathcal{V}}$ in $\mathcal{V}$ in the usual way. We define operators $\mathcal{A}(t)$, $\mathcal{A}^{\pm}$ in $\mathscr{H}$ as follows.

$$(4.7) \qquad \begin{cases} \mathcal{A}(t) = E(x, t)^{-1}\Big\{\sum_{j=1}^{n} A_j(x, t)\frac{\partial}{\partial x_j} + B(x, t)\Big\}\,, \\ \mathcal{A}^{\pm} = E^{\pm}(x)^{-1}\Big\{\sum_{j=1}^{n} A_j^{\pm}(x)\frac{\partial}{\partial x_j} + B^{\pm}(x)\Big\}\,, \\ D(\mathcal{A}(t)) = D(\mathcal{A}^{\pm}) = \mathcal{V}\,. \end{cases}$$

By the condition (H.3), we have the following coercive inequalities.

$$(4.8) \qquad \begin{cases} \|u\|_{\mathcal{V}} \leqq c(\|\mathcal{A}(t)u\|_{\mathscr{H}} + \|u\|_{\mathscr{H}}) \\ \|u\|_{\mathcal{V}} \leqq c(\|\mathcal{A}^{\pm}u\|_{\mathscr{H}} + \|u\|_{\mathscr{H}}) \end{cases} \quad \text{for any} \quad u \in \mathcal{V}$$

where c is independent of t.

LEMMA 4.1. *For any $u \in \mathcal{V}$, we have*

$$(4.9) \qquad |(\mathcal{A}(t)u, u)_{\mathscr{H}(t)} + (u, \mathcal{A}(t)u)_{\mathscr{H}(t)}| \leqq c\phi(t)(u, u)_{\mathscr{H}(t)}\,.$$

PROOF. By the integration by parts, we get for $u\in\mathcal{V}$,

$$(4.10)\quad (\mathcal{A}(t)u, u)_{\mathcal{H}(t)}+(u, \mathcal{A}(t)u)_{\mathcal{H}(t)}$$
$$=\Big(\sum_{j=1}^{n}A_j(x, t)\frac{\partial u}{\partial x_j}+B(x, t)u, u\Big)_{\mathcal{H}}+\Big(u, \sum_{j=1}^{n}A_j(x, t)\frac{\partial u}{\partial x_j}+B(x, t)u\Big)_{\mathcal{H}}$$
$$=\Big(u, \Big(\sum_{j=1}^{n}\frac{\partial A_j(x, t)}{\partial x_j}-B(x, t)-B(x, t)^*\Big)u\Big)_{\mathcal{H}}.$$

Combining (4.3), (H.4) and (H.5) with (4.10), we get (4.9). Q.E.D.

LEMMA 4.2. *There exists a constant $\delta>0$ such that for all λ real and $|\lambda|>\delta$, $\lambda I-\mathcal{A}(t)$ is a bijective mapping from $\mathcal{V}$ onto $\mathcal{H}(t)$. Moreover, we have*

$$(4.11)\qquad \|(\lambda I-\mathcal{A}(t))^{-1}\|_{\mathcal{H}(t)}\leqq(|\lambda|-\delta)^{-1}.$$

PROOF. By (4.8), it is clear that $\mathcal{A}(t)$ is a closed operator in $\mathcal{H}$. Moreover, we have

$$(4.12)\qquad \|(\lambda I-\mathcal{A}(t))u\|_{\mathcal{H}(t)}\geqq(|\lambda|-c')\|u\|_{\mathcal{H}(t)}$$

where c' satisfies $c\phi(t)\leqq c'$.

For the adjoint operator $(\lambda I-\mathcal{A}(t))^*$ of $\lambda I-\mathcal{A}(t)$ in $\mathcal{H}(t)$, we have the inequality of the same type as (4.12). So, we may obtain (4.11). Q.E.D.

This lemma means that (I.1) holds. (I.2) and (I.3) are proved readily by the smoothness of the coefficients.

The conditions (II.a)$\sim$(II.d) are also proved easily by (H.4) and (H.5). We remark here that the operators $\mathcal{A}^{\pm}$ are self-adjoint in $\mathcal{H}^{\pm}$.

DEFINITION 4.3. A function $u^{\pm}(x, t)\in\mathcal{E}_t^0(\mathcal{H})$ is said to be a weak solution of the problem $(4.2)^{\pm}$ if it satisfies the following.

$(4.13)^{\pm}$

$$\int_{R^n}{}^t u^{\pm}(x, t)\cdot E^{\pm}(x)\overline{\Phi(x, t)}dx-\int_{R^n}{}^t u^{\pm}(x, s)\cdot E^{\pm}(x)\overline{\Phi^{\pm}(x, s)}dx$$
$$=\int_s^t dr\int_{R^n}{}^t u^{\pm}(x, r)\cdot\overline{\Big(-E^{\pm}(x)\frac{\partial}{\partial r}\Phi^{\pm}(x, r)-\sum_{j=1}^{n}A_j^{\pm}(x)\frac{\partial}{\partial x_j}\Phi^{\pm}(x, r)-B^{\pm}(x)\Phi^{\pm}(x, r)\Big)}dx$$

for any $s, t\in\boldsymbol{R}^n$ and for any function $\Phi^{\pm}(x, t)\in\mathcal{E}_t^1(\mathcal{H})\cap\mathcal{E}_t^0(\mathcal{V})$.

Then, we have

THEOREM 4.4. *Let $u(t)\in\mathcal{E}_t^1(\mathcal{H})\cap\mathcal{E}_t^0(\mathcal{V})$ be a solution of (4.1) satisfying*

$$(4.14)\qquad \lim_{t\to-\infty}\|u(t)-u^-(t)\|_{\mathcal{H}}=0,$$

where $u^-(t)\in\mathcal{E}_t^0(\mathcal{H})$ be a weak solution of $(4.2)^-$.

Then, there exists uniquely a weak solution $u^+(x, t) \in \mathcal{E}_t^0(\mathcal{H})$ *of* $(4.2)^+$ *such that*

(4.15) $$\lim_{t\to+\infty} \|u(t)-u^+(t)\|_{\mathcal{H}}=0\,.$$

COROLLARY 4.5. *Suppose that* $u^{\pm}(t)$ *decays locally, i.e. for any bounded set* ω *in* $\boldsymbol{R}^n$, *we have*

(4.16) $$\lim_{t\to\pm\infty} \int_\omega |u^{\pm}(x, t)|^2 dx=0\,.$$

Then, we have

(4.17) $$\lim_{t\to\pm\infty} \int_\omega |u(x, t)|^2 dx=0\,.$$

This can be seen from the following inequality:

$$\int_\omega |u(x, t)|^2 dx \leqq 2\int_{R^n} |u(x, t)-u^{\pm}(x, t)|^2 dx+2\int_\omega |u^{\pm}(x, t)|^2 dx\,.$$

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