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# Cohomology of Lie algebras over a manifold, II

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In the present paper, we shall show that the methods developed in [6] also apply to the investigation of the cohomology groups of some Lie algebras over a manifold which have not been treated there. Specifically, we are concerned with two types of Lie algebras: that is, a Lie algebra associated to some differential representation of vector fields and the Lie algebra of vector fields on a complex analytic manifold. In these cases, we shall really establish the results similar to those which we have already obtained in case of vector fields [6].

The outline of this paper is described as follows. In Section 1 we introduce the notion of inductive differential complex which provides a background for our cohomological treatment of jet spaces. In Section 2 we first treat with the Lie algebra consisting of the first order differential operators on $M$. Next, we consider the situation where a differential representation $\varphi$ of $\mathfrak{A}(M)$ on $\Gamma(F)$ is given; here $\mathfrak{A}(M)$ denotes the Lie algebra of vector fields. Then we obtain a Lie algebra $\Gamma(\tau(M)\oplus F)$, the bracket rule being defined by

$$[\xi+v, \xi'+v']=[\xi, \xi']+(\varphi(\xi)v'-\varphi(\xi')v)\,,$$

where $\xi, \xi'\in\Gamma(\tau(M))$ and $v, v'\in\Gamma(F)$. If, moreover, another differential representation $\rho$ of $\mathfrak{A}(M)$ on $\Gamma(W)$ is given, then, associated to the lifting of $\rho$, the complex $\{C^p[\tau(M)\oplus F, W], d\}$ is canonically constructed. The cohomology group of this complex is finite-dimensional in each dimension whenever $\varphi$ and $\rho$ are tensorial and $M$ is compact. Section 3 deals with the complex analytic case. The complexification of the tangent bundle has the canonical splitting $T\oplus\bar{T}$, and thus brings about two Lie algebras $\Gamma(T)$ and $\Gamma(\bar{T})$. We, however, are mainly interested in the former Lie algebra $\Gamma(T)$ and its differential representation $\rho$ on $\Gamma(W)$. Since the jet bundle admits bi-degree according to $\partial$, $\bar{\partial}$-derivatives, the cohomology group $H^*(T, W)$ associated to $\rho$ becomes somewhat complicated. In fact, even under the assumptions that $M$ is compact and $\rho$ satisfies a favourable condition, we can only prove that $H^*(T, W)$ is expressed as an inductive limit space of a sequence of some cohomology groups, each of which possesses finite dimension.

We use the same notations as in [**6**]. In particular, the underlying manifold is always assumed to be smooth and have a countable basis.

## § 1. Inductive differential complexes.

Before entering the discussions on individual cases, we think it better to present a higher view of the notions related to the differential complexes, which will be useful for making our situation clearer. Let $E^0$ be a vector bundle over $M$ and $\{E^p\}$ $(p=1, 2, \cdots)$ a sequence of inductive vector bundles over $M$. Recall that in [**6**] we have introduced a Fréchet nuclear topology to the cross-section space of any inductive vector bundle.

DEFINITION 1.1. A sequence of linear maps

$$0 \longrightarrow \Gamma(E^0) \xrightarrow{d} \Gamma(E^1) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(E^p) \xrightarrow{d} \Gamma(E^{p+1}) \xrightarrow{d} \cdots$$

is called an *inductive differential complex* over $M$ if the following conditions are satisfied:

i) $d$ is a continuous map from $\Gamma(E^p)$ to $\Gamma(E^{p+1})$ $(p=0, 1, \cdots)$,

ii) $\operatorname{supp} dL \subset \operatorname{supp} L$ for any $L \in \Gamma(E^p)$ $(p=0, 1, \cdots)$,

iii) $d \circ d = 0$.

Given an inductive differential complex $\mathcal{E}=\{\Gamma(E^p), d\}$, we denote its cohomology group by $H^*(\mathcal{E})=\sum H^p(\mathcal{E})$. Since $\Gamma(E^p)$ is a nuclear space, in case where $d\Gamma(E^{p-1})$ is closed, each $H^p(\mathcal{E})$ also becomes a nuclear space being inherited from the nuclear structure of $\Gamma(E^p)$. But usually we shall treat $H^p(\mathcal{E})$ only as a vector space without regard to topology. Let

$$\underline{\mathcal{E}}:\ 0 \longrightarrow \underline{E}^0 \xrightarrow{d} \underline{E}^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{E}^p \xrightarrow{d} \underline{E}^{p+1} \xrightarrow{d} \cdots$$

be the sheaf of complex which is naturally obtained from $\mathcal{E}$ by taking germs of cross-sections of $\Gamma(E^p)$ $(p=0, 1, 2, \cdots)$. We denote the sheaf of cohomology of this complex by $\mathcal{H}^*(\underline{\mathcal{E}})=\sum \mathcal{H}^p(\underline{\mathcal{E}})$. Then by the similar discussions to those used in the proof of Lemma in [**6**; Section 4], we can establish

PROPOSITION 1.1. *There is a spectral sequence* $\{E_r^{p,q}, d_r\}$ *which has the following properties:*

i) *when* $r \to \infty$, $\sum E_r^{p,q}$ *converges to a graded module associated to* $H^*(\mathcal{E})$ *with some filtration;*

ii) $E_2^{p,q} \cong H^p(M, \mathcal{H}^q(\underline{\mathcal{E}}))$.

COROLLARY. *If* $\mathcal{H}^*(\underline{\mathcal{E}})$ *is a locally constant sheaf and if the stalk* $\mathcal{H}^*(\underline{\mathcal{E}})_x$ *at each point* $x \in M$ *satisfies* $\dim \mathcal{H}^*(\underline{\mathcal{E}})_x < +\infty$, *then we have* $\dim \mathcal{H}^*(\mathcal{E}) < +\infty$ *whenever* $M$ *is of finite type* (cf. [**6**; Section 4]).

DEFINITION 1.2. Assume that we have an inductive differential complex

$\mathcal{E}=\{\Gamma(E^p), d\}$. A filtration $\varinjlim E_k^p = E^p$, simultaneously given for $p=1, 2, \cdots$, is said to be *admissible* if for $k=0, 1, 2, \cdots$ the subcomplex

$$0 \longrightarrow \Gamma(E_k^0) \xrightarrow{d} \Gamma(E_k^1) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(E_k^p) \xrightarrow{d} \Gamma(E_k^{p+1}) \xrightarrow{d} \cdots$$

is well-defined.

Here recall that, for a vector bundle $E^0$, we put $E_k^0 = E^0$ $(k=0, 1, \cdots)$ and adopt this as a fixed filtration of $E^0$.

PROPOSITION 1.2. *Let $M$ be compact. Then, for any given inductive differential complex $\mathcal{E}=\{\Gamma(E^p), d\}$, we can find a large number of admissible filtrations to $\mathcal{E}$.*

PROOF. Let $E^p = \varinjlim E_k^p$ be any filtration of $E^p$. First note that, if $M$ is compact, we have $\Gamma(E^p) = \varinjlim \Gamma(E_k^p)$. Since $\Gamma(E^0)$ is a Fréchet space and $\Gamma(E^0) = \bigcup_k d^{-1}(\Gamma(E_k^1))$, we have $d\Gamma(E^0) \subset \Gamma(E_{k_0}^1)$ for some $k_0$. Put $E^1(0) = E_{k_0}^1$. Since $\Gamma(E^1(0))$ is a Fréchet space and $\Gamma(E^1(0)) = \bigcup_k d^{-1}(\Gamma(E_k^2)) \cap \Gamma(E^1(0))$, we have $d\Gamma(E^1(0)) \subset \Gamma(E_{k_0'}^2)$ for some $k_0'$. Put $E^2(0) = E_{k_0'}^2$. Repeating this argument, we can get a subcomplex

$$0 \longrightarrow \Gamma(E^0) \xrightarrow{d} \Gamma(E^1(0)) \xrightarrow{d} \Gamma(E^2(0)) \xrightarrow{d} \cdots.$$

Next take $k_1$ such that $E_{k_1}^1 \supsetneqq E^1(0)$ and put $E^1(1) = E_{k_1}^1$. Then, starting from $E^1(1)$, we can again carry out the above procedure to obtain a subcomplex

$$0 \longrightarrow \Gamma(E^0) \xrightarrow{d} \Gamma(E^1(1)) \xrightarrow{d} \Gamma(E^2(1)) \xrightarrow{d} \cdots$$

with $E^p(1) \supsetneqq E^p(0)$ $(p=1, 2, \cdots)$. Repeat the similar argument successively. Then we finally obtain a filtration

$$E^p(0) \subset E^p(1) \subset \cdots \subset E^p(k) \subset \cdots \longrightarrow E^p$$

for each $p$, which gives rise to an admissible filtration to $\mathcal{E}$.

It is clear that each step of such construction admits much arbitrariness, and thus many admissible filtrations are obtained in this way, which completes the proof.

If a filtration $E^p = \varinjlim E_k^p$ $(p=1, 2, \cdots)$ is admissible, then for any $k$ $(k=0, 1, 2, \cdots)$ we can consider the subcomplex $\mathcal{E}_k = \{\Gamma(E_k^p), d\}$ of $\mathcal{E}$. Moreover, each $\mathcal{E}_k$ yields the quotient complex

$$\mathcal{E}/\mathcal{E}_k : 0 \longrightarrow \Gamma(E^0/E_k^0) \xrightarrow{d} \Gamma(E^1/E_k^1) \xrightarrow{d} \cdots$$

which turns out to be an inductive differential complex. We have then the exact sequence of the cohomology groups

$$(1.1)\qquad \cdots \longrightarrow H^p(\mathcal{E}_k) \xrightarrow{(i_k)_*} H^p(\mathcal{E}) \xrightarrow{(\pi_k)_*} H^p(\mathcal{E}/\mathcal{E}_k) \xrightarrow{\delta} H^{p+1}(\mathcal{E}_k) \longrightarrow \cdots,$$

where $i_k$ denotes the injection $\mathcal{E}_k \to \mathcal{E}$ and $\pi_k$ the projection $\mathcal{E} \to \mathcal{E}/\mathcal{E}_k$. In case $M$ is compact, we have $\Gamma(E^p) = \varinjlim \Gamma(E^p_k)$, whence, if we denote $i^{k'}_k$ the injection map $\mathcal{E}_k \to \mathcal{E}_{k'}$ for $k < k'$, it follows

$$(1.2)\qquad H^*(\mathcal{E}) = \varinjlim \{H^*(\mathcal{E}_k)\,;\,(i^{k'}_k)_*\}\,.$$

DEFINITION 1.3. An admissible filtration $\varinjlim E^p_k = E^p$ is said to give the *stable range of cohomology group* if the injections

$$i_k : \mathcal{E}_k = \{\Gamma(E^p_k), d\} \subsetneq \mathcal{E} = \{\Gamma(E^p), d\} \qquad (k = 0, 1, 2, \cdots)$$

as the subcomplexes induce the isomorphism on cohomology level. If there is an admissible filtration satisfying this condition, we say that the inductive differential complex $\mathcal{E}$ (or its cohomology group) has the stable range.

From (1.1) it follows immediately that an admissible filtration $\varinjlim E^p_k = E^p$ $(p = 1, 2, \cdots)$ gives the stable range if and only if

$$(1.3)\qquad H^*(\mathcal{E}/\mathcal{E}_k) = 0$$

holds for $k = 0, 1, 2, \cdots$. In case $M$ is compact, in view of (1.2) the condition (1.3) is fulfilled whenever we have

$$H^*(\mathcal{E}_{k'}/\mathcal{E}_k) = 0 \qquad \text{for} \quad k < k'\,.$$

Observe that $\mathcal{E}_{k'}/\mathcal{E}_k$ is a differential complex in a usual sense, that is, it consists of the cross-section spaces of vector bundles and differential operators.

Localizing Definition 1.3, we can state the similar definition concerning the stable range in terms of the sheaf of complex $\underline{\mathcal{E}}$. More precisely, an admissible filtration $\varinjlim E^p_k = E^p$ is said to give the stable range of $\mathcal{H}^*(\underline{\mathcal{E}})$ if the injection $\underline{i}_k : \underline{\mathcal{E}}_k \subsetneq \underline{\mathcal{E}}$ gives rise to isomorphism on cohomology level for each $k$. We note that a spectral sequence $\{E^{p,q}_r(k), d_r(k)\}$ can be constructed for $\mathcal{E}_k$, corresponding to the one for $\mathcal{E}$ which is given in Proposition 1.1. Moreover, the injection $\underline{i}_k$ induces a homomorphism from $\{E^{p,q}_r(k), d_r(k)\}$ to $\{E^{p,q}_r, d_r\}$. Thus, from Proposition 1.1 we can deduce the following

PROPOSITION 1.3. *If an admissible filtration* $\varinjlim E^p_k = E^p$ $(p = 0, 1, 2, \cdots)$ *gives the stable range of* $\mathcal{H}^*(\underline{\mathcal{E}})$, *then this gives also the stable range of* $H^*(\mathcal{E})$.

Generally, it seems that an inductive differential complex does not necessarily have stable range. But in many cases we shall confront with the following situation: under a suitable choice of a filtration $\varinjlim E^p_k = E^p$, for any $p$ $(p = 1, 2, \cdots)$ we can find an integer $\alpha(p)$ such that a representative cocycle of any cohomology class of $H^p(\mathcal{E})$ is already chosen from $\Gamma(E^p_{\alpha(p)})$.

Let an inductive differential complex $\mathcal{E} = \{\Gamma(E^p), d\}$ be given. Take an admissible filtration $\varinjlim E_k^p = E^p$ $(p=1, 2, \cdots)$ to $\mathcal{E}$. If, for any $k$, the coboundary operators $d$ in the subcomplex

$$\cdots \longrightarrow \Gamma(E_k^p) \xrightarrow{d} \Gamma(E_k^{p+1}) \longrightarrow \cdots$$

are expressed as differential operators with order $r_0$, and $r_0$ is chosen to be independent of $k$, then we say that $E$ has a *finite order* $r_0$. Obviously, this definition does not depend on the choice of admissible filtrations.

A crucial definition on ellipticity of inductive differential complex is really open, because we have not known as yet how to approach the analytical theory on elliptic operators on inductive vector bundles. Nevertheless, many examples obtained hitherto suggests us that the following definition might be relevant to our purpose.

DEFINITION 1.4. An inductive differential complex $\{\Gamma(E^p), d\}$ with finite order $r_0$ is called *elliptic*, if there is a filtration $\varinjlim E_k^p = E^p$ for each $p$ such that

i) for any non-zero cotangent vector $\eta_x$ $(x \in M)$, the symbol sequence

$$\sigma_{r0}^{(k)}: \cdots \longrightarrow E_{k,x}^p \xrightarrow{\sigma^{(k)}(d,\, \eta)} E_{k,x}^{p+1} \longrightarrow \cdots$$

is well-defined and exact;

ii) the following diagram is commutative:

$$\begin{array}{ccccccc} \cdots \longrightarrow & E_{k+1,x}^p & \xrightarrow{\sigma^{(k+1)}(d,\, \eta)} & E_{k+1,x}^{p+1} & \longrightarrow \cdots \\ & \uparrow & & \uparrow & \\ \cdots \longrightarrow & E_{k,x}^p & \xrightarrow{\sigma^{(k)}(d,\, \eta)} & E_{k,x}^{p+1} & \longrightarrow \cdots \end{array} \tag{1.4}$$

where the vertical arrows indicate the injections.

In case where $\mathcal{E} = \{\Gamma(E^p), d\}$ is elliptic, passing to the inductive limit from the diagram (1.4), we can obtain an exact sequence

$$\sigma_{r_0}: \cdots \longrightarrow E_x^p \xrightarrow{\sigma(d,\, \eta)} E_x^{p+1} \longrightarrow \cdots,$$

which might be called the symbol sequence of $\mathcal{E}$. We here make a remark on a serious point of the above definition. We do not assume that a filtration $\varinjlim E_k^p = E^p$ gives rise to an admissible one. In fact, we shall see later that such an assumption is rather restrictive in general case. This, however, leads us to another strict definition.

DEFINITION 1.5. An admissible filtration $\varinjlim E_k^p = E^p$ is said to give the *elliptic range* to $\mathcal{E} = \{\Gamma(E^p), d\}$, if for each $k$ the subcomplex $\mathcal{E}_k$ becomes an elliptic complex over $M$. If there is an admissible filtration satisfying this condition, we say that the inductive differential complex $\mathcal{E}$ has the elliptic range.

It is easy to verify that, if $\mathcal{E}$ has the elliptic range, then $\mathcal{E}$ becomes elliptic. The next proposition is an immediate consequence of the well-known theorem on elliptic complexes [2].

PROPOSITION 1.4. *Let $M$ be compact. Assume that there is an admissible filtration* $\varinjlim E_k^p = E^p$ $(p=1, 2, \cdots)$ *to $\mathcal{E}$ which realizes both the stable range and the elliptic range simultaneously. Then we can conclude* $\dim H^p(\mathcal{E}) < +\infty$ *for* $p=0, 1, 2, \cdots$. *Moreover, if such a filtration satisfies a supplementary condition:*

*) *for any $k$, $E_k^p$ becomes zero when $p$ is sufficiently large, then we have* $\dim H^*(\mathcal{E}) < +\infty$.

According to [6], if we are given a Lie algebra $\Gamma(E)$ over $M$ and a differential representation $\varphi$ of $\Gamma(E)$ on $\Gamma(W)$, we can canonically construct the complex $\{C^p[E, W], d\}$, which gives an example of inductive differential complex introduced above. $C^p[E, W]$ has the jet filtration, which seems to be the standard one. Note that a remarkable property of the jet filtration is to satisfy the condition *) stated in Proposition 1.4. But we shall often need other filtrations of $C^p[E, W]$. Actually, for any positive integer $h$, put

$$C_k^p(h)(E, W) = C_{(p+k)h}^p(E, W),$$

where the subscript $(p+k)h$ in the right side means the degree of the jet filtration of $C^p(E, W)$. We have then a filtration

$$C_0^p(h)(E, W) \subset C_1^p(h)(E, W) \subset \cdots \subset C_k^p(h)(E, W) \subset \cdots \to C^p(E, W).$$

DEFINITION 1.6. The above filtration of $C^p(E, W)$ is called the *jet filtration with height h.*

Notation: $C_k^p(h)[E, W] = \Gamma(C_k^p(h)(E, W))$.

## § 2. Lie algebras associated to differential representations.

We shall first consider the Lie algebra $\boldsymbol{D}(1)$ over $M$, consisting of the first order differential operators on $M$. Specifically, putting $D(1) = \tau(M) \oplus \varepsilon^1$, we have $\boldsymbol{D}(1) = \Gamma(D(1))$ where the bracket operation is given by

$$[\xi+f, \eta+g] = [\xi, \eta] + (\xi g - \eta f), \qquad \xi, \eta \in \mathfrak{A}(M),\ f, g \in \Gamma(\varepsilon^1).$$

$\boldsymbol{D}(1)$ has a differential representation $\rho_0$ on $\Gamma(\varepsilon^1)$, defined by

$$\rho_0(\xi+f)(\phi) = \xi(\phi), \qquad \phi \in \Gamma(\varepsilon^1).$$

We shall make a remark on this differential representation. The space $\boldsymbol{D}(M)$, formed by all the differential operators on $M$, becomes a Lie algebra, the bracket operation being given by

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 .$$

According to our definition, $\boldsymbol{D}(M)$ does not become a Lie algebra over $M$ (cf. [**6**; Definition 2.1]), because we have no canonical method of identifying $\boldsymbol{D}(M)$ with the cross-section space of a certain vector bundle. Nevertheless, the investigation of $\boldsymbol{D}(M)$ seems to be very interesting itself, and our cohomology theory effectively applies to it (cf. [**1**]). From this viewpoint it is natural to regard $\boldsymbol{D}_1(M)$ as a subalgebra of $\boldsymbol{D}(M)$. Generally, for any non-negative integer $k$, we can consider the subspace $\boldsymbol{D}_k(M)$ of $\boldsymbol{D}(M)$, consisting of the differential operators with the order $k$. Note that $\boldsymbol{D}_0(M)$ is abelian and canonically identified with $\Gamma(\varepsilon^1)$. The differential representation $\rho_0$ is then interpreted as the adjoint representation of $\boldsymbol{D}_1(M)$ on $\boldsymbol{D}_0(M)$. We consider in general the adjoint representation $\rho_k$ of $\boldsymbol{D}_1(M)$ on $\boldsymbol{D}_k(M)$ for any $k = 0, 1, 2, \cdots$.

Put $D_k(M) = \mathrm{Hom}\,(J^k(\varepsilon^1), \varepsilon^1)$. Then we have $\boldsymbol{D}_k(M) = \Gamma(D_k(M))$, and we find that for each $k$, $\rho_k$ gives rise to a differential representation of $\boldsymbol{D}_1(M)$ to $\boldsymbol{D}_k(M)$. In view of Theorem 2.1 of [**6**], to these differential representations we can canonically associate differential complexes $\{C^p[D(1), D(k)], d\}$, the cohomology groups of which are completely determined as follows:

THEOREM 2.1. i) *$H^*(D(1), D(0)) \cong H^*(B(\tau^C) \times S^1, \boldsymbol{R})$, where $B(\tau^C)$ denotes the principal $U(n)$-bundle associated to the vector bundle $\tau(M) \otimes \boldsymbol{C}$, and $S^1$ denotes the circle.*

ii) *For each $k$, the injection $\iota_k : D(0) \to D(k)$ induces an isomorphism of the cohomology groups:*

$$(\iota_k)_* : H^*(D(1), D(0)) \cong H^*(D(1), D(k)) .$$

PROOF. i) For any $\xi + f \in \boldsymbol{D}(1)$ ($\xi \in \mathfrak{A}(M)$, $f \in \Gamma(\varepsilon^1)$), set locally

$$\omega_A^0(\xi + f) = \frac{\partial^{|A|} f}{\partial x^A}, \qquad \omega_A^\mu(\xi + f) = \frac{\partial^{|A|} \xi^\mu}{\partial x^A},$$

where $\xi = \sum_{\mu=1}^{n} \xi^\mu \partial/\partial x^\mu$ and $A$ ranges over the multi-indices. Then any $L \in C^p[D(1), D(0)]$ is locally expressed as a linear combination $\omega_{A_1}^{\nu_1} \wedge \cdots \wedge \omega_{A_p}^{\nu_p}$ ($\nu_1, \cdots, \nu_p = 0, 1, \cdots, n$) when we consider $C^p[D(1), D(0)]$ as $C^\infty(M)$-module. Put $\omega^0 = \omega_0^0$. Since $\omega^0$ is globally defined, we can decompose any $L \in C^p[D(1), D(0)]$ in the form

$$L = L_0 + \omega^0 \wedge L_1 , \tag{2.1}$$

where $L_0 \in C^p[D(1), D(0)]$ and $L_1 \in C^{p-1}[D(1), D(0)]$; the local expressions of $L_0$ and $L_1$ contain no exterior factor of $\omega^0$. Let $C_0^p$ be the linear subspace of $C^p[D(1), D(0)]$ spanned by $L_0$'s, and let $C_1^{p-1}$ be the one of $C^{p-1}[D(1), D(0)]$ spanned by $L_1$'s. Then (2.1) implies that we have an isomorphism

$$C^p \cong C_0^p \oplus C_0^{p-1}.$$

For convenience' sake, we write $\hat{d}$ for the coboundary operator of the differential complex $\{C^p[D(1), D(0)], d\}$. Then, locally, we have

$$\hat{d}f = \sum_{\mu=1}^{n} \frac{\partial f}{\partial x^\mu} \omega_0^\mu, \qquad \hat{d}\omega^0 = 0,$$

$$\hat{d}\omega_A^0 = \sum_{\substack{B \leqq A \\ |B| \neq 0}} \sum_{\lambda=1}^{n} \binom{A}{B} \omega_{A-B+\lambda}^0 \wedge \omega_B^\lambda \qquad (|A| \geqq 1),$$

$$\hat{d}\omega_A^\mu = \sum_{\substack{B \leqq A \\ |B| \neq 0}} \sum_{\lambda=1}^{n} \binom{A}{B} \omega_{A-B+\lambda}^\mu \wedge \omega_B^\lambda.$$

From these it follows immediately that $\{C_1^p, \hat{d}\}$ and $\{C_2^{p-1}, \hat{d}\}$ really become subcomplexes of $\{C^p, \hat{d}\}$, so that we have

$$\{C^p, \hat{d}\} = \{C_1^p, \hat{d}\} \oplus \{C_2^{p-1}, \hat{d}\}. \tag{2.2}$$

It is easy to verify that the canonical projection of $\boldsymbol{D}_1(M)$ on $\mathfrak{A}(M)$ naturally induces the injection $\mathcal{L} \to \{C^p, \hat{d}\}$, where $\mathcal{L}$ denotes the Losik complex (cf. [**5**], [**6**]). Consider the compositions of the maps

$$\varphi_1 : \mathcal{L} \longrightarrow \{C^p, \hat{d}\} \xrightarrow{\pi_1} \{C_1^p, \hat{d}\},$$

$$\varphi_2 : \mathcal{L} \longrightarrow \{C^p, \hat{d}\} \xrightarrow{\omega^0 \wedge} \{C^{p+1}, \hat{d}\} \xrightarrow{\pi_2} \{C_2^p, \hat{d}\}$$

where $\pi_i$ $(i=1, 2)$ denote the projections.

Now we can show that $\varphi_i$ $(i=1, 2)$ induces an isomorphism of the corresponding cohomology groups. By virtue of Proposition 1.1, to this aim, it suffices to show that the assertion is valid in the local situation. As in [**6**; Sections 3, 4], first apply Poincaré's Lemma to the complexes $\mathcal{L}$ and $\{C_i^p, \hat{d}\}$ $(i=1, 2)$, and then consider Hochshild-Serre's spectral sequences associated to the abelian subalgebra generated by $\{x^1\partial/\partial x^1, \cdots, x^n\partial/\partial x^n\}$. We compare these spectral sequences and apply the similar arguments to those found in [**6**; Sections 3, 4]. Then we can conclude that we have isomorphisms

$$(\varphi_i)_* : H^*(\mathcal{L}) \cong H^*(\{C_i^p, \hat{d}\}) \qquad (i=1, 2). \tag{2.3}$$

In these procedures, we have only to check the following point. Consider the cochain complex $\{E_0^{p,q}, d_0\}$, where $E_0^{p,q}$ is a linear space over $\boldsymbol{R}$, spanned by the elements

$$\omega_{\nu_1}^{\nu_1} \wedge \cdots \wedge \omega_{\nu_q}^{\nu_q} \wedge \omega_{A_1}^0 \wedge \cdots \wedge \omega_{A_s}^0 \wedge \omega_{A_{s+1}}^{\mu_{s+1}} \wedge \cdots \wedge \omega_{A_p}^{\mu_p} \tag{2.4}$$

$(1 \leqq \nu_1 < \cdots < \nu_q \leqq n;\ |A_i| \geqq 1;\ A_j \neq \mu_j\ (j=s+1, \cdots, p))$, and $d_0$ maps $E_0^{p,q}$ to $E_0^{p,q+1}$; moreover, $d_0$ operates multiplicative and we have

$$d_0\omega_A^0 = -\sum_{\lambda=1}^{n} (A)_\lambda \omega_\lambda^\lambda \wedge \omega_A^0$$

$$d_0\omega_A^\mu = \sum_{\lambda=1}^{n} \{\delta_\lambda^\mu - (A)_\lambda\} \omega_\lambda^\lambda \wedge \omega_A^\mu, \qquad \mu = 1, \cdots, n.$$

We denote by $E_0^{p,q}(A_1, \cdots, A_s)$ the subspace of $E_0^{p,q}$ spanned by the elements with the form (2.4), where multi-indices $A_1, \cdots, A_s$ are fixed. Then, for any $A_1, \cdots, A_s$, $\{E_0^{p,q}(A_1, \cdots, A_s), d_0\}$ becomes a subcomplex of $\{E_0^{p,q}, d_0\}$ and its cohomology groups all vanish when

$$-\sum_{i=1}^{s} (A_i)_\nu + \sum_{j=s+1}^{p} \{\delta_\nu^{\mu_j} - (A_j)_\nu\} = 0, \qquad \nu = 1, \cdots, n$$

do never hold for any choice of $A_{s+1}, \cdots, A_p$. From this it follows that the cohomology groups of $\{E_0^{p,q}(A_1, \cdots, A_s), d_0\}$ vanish when $s > 0$.

In conclusion, by (2.2) and (2.3) we obtain

$$H^p(D(1), D(0)) \cong H^p(L) \oplus H^{p-1}(L),$$

which, in turn, implies

$$H^*(D(1), D(0)) \cong H^*(B(\tau^C) \times S^1, \boldsymbol{R})$$

by virtue of Losik's theorem [**5**]. This completes the proof.

ii) Consider the local expression of $L \in C^p[D(1), D(k)]$. Locally, $L$ is written as

$$L = \sum_{|A| \leqq k} L_A \otimes \partial^A,$$

where $L_A \in C^p[D(1), D(0)]$ and $\partial^A$ denotes symbolically the local basis of the vector bundle $D(k)$ corresponding to the partial differentiation $\partial^{|A|}/\partial x_1^{a_1} \cdots \partial x_n^{a_n}$. The coboundary operator $d$ is then expressed as follows:

$$d\Big(\sum_{|A| \leqq k} L_A \otimes \partial^A\Big) = \sum_{|A| \leqq k} \hat{d} L_A \otimes \partial^A - \sum_{|A| \leqq k} \sum_{\mu=1}^{n} (A)_\mu \omega_\mu^\mu \wedge L_A \otimes \partial^A$$

$$- \sum_{|A| \leqq k} \sum_{\substack{0 < B \leqq A \\ B \neq \mu}} \sum_{\mu=1}^{n} \binom{A}{B} \omega_B^\mu \wedge L_A \otimes \partial^{A-B+\mu}$$

$$- \sum_{|A| \leqq k} \sum_{0 < B \leqq A} \binom{A}{B} \omega_B^0 \wedge L_A \otimes \partial^{A-B}.$$

From this it turns out that we can apply the arguments similar to those we have made in [**6**]. The assertion then follows immediately. This completes the proof.

It is easily checked that $\{C^p[D(1), D(k)], d\}$ has the elliptic jet range $l \geqq \operatorname{Max}\{1, k\}$.

In what follows, we shall give other examples of differential complexes which are constructed in the similar way to the above. Although the geometric meaning of these complexes is not so obvious, we find some interest in these complexes because they furnish us with examples of inductive differential complexes which are elliptic but possess no elliptic ranges.

We assume that we have two differential representations $\varphi$ and $\rho$ of $\mathfrak{A}(M)$ on $\Gamma(F)$ and $\Gamma(W)$, respectively, each of which is induced from a finite-dimensional representation of $L_0$ in the sense of Definition 4.1 in [**6**], where $L_0$ denotes the Lie algebra consisting of the formal vector fields without constant terms. Then we know that $\varphi$ and $\rho$ can be locally expressed in the following way:

$$(\varphi(\xi)v)^{\alpha}=\sum\xi^{\nu}\frac{\partial v^{\alpha}}{\partial x^{\nu}}+\sum H_{\mu\beta}^{X\alpha}\frac{\partial^{|X|}\xi^{\mu}}{\partial x^{X}}v^{\beta}\,,$$

$$(\rho(\xi)w)^{\lambda}=\sum\xi^{\nu}\frac{\partial w^{\lambda}}{\partial x^{\nu}}+\sum K_{\mu\kappa}^{Y\lambda}\frac{\partial^{|Y|}\xi^{\mu}}{\partial x^{Y}}w^{\kappa}\,.$$

Here we adopt the following notational conventions: $(x^1, \cdots, x^n)$ denote local coordinates on an open set $U$; $\sum\xi^{\nu}\partial/\partial x^{\nu}$ is the local expression of a vector field $\xi$ on $U$; $v^{\alpha}$ (resp. $w^{\lambda}$) denotes the $\alpha$-component of $v\in\Gamma(F)$ (resp. $w\in\Gamma(W)$) with reference to a suitable local-triviality of $F$ (resp. $W$) on $U$. Indices $\mu$, $\nu$ range over $1, \cdots, n$; $\alpha$, $\beta$ range over $1, \cdots, s$ ($s=\dim F$); $\lambda$, $\kappa$ range over $1, \cdots, t$ ($t=\dim W$). $H_{\mu\beta}^{X\alpha}$ and $K_{\mu\kappa}^{Y\lambda}$ denote some constants which are almost all zero; multi-indices $X$, $Y$ range over $|X|\geqq 1$ and $|Y|\geqq 1$.

We introduce a Lie algebra structure to $\Gamma(\tau(M)\oplus F)$ by setting

$$[\xi+v, \xi'+v']=[\xi, \xi']+(\varphi(\xi)v'-\varphi(\xi')v)\,,$$

where $\xi$, $\xi'\in\Gamma(\tau(M))$ and $v$, $v'\in\Gamma(F)$. The Lie algebra obtained in this way is called a *Lie algebra associated to the differential representation* $\varphi$. If $\varphi$ is a natural representation of $\mathfrak{A}(M)$ to $\Gamma(\varepsilon^1)$ as differential operators, then the Lie algebra associated to this representation is nothing but $\boldsymbol{D}(1)$. Define then a differential representation $\tilde{\rho}$ of $\Gamma(\tau(M)\oplus F)$ on $\Gamma(W)$, by

$$\tilde{\rho}(\xi+v)=\rho(\xi)\,.$$

We thus obtain a differential complex

$$\{C^p[\tau(M)\oplus F, W], d\}$$

associated to the differential representation $\tilde{\rho}$.

Let $\omega_A^{\mu}$ ($\mu=1, \cdots, n$; $|A|=0, 1, \cdots$) be the local base of $C^1[\tau(M), \varepsilon^1]$ on $U$ which we have already introduced in [**6**]. Similarly, let $\pi_C^{\alpha}$ ($\alpha=1, \cdots, s$; $|C|=0, 1, \cdots$) be the local base of $C^1[F, \varepsilon^1]$ on $U$. Then any $L\in C^p[\tau(M)\oplus F, W]$ can be locally expressed as

$$L=\sum L_{\mu\alpha}^{AC\lambda}(x)\omega_{A_1}^{\mu_1}\wedge\cdots\wedge\omega_{A_a}^{\mu_a}\wedge\pi_{C_1}^{\alpha_1}\wedge\cdots\wedge\pi_{C_b}^{\alpha_b}\otimes e_{\lambda}\,. \tag{2.5}$$

where $a+b=p$ and $L^{AC\lambda}_{\mu\alpha}(x)$ is a smooth function on the reference neighborhood; $e_\lambda$ denotes the local base of $W$. To find the explicit form of the coboundary operator in the complex $\{C^p[\tau(M)\oplus F, W], d\}$, we shall first consider the simple case where $W=\varepsilon^1$ and $\rho$ is a representation of $\mathfrak{A}(M)$ on $\Gamma(\varepsilon^1)$ given by a natural operation $\rho(\sum\xi^\mu\partial/\partial x^\mu)f=\sum\xi^\mu\partial f/\partial x^\mu$. In this case, we denote the coboundary operator by $\tilde{d}$:

$$\tilde{d}: C^p[\tau(M)\oplus F, \varepsilon^1] \longrightarrow C^{p+1}[\tau(M)\oplus F, \varepsilon^1], \qquad p=0, 1, \cdots.$$

Note that $\{C^p[\tau(M)\oplus F, \varepsilon^1], \tilde{d}\}$ is a multiplicative complex. Hence $\tilde{d}$ is completely determined by the way how it acts on $f$ $(\in\Gamma(\varepsilon^1)=C^0[\tau(M)\oplus F, \varepsilon^1])$, $\omega^\mu_A$ and $\pi^\alpha_C$. But this is easily found by direct calculations:

$$\tilde{d}f=\sum_{\mu=1}^n\frac{\partial f}{\partial x^\mu}\omega^\mu_0, \qquad \tilde{d}\omega^\mu_0=0,$$

$$\tilde{d}\omega^\mu_A=\sum_{\nu=1}^n\sum_{0<B\leqq A}\binom{A}{B}\omega^\mu_{A-B+\nu}\wedge\omega^\nu_B \qquad (|A|\geqq 1), \tag{2.6}$$

$$\tilde{d}\pi^\alpha_C=\sum_{\nu=1}^n\sum_{0<D\leqq C}\binom{C}{D}\pi^\alpha_{C-D+\nu}\wedge\omega^\nu_D+\sum_{X,\mu,\beta}\sum_{0\leqq D\leqq C}H^{X\alpha}_{\mu\beta}\binom{C}{D}\pi^\beta_{C-D}\wedge\omega^\mu_{D+X}.$$

Now we come back to the consideration of $\{C^p[\tau(M)\oplus F, W], d\}$. Using $\tilde{d}$ and writing $L=\sum L^\lambda\otimes e_\lambda$ for (2.5), we can express the local form of $dL$ as follows:

$$dL=\sum\tilde{d}L^\lambda\otimes e_\lambda+\sum\Big(\sum_{Y,\mu,\kappa}K^{Y\lambda}_{\mu\kappa}\omega^\mu_Y\wedge L^\kappa\Big)\otimes e_\lambda. \tag{2.7}$$

Here each $L^\lambda$ is regarded as an element of $C^p[\tau(M)\oplus F, \varepsilon^1]|U$.

Let $h_0$ be the smallest integer to satisfy $H^{X\alpha}_{\mu\beta}=0$ when $|X|\geqq h_0$, and let $k_0$ be the smallest integer to satisfy $K^{Y\lambda}_{\mu\kappa}=0$ when $|Y|\geqq k_0$. Then, from (2.6) and (2.7) we can deduce

PROPOSITION 2.1. *The jet filtration with height* $h_0$

$$C^p_l(h_0)[\tau(M)\oplus F, W]\subset C^p_{l+1}(h_0)[\tau(M)\oplus F, W]\subset\cdots\longrightarrow C^p[\tau(M)\oplus F, W]$$

$(p=1, 2, \cdots)$ *gives rise to an admissible filtration of* $\{C^p[\tau(M)\oplus F, W], d\}$ *for* $l\geqq k_0$.

On the other hand, it is immediately follows from (2.6), (2.7) that the coboundary operator $d$ is expressed as the first order differential operator and its principal part is locally given in the form

$$\sum L^{AC\lambda}_{\mu\alpha}(x)\Omega^\mu_A\wedge\Pi^\alpha_C\otimes e_\lambda\longrightarrow\sum\frac{\partial L^{AC\lambda}_{\mu\alpha}}{\partial x^\mu}\omega^0_\mu\wedge\Omega^\mu_A\wedge\Pi^\alpha_C\otimes e_\lambda,$$

where we write

$$\Omega^\mu_A=\omega^{\mu_1}_{A_1}\wedge\cdots\wedge\omega^{\mu_a}_{A_a},$$

$$\Pi^\alpha_C=\pi^{\alpha_1}_{C_1}\wedge\cdots\wedge\pi^{\alpha_b}_{C_b}.$$

Hence, with reference to the jet filtration, for each $k$ the symbol sequence

$$\sigma_k(d, \eta) : \cdots \longrightarrow C_k^p(\tau(M)\oplus F, W)_x \xrightarrow{\sigma_k} C_k^{p+1}(\tau(M)\oplus F, W)_x \longrightarrow \cdots$$

is well-defined and $\sigma_k$ is given by the exterior product of $\eta$, where $\eta$ is a non-zero cotangent vector at $x$. It follows that these sequences satisfy the conditions stated in Definition 1.4, whence we have:

PROPOSITION 2.2. *$\{C^p[\tau(M)\oplus F, W], d\}$ is elliptic in the sense of Definition* 1.4.

Note that, if $h_0 \geqq 1$, the jet filtration with height $h_0$ does never induce the symbol sequence with exactness.

Let $\varphi$ and $\rho$ be the finite-dimensional representations of $G(h)$ $(h \geqq h_0, k_0)$ (as to the definition of $G(h)$, cf. [**6**; Section 4]), each of which induces $\varphi$ and $\rho$ in the sense of Definition 4.1 in [**6**]. We say that $\varphi$ and $\rho$ are tensorial, if $\varphi | \mathfrak{gl}(n ; \boldsymbol{R})$ and $\rho | \mathfrak{gl}(n ; \boldsymbol{R})$ are obtained as tensorial representations on certain tensor spaces. In case where $\varphi$ is tensorial, we say that $\varphi$ is of covariant type, if each irreducible constituent of $\varphi | \mathfrak{gl}(n ; \boldsymbol{R})$ is obtained from a decomposition of some covariant tensor representation. Similarly, we can formulate the definition that $\rho$ is of contravariant type.

THEOREM 2.2. *Assume that $\varphi$ and $\rho$ be tensorial.*

i) *In case where $M$ is of finite type, we have*

$$\dim H^p(\tau(M)\oplus F, W) < +\infty, \qquad p = 0, 1, 2, \cdots;$$

*moreover, if $\varphi$ is of covariant type, we have*

$$\dim H^*(\tau(M)\oplus F, W) < +\infty .$$

ii) *If $\varphi$ is of covariant type and $\rho$ of contravariant type such that $\rho | \mathfrak{gl}(n ; \boldsymbol{R})$ is non-trivial, then*

$$H^*(\tau(M)\oplus F, W) = 0$$

*holds.*

Theorem can be proved by applying the similar methods to those we have often used previously. We only note the following facts: That $\varphi$ and $\rho$ are tensorial means $H_{\mu\alpha}^{\mu\alpha}$ and $K_{\mu\lambda}^{\mu\lambda}$ are all integers; moreover, we have $H_{\mu\alpha}^{\mu\alpha} \geqq 0$ for any $\alpha$, $\mu$ if $\varphi$ is of covariant type and $K_{\mu\lambda}^{\mu\lambda} \leqq 0$ if $\rho$ is of contravariant type. We leave the details to the readers.

## § 3. Complex manifolds.

Let $M$ be a complex manifold with $\dim_C M = n$. The complexification $\tau(M)\otimes \boldsymbol{C}$ of $\tau(M)$ is canonically decomposed as

$$\tau(M)\otimes \boldsymbol{C}=T\oplus\bar{T}$$

according to the complex structure of $M$. Let $E$ be a complex vector bundle over $M$. Then we can introduce the jet bundle $J^{a,b}(E)$ of $E$ with type $(a, b)$ as follows: For any point $P\in M$, set

$$I_P^{a,b}=\{f \mid f\in E,\ \partial^A f(P)=0,\ \bar{\partial}^B f(P)=0$$
$$\text{for}\quad |A|\leqq a,\ |B|\leqq b\}$$

where $A=(\alpha_1, \cdots, \alpha_n)$, $B=(b_1, \cdots, b_n)$ are multi-indices and

$$\partial^A f=\frac{\partial^{|A|}f}{(\partial z^1)^{\alpha_1}\cdots(\partial z^n)^{\alpha_n}},\quad \bar{\partial}^B f=\frac{\partial^{|B|}f}{(\partial \bar{z}^1)^{\beta_1}\cdots(\partial \bar{z}^n)^{\beta_n}}$$

for local analytic coordinates $(z^1, \cdots, z^n)$ around $P$. Put

$$Z_P^{a,b}=I_P^{a,b}\cdot\Gamma(E)$$

and define

$$J^{a,b}(E)_P=\Gamma(E)/Z_P^{a,b}.$$

Then

$$J^{a,b}(E)=\bigcup_{P\in M}J^{a,b}(E)_P$$

admits a structure of smooth vector bundle over $M$. Given a local triviality of $E$ with local basis $e^1, \cdots, e^w$ on an analytic coordinates neighborhood $U$, any $f\in\Gamma(E|U)$ is written as $\sum f_\lambda(z, \bar{z})e^\lambda$, whence a triviality of $J^{a,b}(E)$ on $U$ is canonically induced by arranging the partial derivatives of $f_\lambda$:

$$(\cdots, \partial^A\bar{\partial}^B f_\lambda(z, \bar{z}), \cdots),\qquad |A|\leqq a,\ |B|\leqq b.$$

If $a\leqq a'$ and $b\leqq b'$, then there is a natural injection

$$I_P^{a',b'}\longrightarrow I_P^{a,b},\qquad Z_P^{a',b'}\longrightarrow Z_P^{a,b}\qquad (P\in M),$$

so that we have a bundle homomorphism

$$J^{a',b'}(E)\longrightarrow J^{a,b}(E)\longrightarrow 0\qquad \text{(exact)}.$$

If $E$ is a complex analytic vector bundle, then $J^{a,0}(E)$ becomes a complex analytic vector bundle for each $a$.

Throughout this section we assume that the underlying field is $\boldsymbol{C}$.

Let $W$ be a vector bundle over $M$. Since $J^{a,b}(E)$ $(a, b=0, 1, \cdots; a+b\leqq k)$ gives rise to a bi-filtration of $J^k(E)$, the inductive vector bundle $C^p(E, W)$ has also a bi-filtration

$$C^p(E, W)=\varinjlim_{a,b}\operatorname{Hom}(\Lambda^p J^{a,b}(E), W).$$

Using this expression, for $b=0, 1, 2, \cdots$ put

$$C^p(b)(E, W)=\varinjlim_a \operatorname{Hom}(\Lambda^p J^{a,b}(E), W)$$

and

$$C^p(b)[E, W]=\Gamma(C^p(b)(E, W)).$$

A differential cochain belonging to $C(b)[E, W]=\sum_p C^p(b)[E, W]$ is called a differential cochain with type $b$. Note that $L\in C^p(b)[E, W]$ if and only if for any point $P\in M$ we have

$$L(\xi_1, \cdots, \xi_p)_P=0$$

whenever some $\xi_i$ satisfies $\bar{\partial}^B\xi_i(P)=0$ for $|B|\leqq b$. If $M$ is compact, the filtration

$$C^p(0)[E, W]\subset C^p(1)[E, W]\subset \cdots \subset C^p(b)[E, W]\subset \cdots$$

satisfies $\bigcup_{b=0}^{\infty} C^p(b)[E, W]=C^p[E, W]$.

Set

$$C^p_a(b)[E, W]=\operatorname{Hom}(\Lambda^p J^{a,b}(E), W).$$

Then $\varinjlim_a C^p_a(b)(E, W)=C^p(b)(E, W)$, so that $C^p(b)(E, W)$ becomes an inductive vector bundle; the referred filtration is called the $\partial$-jet filtration of $C^p(b)(E, W)$. We often write

$$C^p_a(b)[E, W]=\Gamma(C^p_a(b)(E, W)).$$

Apart from the general treatment, henceforce we shall consider the Lie algebra $\mathfrak{A}_\partial(M)=\Gamma(T)$ over $M$. Actually, the purpose of this section is to study $\mathfrak{A}_\partial(M)$, by showing how to follow the discussions similar to those which we have already done in case of $\mathfrak{A}(M)$ (cf. [**6**]). As we shall see, new phenomena arise from the fact that the cohomology groups admit, in a sense, bi-filtration.

First consider the local situation. We denote by $\boldsymbol{C}[[z, \bar{z}]]$ the local algebra of formal power series in $z=(z_1, \cdots, z_n)$ and $\bar{z}=(\bar{z}_1, \cdots, \bar{z}_n)$, and by $\boldsymbol{C}[[z]]$ the local algebra of formal power series in $z=(z_1, \cdots, z_n)$. The Lie algebra of formal $\partial$-vector fields is, by definition, the Lie algebra consisting of the elements

$$\sum_{\mu=1}^{n} a^\mu(z, \bar{z})\frac{\partial}{\partial z^\mu}, \qquad a^\mu\in \boldsymbol{C}[[z, \bar{z}]]$$

with the bracket rule

$$\Big[\sum_\mu a^\mu\frac{\partial}{\partial z^\mu}, \sum_\mu b^\mu\frac{\partial}{\partial z^\mu}\Big]=\sum_\mu\Big(\sum_\nu a^\nu\frac{\partial b^\mu}{\partial z^\nu}-\sum_\nu b^\nu\frac{\partial a^\mu}{\partial z^\nu}\Big)\frac{\partial}{\partial z^\mu}.$$

We denote this Lie algebra by $\mathfrak{a}_\partial$. (It seems to be apparently better to use the notation $(\mathfrak{a}_n)_\partial$ instead of $\mathfrak{a}_\partial$. But we wish to simplify our notation.) We regard $\mathfrak{a}_\partial$ as a finitely generated $\boldsymbol{C}[[z, \bar{z}]]$-module so that we may and do introduce the Krull topology to $\mathfrak{a}_{\partial}$. Evidently we have $[\bar{z}^m\xi, \eta]=[\xi, \bar{z}^m\eta]=\bar{z}^m[\xi, \eta]$ for $\xi, \eta\in\mathfrak{a}_{\partial}$. Hence, for any non-negative integer $b$, if we set

$$\mathfrak{a}_{\partial}(b)=\{\xi=\sum_{\mu=1}^{n} a^{\mu}(z, \bar{z})\,\partial/\partial z^{\mu} \mid a^{\mu}(z, \bar{z})=\sum_{|B|\leqq b} a_{AB}z^A\bar{z}^B\}\,,$$

then $\mathfrak{a}_{\partial}(b)$ becomes a subalgebra of $\mathfrak{a}_{\partial}$ and we have a sequence of the canonical surjective homomorphisms

$$\mathfrak{a}_{\partial}(0) \longleftarrow \mathfrak{a}_{\partial}(1) \longleftarrow \cdots \longleftarrow \mathfrak{a}_{\partial}(b) \longleftarrow \cdots .$$

Really we have $\mathfrak{a}_{\partial}=\varprojlim \mathfrak{a}_{\partial}(b)$.

Let $\tilde{L}_0$ be a subalgebra of $\mathfrak{a}_{\partial}$, consisting of elements with the form $\xi=\sum a^{\mu}_{AB}z^A\bar{z}^B\,\partial/\partial\bar{z}^{\mu}$ where $|A|+|B|>0$. That is, any element belonging to $\tilde{L}_0$ does not contain the constant term. Put $\tilde{L}_0(b)=\tilde{L}_0\cap\mathfrak{a}_{\partial}(b)$. Then $\tilde{L}_0(b)$ is a subalgebra of $\mathfrak{a}_{\partial}(b)$, and we have a commutative diagram

$$\begin{array}{ccccccccc}
\tilde{L}_0(0) & \longleftarrow & \tilde{L}_0(1) & \longleftarrow & \cdots & \longleftarrow & \tilde{L}_0(b) & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \\
\mathfrak{a}_{\partial}(0) & \longleftarrow & \mathfrak{a}_{\partial}(1) & \longleftarrow & \cdots & \longleftarrow & \mathfrak{a}_{\partial}(b) & \longleftarrow & \cdots ,
\end{array}$$

where the vertical arrows mean the natural injections. We have $\tilde{L}_0=\varprojlim \tilde{L}_0(b)$.

Since we have canonical isomorphisms

$$\boldsymbol{C}[[z]]\cong\boldsymbol{R}[[x]]\otimes\boldsymbol{C}\,, \qquad x=(x_1, \cdots, x_n)$$

and

$$\mathfrak{a}_{\partial}(0)\cong\mathfrak{a}_n\otimes\boldsymbol{C}\,,$$

the decreasing sequence of subalgebras of $\mathfrak{a}_n$:

$$\mathfrak{a}_n\supset L_0\supset L_1\supset\cdots\supset L_k\supset\cdots$$

is transferred to a sequence of subalgebras of $\mathfrak{a}_{\partial}(0)$ via the above isomorphism. In fact, $\tilde{L}_0(0)$ corresponds to $L_0$. We observe that the results concerning $L_0$, being found in [**6**; Section 3], are also formulated and valid in the same way for $\tilde{L}_0(0)$. In particular, if a finite-dimensional representation $\varphi$ of $\tilde{L}_0(0)$ on $V$ is decomposable (i.e., $\varphi|\mathfrak{gl}(n;C)$ is completely reducible), then the complex $\{C^p(\tilde{L}_0(0), V), d\}$ associated to $\varphi$ has the finite-dimensional cohomology group. Moreover, the stable range is given in the same form as in [**6**; Theorem 3.1].

DEFINITION 3.1. A finite-dimensional representation $\phi$ of $\tilde{L}_0$ on $V$ is said to be *complex analytic* if $\phi$ is obtained as the lifting of some representation $\varphi$ of $\tilde{L}_0(0)$ on $V$; $\phi$ is said to be *decomposable* if $\varphi$ is so.

That is, $\phi$ is complex-analytic if the following diagram is commutative:

$$\begin{tikzcd} \widetilde{L}_0 \arrow[r, "\psi"] \arrow[d, "\pi"'] & \mathrm{Hom}\,(V,\ V) \\ \widetilde{L}_0(0) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Here $\pi$ denotes the canonical projection. Assume that a decomposable complex analytic representation $\psi$ of $\widetilde{L}_0$ on $V$ be given. If we write componentwise $\{a^{\mu}_{AB}\}$ for any element of $\widetilde{L}_0$, then, under a suitable choice of base $\{e_\alpha\}$ of $V$, $\psi$ admits an expression

$$\psi(\{a^{\mu}_{AB}\})=\sum \Psi^{XY\alpha}_{\mu\ \ \beta} a^{\mu}_{XY} e_\alpha \otimes e^{\beta}\,, \tag{3.1}$$

where the constants $\Psi^{XY\alpha}_{\mu\ \ \beta}$ satisfy

$$\Psi^{XY\alpha}_{\mu\ \ \beta}=0\,, \qquad \text{if} \quad |Y|>0\,,$$

$$\Psi^{\mu 0\alpha}_{\mu\ \ \beta}=0\,, \qquad \text{if} \quad \alpha\neq\beta\,,$$

and $\{e^{\beta}\}$ denotes the dual base of $\{e_\alpha\}$. Note that $\psi$ also gives rise to a representation of each $\widetilde{L}_0(b)$ on $V$. Hence, according to the cohomology theory of Lie algebras, we can obtain complexes

$$\{C^p(\widetilde{L}_0(b),\ V),\ d\}\,, \qquad b=0,\ 1,\ 2,\ \cdots$$

$$\{C^p(\widetilde{L}_0,\ V),\ d\}\,,$$

where the cochains to which we refer are assumed to be continuous. The corresponding cohomology groups are denoted by $H^*(\widetilde{L}_0(b),\ V)$ and $H^*(\widetilde{L}_0,\ V)$. For $b<b'$, the surjective map $i^{b'}_b : \widetilde{L}_0(b')\to\widetilde{L}^0(b)$ induces the homomorphism

$$(i^{b'}_b)^* : H^*(\widetilde{L}_0(b),\ V) \longrightarrow H^*(\widetilde{L}_0(b'),\ V)\,.$$

We wish to establish the following

THEOREM 3.1. *We have*

$$\dim H^*(\widetilde{L}_0(b),\ V)<+\infty\,, \qquad b=0,\ 1,\ 2,\ \cdots,$$

*and*

$$H^*(\widetilde{L}_0,\ V)=\varinjlim \{H^*(\widetilde{L}_0(b),\ V),\ (i^{b'}_b)^*\}\,.$$

PROOF. We have already remarked that $\dim H^*(\widetilde{L}_0(0),\ V)<+\infty$ holds. Besides, it is easily seen that $C^p(\widetilde{L}_0,\ V)=\varinjlim C^p(\widetilde{L}_0(b),\ V)$; from this the second assertion immediately follows. Thus it is sufficient to show that, for $b\geqq 1$, $\dim H^*(\widetilde{L}_0(b),\ V)<+\infty$ always holds. Set

$$\theta^{\mu}_{AB}\Big(\sum a^{\nu}(z,\ \bar{z})\frac{\partial}{\partial z^{\nu}}\Big)=\frac{\partial^{|A|+|B|}a^{\mu}}{\partial z^{A}\partial\bar{z}^{B}}(z,\ \bar{z})=0\,.$$

Then any element $\eta$ of $C^p(\widetilde{L}_0(b),\ V)$ is expressed in the following way:

$$\eta=\sum\gamma_{\mu}^{AB}\Theta_{AB}^{\mu}{}^{\alpha}\otimes e_{\alpha}\,,$$

where $\mu=1, \cdots, n$, $A=(A_1, \cdots, A_p)$, $B=(B_1, \cdots, B_p)$ with $|A|+|B|\neq 0$, $|B|\leqq b$, $\gamma_{\mu}^{AB}$ are constants and

$$\Theta_{AB}^{\mu}=\theta_{A_1B_1}^{\mu_1}\wedge\cdots\wedge\theta_{A_pB_p}^{\mu_p}\,.$$

In order to find the explicit form of the coboundary operator $d$ in $\{C^p(\tilde{L}_0(b), V), d\}$, we shall first consider the complex $\{C^p(\tilde{L}_0(b), C), \tilde{d}\}$ associated to the trivial representation. This complex is multiplicative and we have

$$\tilde{d}\theta_{A,B}^{\mu}=\sum_{\substack{C\leqq A, D\leqq B\\ |C|+|D|\neq 0}}\sum_{\nu=1}^{n}\binom{A}{C}\binom{B}{D}\theta_{A-C+\nu,\,B-D}^{\mu}\wedge\theta_{C,D}^{\nu}\,, \tag{3.2}$$

as is easily verified. Let the representation $\psi$ express in the form (3.1) and put $C_{\mu}^{\alpha}=\Psi_{\mu}^{\mu0\alpha}{}_{\alpha}$. Denoting any element of $C^p(\tilde{L}_0(b), V)$ by $\sum_{\alpha=1}^{n}L^{\alpha}\otimes e_{\alpha}$ ($L^{\alpha}\in C^p(\tilde{L}_0(b), C)$), we have then

$$d(\sum_{\alpha=1}^{v}L^{\alpha}\otimes e_{\alpha})=\sum_{\alpha=1}^{v}\tilde{d}L^{\alpha}\otimes e_{\alpha}+\sum_{\alpha=1}^{v}(\sum_{\mu=1}^{n}C_{\mu}^{\alpha}\theta_{\mu,0}^{\mu}\wedge L^{\alpha})\otimes e_{\alpha} \tag{3.3}$$

$$+\sum_{\alpha,\beta=1}^{v}\Sigma'(\Psi_{\mu}^{XY\alpha}{}_{\beta}\theta_{XY}^{\mu}\wedge L^{\beta})\otimes e_{\alpha}$$

where $\Sigma'$ means the summation extended over the indices $Z$, $Y$, $\mu$, omitting the case $(X, Y, \mu)=(\mu, 0, \mu)$ $(\mu=1, \cdots, n)$. Consider Hochshild-Serre's spectral sequences $\{\tilde{E}_r^{p,q}, \tilde{d}_r\}$ and $\{E_r^{p,q}, d_r\}$ of $\{C^p(\tilde{L}_0(b), C), \tilde{d}\}$ and $\{C^p(\tilde{L}_0(b), V), d\}$ respectively, associated to the abelian subalgebra generated by $\{z^1\partial/\partial z^1, \cdots, z^n\partial/\partial z^n\}$ of $\tilde{L}_0(b)$. Then, in view of (3.2) and (3.3), we obtain

$$\tilde{d}_0\theta_{A,B}^{\mu}=\sum_{\nu=1}^{n}(\delta_{\nu}^{\mu}-(A)_{\nu})\theta_{\nu,0}^{\nu}\wedge\theta_{A,B}^{\mu} \tag{3.4}$$

and

$$d_0(\sum_{\alpha=1}^{v}L^{\alpha}\otimes e_{\alpha})=\sum_{\alpha=1}^{v}(\tilde{d}_0L^{\alpha}+\sum_{\mu=1}^{n}C_{\mu}^{\alpha}\theta_{\mu,0}^{\mu}\wedge L^{\alpha})\otimes e_{\alpha}\,.$$

Therefore, in order to get $E_1$-term, we have only to study the respective complexes formed by each component $L^{\alpha}$ occurring in $E_0$-terms. More precisely, we consider the complex $\{C^p(\tilde{L}(b), \boldsymbol{C}), d_0\}$, where $d_0$ is given by

$$d_0L=\tilde{d}_0L+\sum_{\mu=1}^{n}C_{\mu}^{\alpha}\theta_{\mu,0}^{\mu}\wedge L\,.$$

We notice that this situation is utterly similar to what we have treated in [**5**; Section 3].

We should, however, pay attention to a serious distinction arising in this case, which is caused by the fact that in the right side of (3.4) $(A)_{\nu}$ may happen to be all zero for $\nu=1, \cdots, n$. To deal with this situation, we shall introduce the subcomplex

$$\{C^p(\mu_1, B_1; \cdots; \mu_s, B_s), d_0\}$$

of $\{C^p(\tilde{L}(b), C), d_0\}$, which consists of the cochains with the form

$$\theta^{\mu_1}_{0,B_1} \wedge \cdots \wedge \theta^{\mu_s}_{0,B_s} \wedge \eta$$

where $\eta$ does not contain any exterior factor of $\theta^{\mu}_{0,B}$ $(\mu_1, \cdots, n; 0<|B|\leqq b)$. Then, by (3.2) and (3.3), $\{C^p(\mu_1, B_1; \cdots; \mu_s, B_s), d_0\}$ really turns out to become a subcomplex. Moreover, using the arguments similar to those found in [**5**; Section 3], we can show that the $p$-dimensional cohomology group of this subcomplex is identified with a subspace of $C^p(\mu_1, B_1; \cdots; \mu_s, B_s)$, generated by the elements

$$\theta^{\mu_1}_{0,B_1} \wedge \cdots \wedge \theta^{\mu_s}_{0,B_s} \wedge \theta^{\nu_1}_{A_1 B_1} \wedge \cdots \wedge \theta^{\nu_t}_{A_t B_t}, \qquad s+t=p, \ |A_i|\geqq 1,$$

satisfying

$$\delta^{\mu_1}_{\lambda} + \cdots + \delta^{\mu_s}_{\lambda} + \sum_{i=1}^{t} (\delta^{\nu_i}_{\lambda} - (A_i)_{\lambda}) + C^{\alpha}_{\lambda} = 0, \qquad \lambda = 1, 2, \cdots, n.$$

From this we can deduce that the total cohomology group of $\{C^p(\mu_1, B_1; \cdots; \mu_s, B_s), d_0\}$ is finite-dimensional. Note that we have

$$C^p(\tilde{L}(b), C) = \sum C^p(\mu_1, B_1; \cdots; \mu_s, B_s),$$

where the summands appearing in the right side are finite in number since there is only a finite number of $\theta^{\mu}_{B}$ with $\mu=1, \cdots, n$ and $1\leqq|B|\leqq b$.

These results together yield

$$\dim \sum_{p,q} E_1^{p,q} < +\infty,$$

from which the assertion follows. This completes the proof.

Now we turn attention to the global aspects. Let $G_{\partial}(h)$ be the complex Lie group, consisting of the $h$-jets which are induced from complex analytic local-diffeomorphisms of $\boldsymbol{C}^n$ around and fixing the origin. The structural group of the complex analytic vector bundle $J^{h-1,0}(T)$ are reducible to $G_{\partial}(h)$, so that we can consider the principal $G_{\partial}(h)$-bundle $P_{\partial}(h)$, associated to $J^{h-1,0}(T)$. Assume that a finite-dimensional complex representation $\rho$ of $G_{\partial}(h)$ on $V$ be given. Then a complex analytic vector bundle

$$W = P_{\partial}(h) \times_{\rho} V$$

is induced. It is easily checked that the Lie algebra of $P_{\partial}(h)$ is canonically isomorphic to $\tilde{L}_0(0)/\tilde{L}_h(0)$, where $\tilde{L}_h(0)$ denotes the Lie subalgebra of $\tilde{L}_0(0)$ formed by the formal $\partial$-vector fields $\eta$ with $\partial^A\eta/\partial z^A|_{z=0}=0$ for $|A|=0, 1, \cdots, h$. Hence, $\rho$ gives rise to a Lie algebra representation $d\rho$ of $\tilde{L}_0(0)/\tilde{L}_h(0)$ on $V$. The lifting of $d\rho$ to $\tilde{L}_0(0)$ with which we shall be mainly concerned is also denoted by the same notation. We assume that $d\rho$ is explicitly expressed as

$$d\rho(\{a^{\mu}_{\Lambda,0}\})=\sum\Psi^{X\alpha}_{\mu\beta}\,a^{\mu}_{X,0}e_{\alpha}\otimes e^{\beta}\,.$$

Making use of this expression, we shall construct a differential representation of $\Gamma(T)$ on $\Gamma(W)$. Take a local coordinates neighborhood $(U;z^1,\cdots,z^n)$ and a canonical local base $\{\tilde{e}_{\alpha}\}$ of $W$ on $U$. For any $\xi=\sum\xi^{\mu}(z,\bar{z})\,\partial/\partial z^{\mu}\in\Gamma(T|U)$, set

$$\rho^{\#}(\xi)\sigma=\sum_{\mu}\xi^{\mu}(z,\bar{z})\frac{\partial\sigma^{\alpha}}{\partial z^{\mu}}\tilde{e}_{\alpha}+\sum\Psi^{X\alpha}_{\mu\beta}\frac{\partial^{|X|}\xi^{\mu}}{\partial z^{X}}\sigma^{\beta}\tilde{e}_{\alpha}\,,\tag{3.5}$$

where $\sigma=\sum\sigma^{\alpha}\tilde{e}_{\alpha}\in\Gamma(W|U)$. Then, by the similar reasonings as in [**6**; Section 4], we can show

PROPOSITION 3.1. *$\rho^{\#}$ gives rise to a differential representation of $\Gamma(T)$ on $\Gamma(W)$.*

Hence we obtain the complex $\{C^p[T,W],d\}$ associated to $\rho^{\#}$. From the local expression (3.5), it is verified that for each $b$ $(b=0,1,2,\cdots)$ we have

$$d(C^p(b)[T,W])\subset C^{p+1}(b)[T,W]\,,$$

which implies that each $\{C^p(b)[T,W],d\}$ becomes a subcomplex of $\{C^p[T,W],d\}$. Specifically, $\{C^p(b)[T,W],d\}$ is an inductive differential complex. We denote its cohomology group by $H^*(b)(T,W)$.

PROPOSITION 3.2. *For each $b$, $\{C^p(b)[T,W],d\}$ is elliptic.*

In fact, it is easy to see that the symbol sequence

$$\cdots\longrightarrow C^p_a(b)(T,W)_x\xrightarrow{\sigma(d,\,\eta_x)}C^{p+1}_a(b)(T,W)_x\longrightarrow\cdots$$

is exact for $a=0,1,2,\cdots$, where $\eta_x\in\tau^*(M)_x$ $(\eta_x\neq 0)$. Now let $a_0$ be the smallest integer to satisfy $\Psi^{X\alpha}_{\mu\beta}=0$ for $|X|\geqq a_0$. Then, as to the admissible filtrations, we have:

PROPOSITION 3.3. i) *In case $b=0$, the $\partial$-jet filtration $\varinjlim_a C^p_a(0)(T,W)=C^p(0)(T,W)$ gives rise to an admissible filtration for $a\geqq a_0$.*

ii) *In case $b\geqq 1$, the $\partial$-jet filtration with height 1*

$$\varinjlim_a C^p_{p+a}(b)(T,W)=C^p(b)(T,W)$$

*gives rise to an admissible filtration for $a\geqq a_0$.*

The proof is easy. It should be noted that, if $b\geqq 1$, $d$ does never send $C^p_a(b)(T,W)$ to $C^{p+1}_a(b)(T,W)$.

The representation $d\rho$ of $\tilde{L}_0(0)$ to $V$ is further lifted to $\tilde{L}_0$, which furnishes a complex analytic representation of $\tilde{L}_0$ on $V$ (cf. Definition 3.1). Hence, associated to this representation, we can obtain a complex

$$\{C^p(\tilde{L}_0(b),V),d\}$$

for each $b=0, 1, 2, \cdots$. Assume that $\rho$ is decomposable (i.e., $\rho|\mathfrak{gl}(n; \boldsymbol{C})$ is completely reducible). Then by Theorem 3.1 we have

$$\dim H^*(\tilde{L}_0(b), V) < +\infty$$

for $b=0, 1, 2, \cdots$. Let $\bar{\mathcal{O}}$ be the sheaf of germs of anti-analytic functions on $M$, and let

$$\underline{H}^*(\tilde{L}_0(b), V)(\bar{\mathcal{O}}) = \sum \underline{H}^p(\tilde{L}_0(b), V)(\bar{\mathcal{O}})$$

be the sheaf of module over $\bar{\mathcal{O}}$, locally isomorphic to

$$H^*(\tilde{L}_0(b), V) \otimes_C \bar{\mathcal{O}}.$$

Note that $\underline{H}^*(\tilde{L}_0(b), V)(\bar{\mathcal{O}})$ may be identified with the sheaf of germs of anti-analytic cross-sections of some anti-analytic vector bundle.

Our concern lies in establishing a relation between $H^*(b)(T, W)$ and $H^*(\tilde{L}_0(b), V)$ in terms of a spectral sequence. However, using Poincaré's lemma on $\partial$-operators and referring to Proposition 1.1, we find easily that, even in this case, it is possible to apply the arguments similar to those which we have already used in the proof of Theorem 4.1 in [**6**]. This leads us to formulate the following

THEOREM 3.2. *For each $b=0, 1, 2, \cdots$, there is a spectral sequence $\{E_r^{p,q}(b), d_r(b)\}$ which converges to a graded module associated to $H^*(b)(T, W)$ with some filtration. The $E_2$-terms of this spectral sequence have the form*

$$E_2^{p,q} \cong H^p(M, \underline{H}^q(\tilde{L}_0(b), V)(\bar{\mathcal{O}})).$$

Therefore, by virtue of Cartan-Serre's theorem [**3**], combined with Theorem 3.1, we can deduce

THEOREM 3.3. *Let $M$ be compact and let the representation $\rho$ be decomposable. Then the cohomology group $H^*(b)(T, W)$ associated to $\rho$ is finite-dimensional for each $b=0, 1, 2, \cdots$.*

COROLLARY. *$H^*(T, W)$ is expressed as an inductive limit space of finite-dimensional vector spaces. More precisely, we have*

$$H^*(T, W) = \varinjlim \{H^*(b)(T, W), (i_b^{b'})_*\},$$

*where $i_b^{b'}$ denotes the injection $\{C(b)(T, W), d\} \subsetneq \{C(b')(T, W), d\}$ for $b<b'$.*

PROBLEM. Find the value of $\sum(-1)^p \dim H^p(b)(T, W)$ for $b=0, 1, 2, \cdots$.

Finally, we remark that, if the representation $\rho$ is trivial, then we shall be able to follow the similar discussion for the Lie algebra $\Gamma(\bar{T})$, which will yield a result corresponding to the above. This may be regarded as a generalization of a well-known result concerning Dolbeault complex.

## References

[1] I. Amemiya, K. Masuda and K. Shiga, Lie algebras of differential operators, Osaka J. Math. (to appear).

[2] M.F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes, I, Ann. of Math., **86** (1967), 374-407.

[3] H. Cartan and J.P. Serre, Un théorème de finitude concernant les variétés analytiques compactes, C.R. Acad Sci., Paris, **237** (1953), 128-130.

[4] I.M. Gelfand and D.B. Fuks, Cohomologies of Lie algebra of vector fields with non-trivial coefficients, Functional Analysis and their Applications, **4** (1970), 10-25.

[5] M.V. Losik, On the cohomologies of infinite-dimensional Lie algebras of vector fields, Functional Analysis and their Applications, **4** (1970), 127-135.

[6] K. Shiga, Cohomology of Lie algebras over a manifold, I, J. Math. Soc. Japan, **26** (1974), 324-361.

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